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# IDEALS, $\ell$-RINGS AND MV$^\star$-ALGEBRAS

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ABSTRACT. MV$^\star$-algebras constitute a subcategory of perfect MV-algebras categorically equivalent to l-rings. In this paper we study the ideals of MV$^\star$-algebras in connection with the l-ideals of the associated l-ring. The most important results of this paper are concerning with the MV$f$-algebras, a subclass of MV$^\star$-algebras corresponding to $f$-rings.

## 1. Introduction

MV-algebras were introduced in 1958 by C. C. Chang as algebraic models for Lukasiewicz infinite valued logic. In 1986, D. Mundici proved that the category of MV-algebras is equivalent to the category of abelian l-groups with strong unit (see [6]). This result was followed by an impressive growth of the theory of MV-algebras. The best reference on MV-algebras is the book [6].

In [7] A. Di Nola and A. Lettieri established a categorical equivalence between the category of perfect MV-algebras and the category of abelian l-groups. This result was extended in [4] by L. P. Belluce, A. Di Nola and G. Georgescu. They proved that the l-rings are categorically equivalent to the MV$^\star$-algebras, a subcategory of perfect MV-algebras.

The aim of this paper is to study the ideals in MV$^\star$-algebras in connection with the l-ideals in the associated l-rings. We also include some results given in [4] in an outlined form.

Section 2 contains some basic notions and results on $\star$-ideals in a $\star$-algebra. In Section 3 we define $f$-algebras, an important class of $\star$-algebras corresponding to $f$-rings, and in Section 4 we study the $\star$-prime ideals in $f$-algebras. Section 5 is devoted to some MV-versions of some results of M. Henriksen [4] and S. Larson [10], [11], [12], [13], [14], [15], and in Section 6, to chain condition in $f$-algebras. The paper ends with the investigation of two kinds of reticulations associated with an $f$-algebra.

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Let $(A, +, \cdot, {}^*, 0, 1)$ be an MV-algebra. We shall write $xy$ instead of $x \cdot y$. Recall that the lattice operations in $A$ are given by $x \vee y = xy^* + y$ and $x \wedge y = (x + y^*)y$. For $x$, $y$ in $A$ denote $d(x, y) = xy^* + x^*y$. Any ideal $I$ of $A$ induces a congruence on $A$: $x \equiv y \pmod{I}$ if and only if $d(x, y) \in I$. The corresponding quotient MV-algebra will be denoted by $A/I$, and $\mathrm{Id}\, A$ will be the complete lattice of ideals in $A$.

The radical $\mathrm{Rad}\, A$ is the intersection of the maximal ideals in $A$. An MV-algebra $A$ is perfect if $A = \mathrm{Rad}\, A \cup (\mathrm{Rad}\, A)^*$, where $(\mathrm{Rad}\, A)^* = \{x^* : x \in \mathrm{Rad}\, A\}$ (see [7]).

Consider a perfect MV-algebra $A$ and define a congruence $\theta$ on $\mathrm{Rad}\, A \times \mathrm{Rad}\, A$: $(x, y)\, \theta\, (u, v)$ if and only if $x + v = y + u$. Denote by $[x, y]$ the class of $(x, y) \in \mathrm{Rad}\, A \times \mathrm{Rad}\, A$ and $D(A) = (\mathrm{Rad}\, A \times \mathrm{Rad}\, A)/\theta$. Thus $D(A)$ is an abelian l-group with the following properties for $x, y, u, v \in \mathrm{Rad}\, A$:

$$[x, y] + [u, v] = [x + u, y + v],$$
$$[x, y] \leq [u, v] \iff x + v \leq u + y,$$
$$[x, y] \wedge [u, v] = \big[(x + v) \wedge (u + y), y + v\big],$$
$$[x, y] \vee [u, v] = \big[x + u, (x + v) \wedge (u + y)\big].$$

In fact $D$ is a functor from the category of perfect MV-algebras to the category of abelian l-groups.

For any $[x, y] \in D(A)$ one can prove that $[x, y] = [xy^*, x^*y]$, $[x, y]^+ = [xy^*, 0]$, $[x, y]^- = [x^*y, 0]$ and $|[x, y]| = \big[d(x, y), 0\big]$.

For an abelian l-group $G$ consider the lexicographic product $\mathbb{Z} \times G$. $(1, 0)$ is a strong unit in $\mathbb{Z} \times G$, so we can take $\Delta(G) = \Gamma\big(\mathbb{Z} \times G, (1, 0)\big)$, where $\Gamma$ is the Mundici functor (see [6]). Thus $\Delta(G)$ is a perfect MV-algebra and the functors $D$ and $\Delta$ establish a categorical equivalence between perfect MV-algebras and abelian l-groups [7].

An MV$^\star$-algebra $(A, \star)$ ($=$ $\star$-algebra) is a perfect MV-algebra $A$ with a binary operation $\star$ on $\mathrm{Rad}\, A$ fulfilling the following conditions, for $x, y, z \in \mathrm{Rad}\, A$:

(a) $x \star (y \star z) = (x \star y) \star z$;
(b) $x \star (y + z) = x \star y + x \star z$, $(y + z) \star x = y \star x + z \star x$;
(c) $x \star 0 = 0 \star x = 0$.

If $(K, +, \cdot, 0, 1)$ is an l-ring and $K_+ = (K, +, 0)$ the additive l-group of $K$, then the perfect MV-algebra $\Delta(K_+) = \Gamma\big(\mathbb{Z} \times K_+, (1, 0)\big)$ is an MV$^\star$-algebra by putting $(0, x) \star (0, y) = (0, xy)$ for all $x$, $y \geq 0$ in $K$. Conversely, assume $(A, \star)$ is an MV$^\star$-algebra and define a multiplication on the l-group $D(A)$:

$$[a, b] \cdot [c, d] = [a \star c + b \star d,\ a \star d + b \star c].$$

Thus $\big(D(A), \cdot\big)$ is an l-ring and the above constructions give a categorical equivalence between MV$^\star$-algebras and l-rings ([2]).

If $A$ is a perfect MV-algebra and $I$, a proper ideal of $A$, then $D(I) = \{[x,y] : x,y \in I\}$ is a convex l-subgroup of $D(A)$. The map $I \mapsto D(I)$ is a bijection between the proper ideals of $A$ and convex l-subgroups of $D(A)$ (see [3]).

The background for l-rings can be found in [5], [9].

## 2. $\star$-Ideals

This section contains basic notions and results on the $\star$-ideals of an $\star$-algebra.

Let $(A,\star)$ be a $\star$-algebra. A *$\star$-ideal* in $A$ is an ideal $I \subseteq \operatorname{Rad} A$ such that $a \in I$ & $b \in \operatorname{Rad} A \implies a \star b,\ b \star a \in I$.

Similarly, one can define the left and right $\star$-ideals.

**PROPOSITION 2.1.** *For an ideal $I \subseteq \operatorname{Rad} A$ the following are equivalent:*

(1) *$I$ is a $\star$-ideal;*

(2) *$D(I)$ is an $\ell$-ideal in the $\ell$-ring $D(A)$.*

Proof.

(1) $\implies$ (2):

Assume $[a,b] \in D(I)$, $a,b \in I$ and $[c,d] \in D(A)$ $a,b \in D(A)$. Then $a \star c$, $b \star d$, $a \star d$, $b \star c \in I$ and $a \star c + b \star d$, $a \star d + b \star c \in I$. Therefore

$$[a,b] \cdot [c,d] = [a \star c + b \star d,\ a \star d + b \star c] \in D(I)\,.$$

(2) $\implies$ (1):

Assume $a \in I$, $b \in \operatorname{Rad} A$, so $[a,0] \in D(I)$, $[b,0] \in D(A)$, hence $[a \star b, 0] = [a,0] \cdot [b,0] \in D(I)$. It follows that $a \star b \in I$. □

**PROPOSITION 2.2.** *If $J$ is an $\ell$-ideal in the $\ell$-ring $K$, then $\triangle(J^+)$ is a $\star$-ideal in $\star$-algebra $\triangle(K) = \Gamma(Z \times K_+, (1,0))$.*

**LEMMA 2.1.** *If $I$ is a $\star$-ideal and $x_1, x_2, y_1, y_2 \in \operatorname{Rad} A$, then*

$$x_1/I = x_2/I \ \&\ y_1/I = y_2/I \implies (x_1 \star y_1)/I = (x_2 \star y_2)/I\,.$$

Proof. If $x_1/I = x_2/I$, then $x_1 x_2^\star$, $x_1^\star x_2 \in I$ and $x_1 + x_1^\star x_2 = x_1 \vee x_2 = x_2 + x_2^\star x_1$, so there exist $a_1, a_2 \in I$ such that $x_1 + a_1 = x_2 + a_2$. Similarly, $y_1 + b_1 = y_2 + b_2$ for some $b_1, b_2 \in I$. Thus $(x_1+a_1)\star(y_1+b_1) = (x_2+a_2)\star(y_2+b_2)$, so $(x_1 \star y_1) + c_1 = (x_2 \star y_2) + c_2$ for some $c_1, c_2 \in I$ since $I$ is a $\star$-ideal. Thus

$$(x_1 \star y_1)/I = (x_1 \star y_1)/I + c_1/I = (x_2 \star y_2)/I + c_2/I = (x_2 \star y_2)/I\,.$$

□

**Remark 2.1.** It is obvious that $\operatorname{Rad}(A/I) = (\operatorname{Rad} A)/I$. By this Lemma one can define $\star$: $\operatorname{Rad} A/I \star \operatorname{Rad} A/I \to \operatorname{Rad} A/I$ by putting $(x/I)\star(y/I) = (x \star y)/I$. It is easy to prove that $A/I$ becomes a $\star$-algebra.

**PROPOSITION 2.3.** *If $I$ is a $\star$-ideal in $A$, then the $\ell$-rings $D(A/I)$ and $D(A)/D(I)$ are isomorphic.*

P r o o f. We shall prove that, for $a_1, a_2, b_1, b_2 \in \operatorname{Rad} A$, the following holds:

$$[a_1/I, b_1/I] = [a_2/I, b_2/I] \iff [a_1, b_1]/D(I) = [a_2, b_2]/D(I). \qquad (\star)$$

If $[a_1/I, b_1/I] = [a_2/I, b_2/I]$, then $a_1 + b_2 \equiv b_1 + a_2 \pmod I$, so $d(a_1 + b_2, b_1 + a_2) \in I$.

It follows that

$$|[a_1, b_1] - [a_2, b_2]| = |[a_1 + b_2,\, b_1 + a_2]| = [d(a_1 + b_2,\, b_1 + a_2), 0] \in D(I).$$

But $D(I)$ is an $\ell$-ideal, so $[a_1, b_1] - [a_2, b_2] \in D(I)$, i.e. $[a_1, b_1]/D(I) = [a_2, b_2]/D(I)$.

Conversely, if $[a_1, b_1]/D(I) = [a_2, b_2]/D(I)$, then

$$[d(a_1 + b_2,\, b_1 + a_2), 0] = |[a_1, b_1] - [a_2,\, b_2]| \in D(I),$$

therefore $d(a_1 + b_2,\, b_1 + a_2) \in I$, so $a_1 + b_2 \equiv b_1 + a_2 \pmod I$, etc..

Thus one can define a map $[a/I, b/I] \mapsto [a, b]/D(I)$ which is an isomorphism of $\ell$-rings. □

**Remark 2.2.** Any intersection of $\star$-ideals is a $\star$-ideal. Consider a family $I_\lambda$, $\lambda \in \Lambda$, of $\star$-ideals and its supremum $\bigvee I_\lambda$ in $\operatorname{Id} A$. It is easy to prove that $\bigvee I_\lambda$ is a $\star$-ideal. Thus the set $\mathcal{I}\mathrm{d}\, A$ of $\star$-ideals of $A$ is a complete sublattice of $\operatorname{Id} A$.

**PROPOSITION 2.4.** *The map $I \mapsto D(I)$ is a lattice isomorphism between $\mathcal{I}\mathrm{d}\, A$ and the lattice $\mathcal{I}\mathrm{d}\, D(A)$ of the $\ell$-ideals in $D(A)$.*

P r o o f. It is known that $I \mapsto D(I)$ is a lattice isomorphism between $\operatorname{Id} A - \{A\}$ and the lattice $\operatorname{Id} D(A)$ of the convex $\ell$-subgroups of $D(A)$. By Proposition 2.1 one can take the restriction of this isomorphism to $\mathcal{I}\mathrm{d}\, A$.

By [5; 8.2.2] and Proposition 2.3, it follows that we have in $\mathcal{I}\mathrm{d}\, A$:

$$\left(\bigvee I_\lambda\right) \cap J = \bigvee (I_\lambda \cap J).$$

This also follows from the distributivity of $\operatorname{Id} A$. For $M \subseteq \operatorname{Rad} A$ let us denote

$$\operatorname{id}(M) = \text{the ideal generated by } M,$$
$$\langle M \rangle = \text{the } \star\text{-ideal generated by } M.$$

□

**PROPOSITION 2.5.** *We have*

$$\langle M\rangle = \{x \in \operatorname{Rad} A:\ x \le u + t\star u + u\star t + t\star u\star t,\ \ u \in \operatorname{id}(M),\ \ t \in \operatorname{Rad} A\}.$$

Proof. If $J$ is the right member, then it is clear that $J \subseteq \langle M\rangle$ and $M \subseteq J$. We shall prove that $J$ is a $\star$-ideal. If $x_1, x_2 \in J$, then $x_i \le u_i + t_i\star u_i + u_i\star t_i + t_i\star u_i\star t_i$, $u_i \in \operatorname{id}(M)$, $t_i \in \operatorname{Rad} A$, $i = 1,2$. Thus $x_1 + x_2 \le u + t\star u + u\star t + t\star u\star t$ with $u = u_1 + u_2 \in \operatorname{id}(M)$, $t = t_1 + t_2 \in \operatorname{Rad} A$.

If $x \le u + t\star u + u\star t + t\star u\star t$ and $a \in \operatorname{Rad} A$, then $a\star x \le s\star u + s\star u\star s$ with $s = a + a\star t + t \in \operatorname{Rad} A$. □

**COROLLARY 1.** $\langle a\rangle = \{x:\ x \le na + s\star a + a\star s + s\star a\star s,\ \ s \in \operatorname{Rad} A\}$ *for* $a \in \operatorname{Rad} A$.

**LEMMA 2.2.** $D(\langle a\rangle) = \langle [a,0]\rangle$ *for* $a \in \operatorname{Rad} A$.

Proof. Assume $u \in D(\langle a\rangle)^+$. Then $u = [x,0]$ with $x \in \langle a\rangle$. Hence $x \le na + s\star a + a\star s + s\star a\star s$ with $s \in \operatorname{Rad} A$.

It follows that $u = [x,0] \le n[a,0] + [s,0]\cdot[a,0] + [a,0]\cdot[s,0] + [s,0]\cdot[a,0]\cdot[s,0]$. So $u \in \langle [a,0]\rangle$ by [4; 8.2.7]. The converse inclusion is similar. □

**COROLLARY 2.** *For* $x, y \in \operatorname{Rad} A$ *we have:*

(1) $\langle x\star y\rangle \subseteq \langle x\rangle \cap \langle y\rangle$,
(2) $\langle x\rangle \vee \langle y\rangle = \langle x \vee y\rangle = \langle x + y\rangle$.

Proof. By [5; 8.2.8] and Lemma 2.2, or directly, using Corollary 1. □

For any $\star$-ideals $I$, $J$ define

$$I\star J = \langle \{a\star b:\ a \in I,\ \ b \in J\}\rangle.$$

**PROPOSITION 2.6.** *For any* $\star$*-ideals* $I_1$, $I_2$ *we have*

$$I_1\star I_2 = \{x \in \operatorname{Rad} A:\ x \le a\star b,\ \ a \in I_1,\ \ b \in I_2\}.$$

Proof. If $J$ is the right member, then $J \subseteq I_1\star I_2$, and $a \in I_1$, $b \in I_2$ imply $a\star b \in J$. Thus it suffices to prove $J$ is a $\star$-ideal. For example, if $x_i \le a_i\star b_i$, $a_i \in I_1$, $b_i \in I_2$, $i = 1,2$, then $x_1 + x_2 \le a_1\star b_1 + a_2\star b_2 \le (a_1 + a_2)\star(b_1 + b_2)$, $a_1 + a_2 \in I_1$, $b_1 + b_2 \in I_2$, so $x_1 + x_2 \in J$. □

**PROPOSITION 2.7.** $D(I_1\star I_2) = D(I_1)\cdot D(I_2)$.

Proof. Assume $u \in D(I_1\star I_2)^+$, so $u = [x,0]$ with $x \in I_1\star I_2$, i.e. $x \le a_1\star a_2$ with $a_1 \in I_1$, $a_2 \in I_2$, therefore $u = [x,0] \le [a_1\star a_2, 0] = [a_1,0]\cdot[a_2,0]$ and $[a_1,0] \in D(I_1)$, $[a_2,0] \in D(I_2)$, hence $u \in D(I_1)\cdot D(I_2)$ by [5; 8.2.11]. The converse inclusion is similar. □

**COROLLARY 3.**

(1) $I \star (J \star K) = (I \star J) \star K$;

(2) $I \star (\bigvee I_\lambda) = \bigvee (I \star I_\lambda)$, $(\bigvee I_\lambda) \star I = \bigvee (I_\lambda \star I)$.

Proof. By [5; 8.2.12] and Proposition 2.3, 2.7 or directly, by Proposition 2.6. Thus $(\mathcal{I}\mathrm{d}\, A, \star)$ is a quantale. □

**PROPOSITION 2.8.** *The map $I \mapsto D(I)$ is a quantale isomorphism between $(\mathcal{I}\mathrm{d}\, A, \star)$ and $(\mathcal{I}\mathrm{d}\, D(A), \cdot)$.*

Proof. By Propositions 2.4 and 2.7. □

One can define $I^{(n)} = \underbrace{I \star \cdots \star I}_{n\text{-times}}$. Thus $D(I^{(n)}) = (D(I))^n$.

For $a \in \mathrm{Rad}\, A$ let us denote $a^{(n)} = \underbrace{a \star \cdots \star a}_{n\text{-times}}$.

**PROPOSITION 2.9.** $I^{(n)} = \{x \in \mathrm{Rad}\, A : \ x \le a^{(n)}, \ a \in I\}$.

**PROPOSITION 2.10.** *For a $\star$-ideal $I \ne \mathrm{Rad}\, A$ the following are equivalent:*

(1) $(\forall I_1, I_2 \in \mathcal{I}\mathrm{d}\, A)(I_1 \cap I_2 = I \implies (I = I_1 \text{ or } I = I_2))$;

(2) $(\forall I_1, I_2 \in \mathcal{I}\mathrm{d}\, A)(I_1 \cap I_2 \subseteq I \implies (I_1 \subseteq I \text{ or } I_2 \subseteq I))$;

(3) $(\forall a, b \in \mathrm{Rad}\, A)(\langle a \rangle \cap \langle b \rangle \subseteq I \implies (a \in I \text{ or } b \in I))$.

A $\star$-star ideal $I \ne \mathrm{Rad}\, A$ satisfying these properties will be called *irreducible.* It is easy to see that any maximal $\star$-ideal is irreducible.

**LEMMA 2.3.** *For $I \in \mathcal{I}\mathrm{d}\, A$, $I$ is irreducible if and only if $D(I)$ is an irreducible $\ell$-ideal.*

Proof. By Propositions 2.4 and 2.10. □

**LEMMA 2.4.** *For $I \in \mathcal{I}\mathrm{d}\, A$ and $a \in \mathrm{Rad}\, A - I$ there is an irreducible $\star$-ideal $P$ such that $I \subseteq P$ and $a \notin P$.*

Proof. Let $P$ be a $\star$-ideal maximal with respect $I \subseteq P$, $a \notin P$. Assume $\langle a \rangle \cap \langle b \rangle \subseteq P$, $a, b \notin P$. Therefore $P \vee \langle a \rangle = P \vee \langle b \rangle = \mathrm{Rad}\, A$, so $P = P \vee (\langle a \rangle \cap \langle b \rangle) = (P \vee \langle a \rangle) \cap (P \vee \langle b \rangle) = \mathrm{Rad}\, A$, which is a contradiction. □

**PROPOSITION 2.11.** *Any proper $\star$-ideal is an intersection of irreducible $\star$-ideals.*

Proof. By Lemma 2.4. □

Particularly, the intersection of all irreducible $\star$-ideals in $A$ is $\{0\}$.

An element $x \in \mathrm{Rad}\, A$ is *$\star$-nilpotent* if $x^{(n)} = 0$ for some integer $n \ge 1$.

$A$ is *$\star$-semiprime* if there is no non-zero $\star$-nilpotent element of $A$.

$A$ is a *$\star$-domain* if $x \star y = 0$ implies $x = 0$ or $y = 0$.

**LEMMA 2.5.** *Any totally-ordered $\star$-semiprime $\star$-algebra $A$ is a $\star$-domain.*

Proof. Assume $x \star y = 0$. If $x \leq y$, then $x^2 \leq x \star y = 0$, so $x^2 = 0$, hence $x = 0$. □

A $\star$-ideal $I$ is *$\star$-nilpotent* if $I^{(n)} = \{0\}$ for some integer $n \geq 1$.

A $\star$-ideal $I$ is *$\star$-semiprime* if $A/I$ is a $\star$-semiprime $\star$-algebra. One can see that $I$ is $\star$-semiprime if and only if $x^{(2)} \in I$ implies $x \in I$ for any $x \in \operatorname{Rad} A$.

## 3. $f$-Algebras

In this section we shall study the MV$f$-algebras, a subclass of MV*-algebras corresponding to the $f$-rings.

For any subset $M \subseteq \operatorname{Rad} A$, $M^{\perp} = \{a \in A : a \wedge m = 0, m \in M\}$ is an ideal included in $\operatorname{Rad} A$, $x < y$ for any $x \in \operatorname{Rad} A$, $y \in (\operatorname{Rad} A)^{\star}$.

**LEMMA 3.1.** *If $P$ is a minimal prime ideal in $A$, then $P = \bigcup\{x^{\perp} : x \notin P\}$.*

Proof. Assume $x \in P$, so $x \wedge y = 0$ for some $y \notin P$ since $P$ is minimal prime. Thus $y \in x^{\perp}$. The converse is obvious. □

**PROPOSITION 3.1.** *For a $\star$-algebra $A$ the following are equivalent:*

(1) $a, b, x \in \operatorname{Rad} A$ & $a \wedge b = 0 \implies a \wedge (b \star x) = a \wedge (x \star b) = 0$;
(2) *for any $I \subseteq \operatorname{Rad} A$, $I^{\perp}$ is a $\star$-ideal;*
(3) *any $P \in \operatorname{Min} A$ is a $\star$-ideal.*

Proof.

(1) $\implies$ (2):
If $a \in \operatorname{Rad} A$, $b \in I^{\perp}$, then $b \wedge x = 0$ for $x \in I$, therefore $(a \star b) \wedge x = 0$, i.e. $a \star b \in I^{\perp}$.

(2) $\implies$ (3):
By Lemma 3.1.

(3) $\implies$ (1):
Consider $a, b, x \in \operatorname{Rad} A$, $a \wedge b = 0$ and $P \in \operatorname{Min} A$, hence $a \in P$ or $b \in P$. If $a \in P$, then $a \wedge (b \star x) \in P$ because $a \wedge (b \star x) \leq a$. If $b \in P$, then $b \star x \in P$, so $a \wedge (b \star x) \in P$. It follows that $a \wedge (b \star x) \in \bigcap \operatorname{Min} A = \{0\}$. □

A $\star$-algebra $A$ satisfying these properties will be called an *MV$f$-algebra* (= *$f$-algebra*).

**PROPOSITION 3.2.** *For a $\star$-algebra $A$ the following are equivalent:*

(1) *$A$ is an $f$-algebra;*
(2) *$A$ is a subdirect product of totally-ordered $\star$-algebra.*

Proof.

(1) $\Longrightarrow$ (2):
In accordance to Proposition 3.1(3) and $\bigcap \operatorname{Min} A = \{0\}$, $A \hookrightarrow \Pi\{A/P : P \in \operatorname{Min} A\}$ is the desired representation of $A$.

(2) $\Longrightarrow$ (1):
Any totally-ordered $\star$-algebra is an $f$-algebra. □

**PROPOSITION 3.3.** *The following stationes are equivalent:*

(1) *$A$ is an $f$-algebra;*
(2) *$D(A)$ is an $f$-ring.*

Proof.

(1) $\Longrightarrow$ (2):
Consider $u, v, w \geq 0$ in $D(A)$ such that $u \wedge v = 0$, so $u = [a, 0]$, $v = [b, 0]$, $w = [x, 0]$ with $a, b, x \in \operatorname{Rad} A$. Thus $[a \wedge b, 0] = [a, 0] \wedge [b, 0] = [a, 0]$, so $a \wedge b = 0$, hence $a \wedge (b \star x) = 0$. We get

$$u \wedge (v \cdot w) = [a, 0] \wedge \big([b, 0] \cdot [x, 0]\big) = \big[a \wedge (b \star x), 0\big] = [0, 0].$$

(2) $\Longrightarrow$ (1):
Similarly. □

**PROPOSITION 3.4.** *If $A$ is an $f$-algebra, then, for $a, b, a', b', x \in \operatorname{Rad} A$, we have:*

$$\begin{aligned} x \star (a \vee b) &= (x \star a) \vee (x \star b), \qquad & x \star (a \wedge b) &= (x \star a) \wedge (x \star b); \\ (a \vee b) \star x &= (a \star x) \vee (b \star x), \qquad & (a \wedge b) \star x &= (a \star x) \wedge (b \star x); \end{aligned} \tag{a}$$

$$a \wedge b = 0 \Longrightarrow a \star b = 0; \tag{b}$$

$$d(a, b) \star d(a', b') = d(a \star a' + b \star b',\, a \star b' + b \star a'). \tag{c}$$

Proof. We shall prove only (c). By [5; 9.1.10(iii)]:

$$|[a, b] \cdot [a,' b']| = |[a, b]| \cdot |[a', b']|,$$

therefore

$$\begin{aligned} \big[d(a \star a' + b \star b',\, a \star b' + b \star a'), 0\big] &= |[a \star a' + b \star b',\, a \star b' + b \star a']| \\ &= |[a, b] \cdot [a', b']| = |[a, b]| \cdot |[a', b']| \\ &= \big[d(a, b), 0\big] \cdot \big[d(a', b'), 0\big] \\ &= \big[d(a, b) \star d(a', b'), 0\big]. \end{aligned}$$

□

**PROPOSITION 3.5.** *For a $\star$-algebra $A$ the following are equivalent:*

(1) *$A$ is an $f$-algebra.*

(2) *For any irreducible $\star$-ideal $P$ of $A$, $A/P$ is a totally-ordered $\star$-algebra.*

Proof.

(1) $\Longrightarrow$ (2):

Assume $A/P$ is not totally-ordered for some irreducible $P$. Thus there exist $a/P, b/P \in A/P$, $a/P \not\leq b/P$ and $b/P \not\leq a/P$. One can assume $a, b \in \operatorname{Rad} A$. We have $ab^\star/P \neq 0/P$, $ba^\star/P \neq 0/P$. Denoting $x = ab^\star$, $y = a^\star b$ we have $x, y \notin P$, $x \wedge y = 0$.

Thus $x^\perp, x^{\perp\perp} \not\subseteq P$ and $x^\perp \cap x^{\perp\perp} = \{0\}$. But $x^\perp$, $x^{\perp\perp}$ are $\star$-ideals since $A$ is an $f$-algebra. This contradicts the fact that $P$ is irreducible.

(2) $\Longrightarrow$ (1):

The intersection of all irreducible $\star$-ideals of $A$ is $\{0\}$, so $A$ is a subdirect product of totally-ordered $\star$-algebra, hence $A$ is an $f$-algebra by Proposition 3.2.

$\square$

**PROPOSITION 3.6.** *If $A$ is an $f$-algebra, then $\langle a\rangle \cap \langle b\rangle = \langle a \wedge b\rangle$ for any $a, b \in \operatorname{Rad} A$.*

Proof. $D(A)$ is an $f$-ring, so, by [5; 9.1.8] and Lemma 2.2:

$$\begin{aligned} D\big(\langle a\rangle \cap \langle b\rangle\big) = D\big(\langle a\rangle\big) \cap \big(\langle b\rangle\big) &= \big\langle [a,0]\big\rangle \cap \big\langle [b,0]\big\rangle \\ &= \big\langle [a,0]\big\rangle \cap \big\langle [b,0]\big\rangle = \big\langle [a \wedge b, 0]\big\rangle = D\big(\langle a \wedge b\rangle\big)\,. \end{aligned}$$

By Proposition 2.4, $\langle a\rangle \cap \langle b\rangle = \langle a \wedge b\rangle$. $\square$

## 4. $\star$-Prime ideals in $f$-algebras

In this section we shall introduce the $\star$-prime ideals in an $f$-algebra. They correspond to prime ideals in an $f$-ring and will have a main role in this paper.

**LEMMA 4.1.** *If $A$ is an $f$-algebra and $x, y \in \operatorname{Rad} A$, then we have:*

$$x \star y \leq y \star x \implies x^{(n)} \star y^{(n)} \leq (x \star y)^{(n)} \leq (y \star x)^{(n)} \leq y^{(n)} \star x^{(n)}\,.$$

Proof. By [5; 9.2.1] we have in $D(A)$:

$$\begin{aligned} x \star y \leq y \star x \implies [x,0] \star [y,0] &\leq [y,0] \cdot [x,0] \\ \implies [x,0]^n \cdot [y,0]^n &\leq \big([x,0] \cdot [y,0]\big)^n \leq \big([y,0] \cdot [x,0]\big)^n \\ &\leq [y,0]^n \cdot [x,0]^n \\ \implies \big[x^{(n)} \star y^{(n)}, 0\big] &\leq \big[(x \star y)^{(n)}, 0\big] \leq \big[(y \star x)^{(n)}, 0\big] \\ &\leq \big[y^{(n)} \star x^{(n)}, 0\big]\,, \end{aligned}$$

which gives the inequality of the lemma. □

Now consider $A$ totally-ordered. Define $U_n = \{x \in \operatorname{Rad} A :\ x^{(n)} = 0\}$.

**LEMMA 4.2.**

(1) $x, y \in U_n \implies x + y \in U_n$;
(2) $x \in U_n$ & $y \in \operatorname{Rad} A \implies x \star y,\ y \star x \in U_n$;
(3) $x \le y \in U_n \implies x \in U_n$.

Proof.
(1) If $x \le y$, then $(x+y)^{(n)} \le 2^{(n)} \cdot y^{(n)} = 0$.
(2) Assume $x \star y \le y \star x$. By Lemma 4.1,

$$(x \star y)^{(n)} \le (y \star x)^{(n)} \le y^{(n)} \star x^{(n)} = 0\,.$$

(3) $x \le y \in U_n \implies x^{(n)} \le y^{(n)} = 0$. □

**COROLLARY 4.** *If $A$ is totally-ordered and $x \in \operatorname{Rad} A$, then*

$$x^{(n)} = 0 \implies \langle x \rangle^{(n)} = \{0\}\,.$$

Proof. (For $n = 2$) Assume $x^{(2)} = 0$. We have $\langle x \rangle^{(2)} = \{y \in \operatorname{Rad} A :\ y \le a^{(2)},\ a \in \langle x \rangle\}$. If $y \le a^{(2)}$ with $a \le nx + u \star x + x \star u + u \star x \star u$, then, by the previous lemma, $a \in U_2$, so $y \le a^{(2)} = 0$, i.e. $y = 0$. □

**DEFINITION 4.1.** A $\star$-ideal $P \ne \operatorname{Rad} A$ is *$\star$-prime* (resp. *completely $\star$-prime*) if $I \star J \subseteq P \implies I \subseteq P$ or $J \subseteq P$ (resp. $x \star y \in P \implies x \in P$ or $y \in P$) for any $I, J \in \mathcal{I}\mathrm{d}\, A$ (resp. $x, y \in \operatorname{Rad} A$).

**Remark 4.2.** Any $\star$-prime $\star$-ideal is irreducible since $I \star J \subseteq I \cap J$.

**PROPOSITION 4.1.** *For any $\star$-ideal $P$ of an $f$-algebra $A$ the following are equivalent:*

(i) *$P$ is completely $\star$-prime;*
(ii) *$P$ is $\star$-prime;*
(iii) *$A/P$ is a totally-ordered $\star$-domain.*

Proof.
(i) $\implies$ (ii):
Assume $I \star J \subseteq P$ and there is $y \in P - J$. Thus, for each $x \in I$, $x \star y \in I \star J$, so $x \in P$, hence $I \subseteq P$.
(ii) $\implies$ (iii):
If $P$ is $\star$-prime, then $P$ is irreducible, so, by Proposition 3.5, $A/P$ is totally-ordered. We shall prove that $A/P$ has no non-zero $\star$-nilpotent $\star$-ideal. If $J$ is a $\star$-ideal in $A/P$, then $J = I/P$ for some $\star$-ideal $I$ in $A$ and

$$J^{(n)} = \{0\} \implies I^{(n)} \subseteq P \implies J = I/P = \{0/P\}\,.$$

By Corollary 4, $A/P$ has no non-zero $\star$-nilpotent element, hence, by Lemma 2.5, $A/P$ is a $\star$-domain.

(iii) $\Longrightarrow$ (i):
Obvious. □

**PROPOSITION 4.2.** *Let $P$ be a $\star$-ideal of an $f$-algebra $A$. Then the following are equivalent:*

(1) *$P$ is completely $\star$-prime;*
(2) *$D(P)$ is a completely prime ideal in $D(A)$.*

Proof.
(1) $\Longrightarrow$ (2):
Assume $[a,b]\cdot[a',b'] \in D(P)$ with $a,a',b,b' \in \operatorname{Rad} A$.

By Proposition 3.4(c) we have:

$$\begin{aligned}\big[d(a,b)\star d(a',b'),0\big] &= \big[d(a\star a' + b\star b,\ a\star b' + b\star a'),0\big] \\ &= |[a\star a' + b\star b',\ a\star b' + b\star a']| \\ &= |[a,b]\cdot[a',b']| \in D(P)\,,\end{aligned}$$

hence $d(a,b)\star d(a',b') \in P$. It follows that $d(a,b)$ or $d(a',b') \in P$, so $|[a,b]| = \big[d(a,b),0\big] \in D(P)$ or $|[a',b']| = \big[d(a',b'),0\big] \in D(P)$, so $[a,b] \in D(P)$ or $[a',b'] \in D(P)$.

(2) $\Longrightarrow$ (1):
Similarly. □

**Remark 4.3.** Since $P \mapsto D(I)$ is a quantale isomorphism between $\mathcal{I}\mathrm{d}\,A$ and $\mathcal{I}\mathrm{d}\,D(A)$ it follows that a $\star$-ideal $P$ of $A$ is $\star$-prime if and only if $D(P)$ is prime in $D(A)$. Thus Proposition 4.2 implies the equivalence $(i) \iff (ii)$ of Proposition 4.1 and, conversely, Proposition 4.2 follows from Proposition 4.1.

Denote by $\operatorname{Spec} A$ the set of $\star$-prime ideals in $A$ and $\operatorname{Spec} D(A)$ the set of l-prime l-ideals in the $f$-ring $D(A)$. For $I \in \mathcal{I}\mathrm{d}\,A$ set $d(I) = \{P \in \operatorname{Spec} A : I \nsubseteq P\}$.

In this way, $\operatorname{Spec} A$ becomes a topological space.

**COROLLARY 5.** *The map $P \mapsto D(P)$ is a homeomorphism between* $\operatorname{Spec} A$ *and* $\operatorname{Spec} D(A)$.

Let $A$ be an $f$-algebra and $I$ a $\star$-ideal. Denote $\sqrt{I} = \bigcap\{P \in \operatorname{Spec} A : I \subseteq P\}$.

**PROPOSITION 4.3.** $\sqrt{I} = \big\{x \in \operatorname{Rad} A :\ x^{(k)} \in I$ *for some integer* $k \geq 1\big\}$.

Proof. Denote by $J$ the right member and assume $x \notin J$, so $x^{(k)} \notin I$ for $k = 1,2,\dots$. Consider a $\star$-ideal $P$ maximal with respect to $x^{(k)} \notin P$,

$k = 1, 2, \ldots$ and $I \subseteq P$. We shall prove that $P$ is $\star$-prime. Assume there exist two $\star$-ideals $K_1$, $K_2$ such that $K_1 \star K_2 \subseteq P$, $K_1 \nsubseteq P$ and $K_2 \nsubseteq P$, so $x^{(m)} \in P \vee K_1$ and $x^{(n)} \in P \vee K_2$ for some integers $m, n \geq 1$. It follows that

$$x^{(m+n)} \in (P \vee K_1) \star (P \vee K_2) \subseteq P \vee (K_1 \star K_2) \subseteq P .$$

Contradiction, hence $x \notin \sqrt{I}$. The converse inclusion is obvious. □

**COROLLARY 6.** *An $f$-algebra $A$ is $\star$-semiprime if and only if $\sqrt{\langle 0 \rangle} = \{0\}$. A $\star$-ideal $I$ of an $f$-algebra $A$ is $\star$-semiprime if and only if $\sqrt{I} = I$.*

**PROPOSITION 4.4.** *If $A$ is a $\star$-algebra, then the following are equivalent:*

(1) *$A$ is a $\star$-semiprime $f$-algebra;*
(2) *$A$ is a subdirect product of totally-ordered $\star$-domains.*

P r o o f . By Propositions 3.2, 4.1 and Corollary 5. □

**PROPOSITION 4.5.** *For a $\star$-algebra $A$ the following are equivalent:*

(1) *$A$ is a $\star$-semiprime $f$-algebra.*
(2) *For any $a, b \in \operatorname{Rad} A$, $a \wedge b = 0$ if and only if $a \star b = 0$.*
(3) *Any $P \in \operatorname{Min} A$ is a completely $\star$-prime $\star$-ideal.*

P r o o f .

(1) $\Longrightarrow$ (2):
If $a \star b = 0$, then $(a \wedge b)^{(2)} \leq a \star b = 0$, so $(a \wedge b)^{(2)} = 0$, so $a \wedge b = 0$. The converse implication holds in any $f$-algebra.

(2) $\Longrightarrow$ (3):
For any $a, b, x \in \operatorname{Rad} A$ we have:

$$a \wedge b = 0 \Longrightarrow a \star b = 0 \Longrightarrow a \star (b) = 0 \Longrightarrow a \wedge (b \star x) = 0 ,$$

so $A$ is an $f$-algebra. By Proposition 3.1, $a^{\perp}$ is a $\star$-ideal for any $a \in \operatorname{Rad} A$. We shall prove that $a^{\perp}$ is $\star$-semiprime. If $x^{(2)} \in a^{\perp}$, then $(x \star x) \wedge a = 0$, so $x \star x \star a = 0$. Thus $x \star (x \star a) = (x \star x) \wedge (x \star a) = 0$ because $(x \star x) \star (x \star a) = 0$, therefore $x \wedge x \wedge a = 0$, i.e. $x \in a^{\perp}$. Thus $a^{\perp}$ is $\star$-semiprime. Consider now $P \in \operatorname{Min} A$, so $P$ is a $\star$-ideal by Proposition 3.1 and $P = \bigcup \{a^{\perp} : a \notin P, a \in \operatorname{Rad} A\}$ The previous remark shows that $P$ is $\star$-semiprime, so $A/P$ is $\star$-semiprime and totally-ordered. By Lemma 2.5, $A/P$ is a $\star$-domain for each $P \in \operatorname{Min} A$, hence $A$ is a subdirect product of totally-ordered $\star$-domains.

(3) $\Longrightarrow$ (1):
By Propositions 3.2 and 4.4, $A$ is a $\star$-semiprime $f$-algebra. □

**PROPOSITION 4.6.** *If $P$ is a $\star$-ideal in an $f$-algebra $A$, then the following hold:*

(1) *$P$ $\star$-prime $\implies$ $P$ is prime.*
(2) *If $A$ is $\star$-semiprime, then $P$ is $\star$-prime if and only if $P$ is prime.*
(3) *The set of all $\star$-ideals containing a $\star$-prime $\star$-ideal $P$ forms a chain.*

Proof.

(1) Assume $P$ $\star$-prime, so $A/P$ is totally-ordered, hence $P$ is prime.

(2) If $A$ is $\star$-semiprime, then $A/P$ is also $\star$-semiprime, so one can apply Proposition 4.5(2) for any $P$ prime:

$$\begin{aligned} x \star y \in P &\implies x/P + y/P = 0/P \implies x/P \wedge y/P = 0/P \\ &\implies x \wedge y \in P \implies x \in P \text{ or } y \in P, \end{aligned}$$

so $P$ is $\star$-prime.

(3) By (1). □

By this proposition, in a $\star$-semiprime $f$-algebra, any $\star$-prime $\star$-ideal is included in a unique maximal $\star$-ideal.

**PROPOSITION 4.7.** *For any $\star$-ideal $I$, $D(\sqrt{I}\,) = \sqrt{D(I)}$.*

Proof. By Proposition 2.8 and Corollary 5. □

If $N(A) = \sqrt{\langle 0 \rangle}$ and with the same notation in $\ell$-groups, we have $D(N(A)) = N(D(A))$.

## 5. $\star$-Semiprime and $\star$-pseudoprime $\star$-ideals in $f$-algebras

A $\star$-ideal $I$ in a $\star$-algebra $A$ is *$\star$-pseudoprime* if

$$x \star y = 0 \ \& \ x, y \in \operatorname{Rad} A \implies x \in I \text{ or } y \in I .$$

**PROPOSITION 5.1.** *A $\star$-ideal $P$ of an $f$-algebra $A$ is $\star$-prime if and only if it is $\star$-semiprime and $\star$-pseudoprime.*

Proof. Assume $P$ is $\star$-semiprime and $\star$-pseudoprime. Consider $x \star y \in P$, so $(x \wedge y)^{(2)} \in P$ since $(x \wedge y)^{(2)} \le x \star y$, therefore $x \wedge y \in \sqrt{P} = P$. We stress that

$$x(x \wedge y)^\star \wedge y(x \wedge y)^\star = (x \wedge y)(x \wedge y)^\star = 0 ,$$

so $\big(x(x \wedge y)^\star\big) \star \big(y(x \wedge y)^\star\big) = 0$, $A$ being an $f$-algebra. Since $P$ is $\star$-pseudoprime, $x(x \wedge y)^\star \in P$ or $y(x \wedge y)^\star \in P$. If $x(x \wedge y)^\star \in P$, then

$$x = (x \wedge y) \vee x = \big(\big(x(x \wedge y)^\star\big) + (x \wedge y)\big) \in P .$$

Thus $P$ is $\star$-prime. The converse implication is trivial. □

In what follows we will assume that $A$ is an $f$-algebra.

In accordance to Proposition 4.7, $P = \sqrt{P}$ if and only if $D(P) = \sqrt{D(P)}$, so a $\star$-ideal $P$ is $\star$-semiprime if and only if $D(P)$ is semiprime in $D(A)$.

**PROPOSITION 5.2.** *A $\star$-ideal $P$ is $\star$-pseudoprime if and only if $D(P)$ is pseudoprime in $D(A)$.*

Proof. Assume $P$ $\star$-pseudoprime and $[x,y]\cdot[u,v] = [0,0]$. Thus

$$\begin{aligned}\big[d(x,y)\star d(u,v),0\big] &= \big[d(x,y),0\big]\cdot\big[d(u,v),0\big]\\ &= |[x,y]|\cdot|[u,v]| = |[x,y]\cdot[u,v]| = [0,0]\end{aligned}$$

because $D(A)$ is an $f$-ring. Thus $d(x,y)\star d(u,v) = 0$, so $d(x,y)\in P$ or $d(u,v)\in P$. It follows that $|[x,y]| = \big[d(x,y),0\big]\in D(P)$ or $|[u,v]| = \big[d(u,v),0\big]\in D(P)$, i.e. $[x,y]\in D(P)$ or $[u,v]\in D(P)$. Then $D(P)$ is pseudoprime.

The converse implication is very similar. □

**LEMMA 5.1.** *Assume $A$ $\star$-semiprime and $a,b\in \operatorname{Rad} A$. Then*

(i) $a\le b \iff a^{(2)}\le b^{(2)}$;

(ii) $(a+b)^{(2)}\le 2\big(a^{(2)}+b^{(2)}\big)$;

(iii) $(a\star b)^{(2)}\le \big(b^{(2)}\star a^{(2)}\big)\vee\big(a^{(2)}\star b^{(2)}\big)$.

Proof.

(i) By [8; 2.3] we have, because $D(A)$ is $\star$-semiprime,

$$\begin{aligned}a\le b &\iff [a,0]\le[b,0] \iff [a,0]^2\le[b,0]^2\\ &\iff [a^{(2)},0]\le[b^{(2)},0] \iff a^{(2)}\le b^{(2)}.\end{aligned}$$

(ii) By [8; 2.4] we also have

$$\begin{aligned}\big[(a+b)^{(2)},0\big] &= [a+b,0]^2 = \big([a,0]+[b,0]\big)^2 \le 2\big([a,0]^2+[b,0]^2\big)\\ &= \big[2(a^{(2)}+b^{(2)}),0\big],\end{aligned}$$

therefore $(a+b)^{(2)}\le 2\big(a^{(2)}+b^{(2)}\big)$.

(iii) Similarly, using [8; 2.5]. □

For a $\star$-ideal $I$ in $A$, denote

$$S(I) = \big\{a\in\operatorname{Rad} A:\ a\le x^2 \text{ for some } x\in\operatorname{Rad} A \text{ such that } x^{(2)}\in I\big\}.$$

**LEMMA 5.2.** *$S(I)$ is a $\star$-ideal of $A$.*

Proof. For $a,b \in \operatorname{Rad} A$ we have:

$$\begin{aligned} a,b \in S(I) &\implies a \le x^{(2)} \in I \ \&\ b \le y^{(2)} \in I \\ &\implies a+b \le x^{(2)} + y^{(2)} \le (x+y)^{(2)} \le 2\big(x^{(2)} = y^{(2)}\big) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 5.2(ii))} \\ &\implies a+b \in S(I). \end{aligned}$$

$$\begin{aligned} a \in \operatorname{Rad} A \ \&\ b \in S(I) &\implies b \le x^{(2)} \in I \\ &\implies a \star b \le a \star x^{(2)} \le (a\star x + x)^{(2)} \le 2\big((a\star x)^{(2)} + x^{(2)}\big) \\ &\qquad \le 2\big(x^{(2)} + (a^{(2)} \star x^{(2)}) \vee (x^{(2)} \star a^{(2)})\big) \in I \\ &\implies a \star b \in S(I) \end{aligned}$$

in accordance to Lemma 5.2(ii) and (iii). □

**LEMMA 5.3.** *For any $\star$-ideal $I$, $I^{(2)} \subseteq S(I) \subseteq I$ and $S\big(S(I)\big) = S(I)$.*

Proof. By Proposition 2.11, we have

$$I^{(2)} = \{a \in \operatorname{Rad} A :\ a \le x^2 \ \text{for some } x \in I\},$$

therefore: $a \in I^{(2)} \implies a \le x^{(2)},\ x \in I \implies a \le x^{(2)} \in I \implies a \in S(I)$.

We also have:

$$a \in S(I) \implies a \le x^{(2)} \in I \implies a \le x^{(2)} \in S(I)$$

because

$$x^{(2)} \le x^{(2)} \in I \implies a \in S\big(S(I)\big).$$

The rest of the proof is obvious. □

**Remark 5.1.** If $I$ is an $\ell$-ideal in an $\ell$-ring $R$, there exist two notations for the same notion

$$I^n = \langle I^n \rangle = \{a \in R :\ |a| \le x^n \ \text{for some } x \in I^+\}$$

$I^n$: in [5; p. 158] (we adopt this notation).
$\langle I^n \rangle$: in [4; 2.1].

**LEMMA 5.4.** *We have $D\big(S(I)\big) = S\big(D(I)\big)$.*

Proof. Consider $u = [a,0] \in D\big(S(I)\big)^+$ with $a \in S(I)$, so $a \le x^{(2)} \in I$ for some $x \in \operatorname{Rad} A$, therefore $v^2 = [x^{(2)},0] \in D(I)$ and $u \le v^{(2)}$. This yields $u \in S\big(D(I)\big)$.

Conversely, assume $u = [a,0] \in S\big(D(I)\big)^+$, hence $u \le v^2 \in D(I)$, $v = [x,0]$ with $x \in \operatorname{Rad} A$. Thus $v^2 = [x^{(2)},0] \in D(I)$, hence $x^{(2)} \in I$, hence $a \le x^{(2)} \in I$. Thus $a \in S(I)$ and $u \in D\big(S(I)\big)$. □

**PROPOSITION 5.3.** *For a $\star$-ideal $I$ in an $f$-algebra $A$ the following are equivalent:*

(1) *$I$ is $\star$-semiprime;*
(2) *$N(A) \subseteq I$ and $a \in \mathrm{Rad}\, A$, $a^{(2)} \in I$ implies $a^{(2)} \in I$;*
(3) *$N(A) \subseteq I$ and $S(I) = I^{(2)}$.*

Proof. By [8; Theorem 3.2], Lemma 5.4 and other transfer properties. □

**PROPOSITION 5.4.** *If $S(I)$ is $\star$-semiprime, then $I = S(I)$.*

Proof. By [8; Theorem 3.3] and Lemma 5.4. □

A $\star$-ideal $I$ is *$\star$-square dominated* if $S(I) = I$. $I$ is called *$\star$-square-root closed* if for any $a \in I$ there exists $x \in I$ such that $x^{(2)} = a$.

**PROPOSITION 5.5.** *Let $I$ be a $\star$-ideal in $A$.*

(1) *$I$ is $\star$-square dominated in $A$ $\Longleftrightarrow$ $D(I)$ is square dominated in $D(A)$* ([15]).
(2) *$I$ is $\star$-square-root closed in $A$ $\Longleftrightarrow$ $D(I)$ is square-root closed in $D(A)$* ([15]).

Proof.

(i) By Lemma 5.4.

(ii) Assume $I$ is $\star$-square-root closed and $[a,b] \in D(I)$, so $a,b \in I$ and $d(a,b) \in I$. Thus $d(a,b) = x^{(2)}$ for some $x \in I$, hence $\|[a,b]\| = \big[d(a,b),0\big] = \big[x^{(2)},0\big] = [x,0]^2$ and $[x,0] \in D(I)$. Thus $D(I)$ is square-root closed.

Assume now $D(I)$ is square-root closed and $a \in I$. Thus $[a,0] \in D(I)$, so $[a,0] = [x,0]^2 = \big[x^{(2)},0\big]$ with $[x,0] \in D(I)$, therefore $x \in I$ and $a = x^{(2)}$, i.e. $I$ is $\star$-square-root closed.

If $x_1^{(2)} = x_2^{(2)} = a$, $x_1, x_2 \in \mathrm{Rad}\, A$ in a $\star$-semiprime $f$-algebra, then $x_1 = x_2$ by Lemma 5.1(i). The unique solution of $x^{(2)} = a$ will be denoted by $a^{(1/2)}$.

It is clear that $[a^{1/2},0] = [a,0]^{(1/2)}$ with usual notation in $f$-rings (see [8; p. 404]). □

**PROPOSITION 5.6.** *Let $I$ be a $\star$-ideal in a $\star$-semiprime $f$-algebra $A$.*

(i) *$I = I^{(2)}$ if and only if $I$ is $\star$-semiprime and $\star$-square dominated.*

(ii) *If $I$ is $\star$-square-root closed, then*

$$I^{(2)} = \big\{a \in \mathrm{Rad}\, A : \ a^{(1/2)} \in I\big\} = \big\{a \in \mathrm{Rad}\, A : \ a = b \star c \ \textit{for some}\ b, c \in I\big\}.$$

(iii) *If $I$ is $\star$-square-root closed, then $I = I^{(2)}$ if and only if $I$ is $\star$-semiprime.*

Proof.

(i) By [8; Theorem 3.4(a)] and Proposition 5.5(i) using some other well-known fact.

(ii) Assume $a^{(1/2)} \in I$, so $[a,0]^{1/2} = [a^{(1/2},0] \in D(I)$ and $D(I)$ is square-root closed. By [8; Theorem 3.4(b)] there are $u, v \in D(I)$ such that $[a,0] = u \circ v$. But $A$ is an $f$-algebra, so $D(A)$ is an $f$-ring, hence $[a,0] = |u \circ v| = |u| \circ |v|$, so one can assume that $u, v \geq 0$. Thus $u = [b,0]$, $v = [c,0]$, $b, c \in \operatorname{Rad} A$ and $[a,0] = [b \star c,\, 0]$, so $a = b \star c$ with $b, c \in I$.

Conversely, assume $a = b \star c$, $b, c \in I$, hence $[a,0] = [b,0] \cdot [c,0]$, $[b,0], [c,0] \in D(I)$, hence, by [8; Theorem 3.4(b)], $[a^{(1/2)},0] = [a,0]^{1/2} \in D(I)$.

It follows that $a^{(1/2)} \in I$.

We have proved the equality of the last sets in (ii).

If $a \in I^{(2)}$, then $a \leq x^{(2)}$, $x \in I$, hence $a^{(1/2)} \leq x \in I$, by Lemma 5.1(i), therefore $a^{(1/2)} \in I$. It is clear that the third set in (ii) is included in $I^{(2)}$.

(iii) By (i) and (ii), because any $\star$-square-root closed $\star$-ideal is $\star$-square dominated. □

**PROPOSITION 5.7.** *If $I$ is a $\star$-ideal in a $\star$-semiprime $f$-algebra, then $\bigcap_{n=1}^{\infty} I^{(n)}$ is $\star$-semiprime.*

Proof. By Proposition 2.8, $D\Big(\bigcap_{n=1}^{\infty} I^{(n)}\Big) = \bigcap_{n=1}^{\infty} \big(D(I)\big)^n$ and by [8; Theorem 3.5] this is semiprime in $D(A)$, so $\bigcap_{n=1}^{\infty} I^{(n)}$ is $\star$-semiprime in $A$. □

Assume $A$ is a $\star$-commutative $f$-algebra. If $M \subseteq \operatorname{Rad} A$, then

$$M^d = \big\{x \in \operatorname{Rad} A :\ x \star m = 0 \text{ for } m \in M\big\}$$

is a $\star$-ideal.

**LEMMA 5.5.** *$D(M^d) = D(M)^d$ for each $\star$-ideal $M$ in $A$.*

Proof. Assume $u = [a,0] \in D(M^d)^+$, so $a \star m = 0$, $m \in M$. Take $[x,y] \in D(M)$, so $x, y \in M$ and $d(x,y) \in M$. Since $D(A)$ is an $f$-ring, we have

$$|u \cdot [x,y]| = u \cdot |[x,y]| = [a,0] \cdot \big[d(x,y),0\big] = \big[a \star d(x,y),0\big] = [0,0],$$

hence $u \circ [x,y] = [0,0]$. This shows that $u \in D(M)^d$. The converse inclusion is similar. □

**COROLLARY 7.** $D(\{a\}^d) = \big\{[a,0]\big\}^d$ *for any* $a \in \operatorname{Rad} A$.

Proof. $D(\{a\}^d) = D(\langle a\rangle^d) = D(\langle a\rangle)^d = \langle [a,0]\rangle^d = \big\{[a,0]\big\}^d$ in accordance to Lemma 2.2 and Lemma 5.5. □

A (commutative) $f$-algebra $A$ is *$\star$-normal* if $\operatorname{Rad} A = \{ab^\star\}^d \vee \{a^\star b\}^d$ for any $a, b \in \operatorname{Rad} A$.

**PROPOSITION 5.8.** *The following are equivalent:*

(1) *$A$ is $\star$-normal;*
(2) *$D(A)$ is a normal $f$-ring (in the sense of* [11; p. 686]).

Proof.
(1) $\Longrightarrow$ (2):
Consider $[a,b] \in D(A)$. We have in $D(A)$:

$$\begin{aligned}&\{[a,b]^+\}^d \vee \{[a,b]^-\}^d\\ &= \{[ab^\star,0]\}^d \vee \{[ba^\star,0]\}^d = D(\{ab^\star\}^d) \vee D(\{ba^\star\}^d)\\ &= D(\{ab^\star\}^d \vee \{ba^\star\}^d) = D(\operatorname{Rad} A) = D(A),\end{aligned}$$

so $D(A)$ is normal.
(2) $\Longrightarrow$ (1):
Similarly. □

**PROPOSITION 5.9.** *For a $\star$-commutative $f$-algebra $A$ the following are equivalent:*

(1) *$A$ is $\star$-normal.*
(2) *For $a, b \in \operatorname{Rad} A$, $a \wedge b = 0$ implies $\{a\}^d \vee \{b\}^d = \operatorname{Rad} A$.*

Proof.
(2) $\Longrightarrow$ (1):
Obvious.
(1) $\Longrightarrow$ (2):
Assume $a \wedge b = 0$, so $[a,0] \wedge [b,0] = [0,0]$. Since $D(A)$ is normal, $\{[a,0]\}^d \vee \{[b,0]\}^d = D(A) = D(\operatorname{Rad} A)$ (see [11; p. 686]). By Corollary 7 we have

$$D(\{a\}^d \vee \{b\}^d) = D(\{a\}^d) \vee D(\{b\}^d) = \{[a,0]\}^d \vee \{[b,0]\}^d = D(\operatorname{Rad} A).$$

By the injectivity of $D$ on $\star$-ideals, $\{a\}^d \vee \{b\}^d = \operatorname{Rad} A$. □

**PROPOSITION 5.10.** *Consider a $\star$-commutative $\star$-semiprime $f$-algebra $A$.*

(1) *A $\star$-semiprime $\star$-ideal $I$ in $A$ is $\star$-square dominated if any $\star$-prime $\star$-ideal minimal with respect to containing $I$ is $\star$-square dominated.*
(2) *Every minimal $\star$-prime $\star$-ideal of $A$ is $\star$-square dominated if and only if for any $a \in \operatorname{Rad} A$, $\{a\}^d$ is $\star$-square dominated.*

Proof. This is a translation of [11; Lemma 2.1] using the above transfer properties. □

If $I$, $J$ are two $\star$-ideals in $A$, then $I : J = \{a \in \operatorname{Rad} A : x \in J \Longrightarrow a \star x \in I\}$ is a $\star$-ideal in $A$.

**Lemma 5.6.** *We have $D(I:J) = D(I):D(J)$.*

Proof. Assume $[a,0] \in D(I:J)^+$, so $a \in I:J$, so $a \star b = I$, so $a \star b = I$ for $b \in J$. Consider $[x,y] \in D(J)$, so $x,y \in J$, so $d(x,y) \in J$, hence $a \star d(x,y) \in I$. Thus $|[a,0] \circ [x,y]| = [a \star d(x,y), 0] \in D(I)$, so $[a,0] \circ [x,y] \in D(I)$, i.e. $[a,0] \circ D(I) : D(J)$.

Conversely, assume $[a,0] \in \big(D(I) : D(J)\big)^+$ and $x \in J$, therefore $[a,0] \star [x,0] \in D(I)$, so $a \star x \in I$, i.e. $a \in I:J$. Thus $[a,0] \in D(I:J)$. □

**Proposition 5.11.** *Let $A$ be a $\star$-commutative and $\star$-semiprime $f$-algebra with $\star$-identity element and in which every minimal $\star$-prime $\star$-ideal is $\star$-square dominated. For any $\star$-ideal $I$ the following are equivalent:*

(1) *$I$ is $\star$-pseudoprime;*
(2) *$\bigcap\limits_{n=1}^{\infty} I^{(n)}$ is $\star$-prime;*
(3) *$I \star \sqrt{I}$ is $\star$-pseudoprime;*
(4) *$I = \sqrt{I}$ is $\star$-pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is $\star$-prime.*

Proof. By [11; Theorem 2.2] and some transfer properties. □

For a $\star$-prime $\star$-ideal $P$ in a $\star$-commutative $f$-algebra $A$ denote

$$O_P = \{a \in \operatorname{Rad} A : \ a \star b = 0 \ \text{for some} \ b \notin P\}.$$

A similar notation will be used for $f$-rings.

**Lemma 5.7.** *$O_P$ is a $\star$-ideal and $D(O_P) = O_{D(P)}$.*

Proof. If $[a,0] \in D(O_P)^+$, then $a \in O_P$, so $a \star b = 0$ for some $b \notin P$. Thus $[a,0] \star [b,0] = [0,0]$ and $[b,0] \notin D(P)$, i.e. $[a,0] \in O_{D(P)}$.

Conversely, assume $[a,0] \in O^+_{D(P)}$, so $[a,0] \cdot [x,y] = [0,0]$ for some $[x,y] \notin D(P)$. Thus $[a,0] \cdot |[x,y]| = [0,0]$ and $|[x,y]| \notin D(P)$. But $|[x,y]| = [d(x,y),0]$, so $a \star d(x,y) = 0$ and $d(x,y) \notin P$. Thus $a \in O_P$ and $[a,0] \in D(O_P)$. □

**Proposition 5.12.** *If $A$ is a $\star$-commutative and $\star$-semiprime $f$-algebra with $\star$-identity element, then the following are equivalent:*

(1) *$A$ is $\star$-normal;*
(2) *for any $\star$-prime $\star$-ideal $P$ in $A$, $O_P$ is $\star$-prime;*
(3) *for any maximal $\star$-ideal $P$ in $A$, $O_P$ is $\star$-prime.*

Proof. By [11; Theorem 2.4], Proposition 5.8, Lemma 5.7 and some other transfer properties. □

**PROPOSITION 5.13.** *Let $A$ be a $\star$-commutative, $\star$-semiprime and $\star$-normal $f$-algebra with $\star$-identity element. For a $\star$-ideal $I$ the following are equivalent:*

(1) *$I$ is $\star$-pseudoprime;*
(2) *the $\star$-prime $\star$-ideals containing $I$ form a chain;*
(3) *$\sqrt{I}$ is $\star$-prime.*

Proof. We apply [11; Theorem 2.6] and Propositions 4.7, 5.2, 5.8 and other transfer properties. □

A $\star$-ideal $I$ in a ($\star$-commutative) $f$-algebra $A$ is *$\star$-primary* if for $a,b \in \operatorname{Rad} A$, $a \star b \in I$ and $a \notin I$ imply $b^{(n)} \in I$ for some $n \geq 1$. For the definition of primary $\ell$-ideal in $f$-rings, see e.g. [10; p. 106].

**PROPOSITION 5.14.** *The following are equivalent:*

(1) *$I$ is $\star$-primary in $A$;*
(2) *$D(I)$ is primary in $D(A)$.*

Proof.
(1) $\Longrightarrow$ (2):
For $[a,b],[x,y] \in D(I)$ we shall prove that

$$[a,b]\cdot[x,y] \in D(I) \ \ \& \ \ [a,b] \notin D(I) \Longrightarrow [x,y]^n \in D(I) \text{ for some } n \geq 1 .$$

If $[a,b]\cdot[x,y] \in D(I)$, then we have (because $D(A)$ is an $f$-ring):

$$\begin{aligned}\big[d(a,b) \star d(x,y), 0\big] &= \big[d(a,b),0\big]\cdot\big[d(x,y),0\big] \\ &= |[a,b]|\cdot|[x,y]| = |[a,b]\cdot[x,y]| \in D(I),\end{aligned}$$

so $d(a,b) \star d(x,y) \in D(I)$. From $[a,b] \notin I$ we get $\big[d(a,b),0\big] = |[a,b]| \notin D(I)$, so $d(a,b) \notin I$. Thus $d(x,y)^{(n)} \in I$ for some $n \geq 1$, since $I$ is $\star$-primary, therefore $|[x,y]|^n = \big[d(x,y)^{(n)},0\big] \in D(I)$, hence $[x,y]^n \in D(I)$. Then $D(I)$ is primary.
(2) $\Longrightarrow$ (1):
If $a \star b \in I$, $a \notin I$, then $[a,0]\cdot[b,0] = [a \star b, 0] \in D(I)$ and $[a,0] \notin D(I)$, so $\big[b^{(n)},0\big] = [b,0]^n \in D(I)$ for some $n \geq 1$, hence $b^{(n)} \in I$. □

**PROPOSITION 5.15.** *Let $A$ be a $\star$-commutative and $\star$-semiprime $f$-algebra with $\star$-identity element and $I$, a $\star$-ideal in $A$.*

(1) *If $I$ is $\star$-pseudoprime and it is an intersection of $\star$-primary $\star$-ideals, then $I$ is itself.*
(2) *If $I = I \star \sqrt{I}$ or $I = I : \sqrt{I}$, then $I$ is an intersection of $\star$-primary $\star$-ideals.*
(3) *If $I$ is a $\star$-pseudoprime ideal satisfying $I = I \star \sqrt{I}$ or $I = I : I\sqrt{I}$, then $I$ is $\star$-primary.*

Proof.

(1) By [10; 3.6] and Propositions 5.2, 5.14.

(2) and (3) By [10; 3.5, 3.6], Propositions 2.2, 4.7, 5.2, 5.14, and Lemma 5.6. □

**PROPOSITION 5.16.** *Let $A$ be an $f$-algebra.*

(1) *The join of a $\star$-semiprime $\star$-ideal and a $\star$-square dominated and $\star$-semiprime $\star$-ideal is $\star$-semiprime.*

(2) *Assume that any minimal $\star$-prime $\star$-ideal in $A$ is $\star$-square dominated. Then the join of any two $\star$-prime (resp. $\star$-semiprime) $\star$-ideals in $A$ is $\star$-prime (resp. $\star$-semiprime).*

Proof. By [13; 2.1, 2.2, 2.3] and the fact that these properties are transferable from $\ell$-rings in $\star$-algebras and vice-versa. □

An (arbitrary) $f$-algebra $A$ has the left *n*th-*convexity property* if for any $a, b \in \operatorname{Rad} A$ we have

$$a \leq b^{(n)} \implies (\exists\, c \in \operatorname{Rad} A)(a = c \star b)\,.$$

Similarly, one can define the right and *n*th-*convexity property.*

**PROPOSITION 5.17.** *The following are equivalent:*

(1) *$A$ has the left nth-convexity property.*

(2) *$D(A)$ has the left nth-convexity property (in sense of [10]).*

Proof.

(1) $\implies$ (2):

Assume $0 \leq u \leq v^n$, $v \geq 0$ in $D(A)$, so $u = [a,0]$, $v = [b,0]$, so $[a,0] \leq [b,0]^n = \big[b^{(n)},0\big]$, i.e. $a \leq b^{(n)}$. Thus there is $c \in \operatorname{Rad} A$, $a = c \star b$, hence $u = w \cdot v$ for $w = [c,0]$.

(2) $\implies$ (1):

Assume $a \leq b^{(n)}$, so $[a,0] \leq \big[b^{(n)},0\big] = [b,0]^n$, hence there is $[x,y] \in D(A)$ such that $[a,0] = [x,y] \cdot [b,0]$. $D(A)$ is an $f$-ring, hence

$$\begin{aligned}[a,0] &= |[x,y]\cdot[b,0]| = |[x,y]|\cdot[b,0] = [d(x,y),0]\cdot[b,0] \\ &= \big[d(x,y)\star b, 0\big]\,.\end{aligned}$$

Thus $a = d(x,y) \star b$, $d(x,y) \in \operatorname{Rad} A$. □

**COROLLARY 8.** *Assume $A$ has the left nth-convexity property. Thus any homomorphic image of $A$ has the nth-convexity property.*

Proof. By [10; 2.3] and Proposition 5.17. □

**COROLLARY 9.** *If $A$ satisfies the left nth-convexity property, then for any $a, b \in \operatorname{Rad} A$ there exist $x, y \in \operatorname{Rad} A$ such that*

$$a + \big(y \star d(a,b)\big) = b + \big(x \star d(a,b)\big)\,,$$
$$d(a,b) + (x \star b) + (y \star a) = (x \star a) + (y \star b)\,.$$

*Particularly, for any $a \in \operatorname{Rad} A$ there is $x \in \operatorname{Rad} A$ such that $a = x \star b$.*

Proof. By [10; Theorem 2.4(1)] and Proposition 5.17 there exist $x, y \in \operatorname{Rad} A$ such that

$$[a,b] = [x,y] \cdot |[a,b]| = [x,y] \cdot \big[d(a,b), 0\big] = \big[x \star d(a,b), y \star d(a,b)\big]\,,$$
$$\big[d(a,b), 0\big] = |[a,b]| = [x,y] \cdot [a,b] = [x \star a + y \star b, x \star b + y \star a]\,.$$

From these one gets the desired properties. □

Let $A$ be an $f$-algebra. An *$n$-convexity cover* of $A$ is an $f$-algebra $B$ such that there is an embedding $A \leq B$ and $B$ has the *nth-convexity property.*

**PROPOSITION 5.18.** *Let $A$ be a $\star$-commutative and $\star$-semiprime $f$-algebra. Then there is a unique $\star$-commutative and $\star$-semiprime $f$-algebra $K_n(A)$ such that*

(a) *$K_n(A)$ is an $n$-convexity cover of $A$.*
(b) *For any embedding (resp. $\star$-morphism) $f\colon A \to B$ with $B$ a $\star$-semiprime $f$-algebra satisfying the nth-convexity property, there is an embedding (resp. a $\star$-morphism) $\bar{f}\colon K_n(A) \to B$ such that $\bar{f}|_A = f$.*

Proof. We apply [12; Theorem 2.4] for $D(A)$, so one can take the minimal $n$-convexity cover $K_n\big(D(A)\big)$ of $D(A)$. Thus $\Delta\big(K_n\big(D(A)\big)\big)$ satisfies the above conditions (a), (b). □

## 6. Chain conditions in $f$-algebras

Let $A$ be an $f$-algebra. Recall that for $S \subseteq \operatorname{Rad} A$, $S^{\perp}$ is a $\star$-ideal.

A *polar $\star$-ideal* is a $\star$-ideal $I$ such that $I^{\perp\perp} = I$. It is easy to see that the set $\operatorname{Pol}(A)$ of polar $\star$-ideals of $A$ is a complete Boolean algebra with respect to:

$$\bigsqcap I_\lambda = \bigcap I_\lambda \qquad \text{and} \qquad \bigsqcup I_\lambda = \Big(\bigcup I_\lambda\Big)^{\perp\perp} = \Big(\bigcap I_\lambda^{\perp}\Big)^{\perp}.$$

A *polar $\ell$-ideal* in an $f$-ring $R$ is an $\ell$-ideal $K$ such that $K^{\perp\perp} = K$ in $R$.

In [1] the polar $\ell$-ideals are known under the name of *closed* $\ell$-ideals. Similarly, the set $\operatorname{Pol}(R)$ of polar $\ell$-ideals in $R$ is a complete Boolean algebra.

**Lemma 6.1.** *For any $\star$-ideal $I$ in an $f$-algebra $A$, $D(I^{\perp}) = D(I)^{\perp}$.*

Proof. Straightforward. □

**Lemma 6.2.** *The map $I \mapsto D(I)$ is an isomorphism between* $\mathrm{Pol}(A)$ *and* $\mathrm{Pol}(D(A))$.

Proof. By Lemma 6.1. □

**Lemma 6.3.** *For any $\star$-ideal $I$, $I$ is totally-ordered if and only if $D(I)$ is totally-ordered.*

**Lemma 6.4.** *For any non-zero $\star$-ideal $I$ in $A$ the following are equivalent:*

(1) *$I$ is totally-ordered;*
(2) *$I^{\perp}$ is a maximal polar $\star$-ideal;*
(3) *$A/I^{\perp}$ is totally-ordered $\star$-algebra.*

Proof. By [1; Lemma 1] and the previous lemmas. □

We shall write ACC for "ascending chain condition" and DCC for "descending chain condition".

**Lemma 6.5.** *For an $f$-algebra $A$ the following are equivalent:*

(1) *$A$ has* ACC (*resp.* DCC) *for polar $\star$-ideal;*
(2) *$D(A)$ has* ACC (*resp.* DCC) *for polar $\ell$-ideal.*

Proof. By the boolean isomorphism $\mathrm{Pol}(A) \xrightarrow{\sim} \mathrm{Pol}(D(A))$. □

For an $f$-algebra $A$ denote by $\mathcal{M}_A$ the maximal polar $\star$-ideals in $A$. Similarly, for an $f$-ring $R$ denote by $\mathcal{M}_R$ the maximal polar $\ell$-ideals in $R$.

**Lemma 6.6.** *For an $f$-algebra $A$, $\bigcap \mathcal{M}_A = \{0\}$ if and only if $\bigcap \mathcal{M}_{D(A)} = \{0\}$.*

Proof. The map $I \mapsto D(I)$ is an order-preserving bijection between $\mathcal{M}_A$ and $\mathcal{M}_{D(A)}$. □

**Proposition 6.1.** *For an $f$-algebra $A$ the following are equivalent:*

(1) *$A$ has* ACC *for polar $\star$-ideals;*
(2) *$A$ has* DCC *for polar $\star$-ideals;*
(3) *$A$ is isomorphic to a subdirect product of a finite family of totally-ordered $\star$-algebras.*

Proof.

(1) $\iff$ (2):
By [1; Theorem 1] and Lemma 6.6.

(3) $\implies$ (1):
Obvious, because any totally-ordered $\star$-algebra has ACC for polar $\star$-ideals.

(1) $\Longrightarrow$ (3):
By Lemma 6.6 and [1; Lemma 4] we have $\bigcap \mathcal{M}_A = \{0\}$ and $\mathcal{M}_A$ is finite. Thus $A \hookrightarrow \Pi\{A/P : P \in \mathcal{M}_A\}$ is the desired representation of $A$ in accordance to Lemma 6.4. □

**PROPOSITION 6.2.** *For a $\star$-semiprime $f$-algebra the following are equivalent:*

(1) *$A$ has* ACC *for polar $\star$-ideals;*
(2) *$A$ has* DCC *for polar $\star$-ideals;*
(3) *$A$ is isomorphic to a subdirect product of a finite family of totally-ordered $\star$-domains.*

P r o o f. By [1; Theorem 2], Proposition 6.1 and the $\star$-version of [1; Lemma 5]. □

## 7. Reticulations of an $f$-algebra

Denote by $K\,\mathrm{Spec}\,A$ the set of irreducible $\star$-ideals in an $f$-algebra $A$. If $R$ is an $f$-ring, $K\,\mathrm{Spec}\,R$ will be the set of irreducible $\ell$-ideals in $R$ (Keimel spectrum).

For any $\star$-ideal $I$ in $A$ denote $d(I) = d_A(I) = \{P \in K\,\mathrm{Spec}\,A : I \not\subseteq P\}$. It is easy to see that $d(I \cap J) = d(I) \cap d(J)$; $d(\bigvee I_\lambda) = \bigcup d(I_\lambda)$; $d(a \vee b) = d(a) \cup d(b)$· $d(a \wedge b) = d(a) \cap d(b)$. $K\,\mathrm{Spec}\,A$ becomes a topological space.

**LEMMA 7.1.** *The map $P \mapsto D(P)$ is a homeomorphism between $K\,\mathrm{Spec}\,A$ and $K\,\mathrm{Spec}\,D(A)$.*

P r o o f. For any $\star$-ideal $I$ we have $D(d_A(I)) = d_{D(A)}(D(I))$. □

It is easy to see that $D(d_A(a)) = d_{D(A)}([A, 0])$, so any $d_A(a)$ is a compact set in $K\,\mathrm{Spec}\,A$. An element $a \neq 0$ in $\mathrm{Rad}\,A$ is a *formal $\star$-unit* if $d(a) = K\,\mathrm{Spec}\,A$. An element $a$ is a formal $\star$-unit if and only if $[a, 0]$ is a formal unit in $D(A)$ (see [9]). $K\,\mathrm{Spec}\,A$ is compact if and only if $A$ has a formal $\star$-unit.

Consider the following equivalence relation: $x \sim y \iff d(x) = d(y)$ on $\mathrm{Rad}\,A$.

Denote $\gamma(A) = \mathrm{Rad}\,A/\sim$ and let $\gamma(x)$ be the equivalence class of $x \in \mathrm{Rad}\,A$. Setting

$$\left.\begin{aligned} \gamma(x) \vee \gamma(y) &= \gamma(x + y) \\ \gamma(x) \wedge \gamma(y) &= \gamma(x \wedge y) \end{aligned}\right. \qquad \text{for} \quad x, y \in \mathrm{Rad}\,A,$$

$(\gamma(A), \vee, \wedge, \gamma(0), \gamma(1))$ becomes a bounded distributive lattice. For a $\star$-ideal $I$ of $A$, $\gamma(I) = \{\gamma(x) : x \in I\}$ is an ideal of the lattice $\gamma(A)$. For any ideal $J$ in $\gamma(A)$, $\gamma^{-1}(J)$ is a $\star$-ideal in $A$.

**LEMMA 7.2.** *The maps $I \mapsto \gamma(I)$, $J \mapsto \gamma^{-1}(J)$ establish a lattice isomorphism between $\mathcal{I}\mathrm{d}\, A$ and the lattice $\mathrm{Id}\,\gamma(A)$ of the ideals in $\gamma(A)$.*

Proof. For $I \in \mathcal{I}\mathrm{d}\, A$ we have:

$$\gamma^{-1}\big(\gamma(I)\big) = \big\{a \in \mathrm{Rad}\, A : \ d(a) = d(x) \ \text{for some } x \in I\big\} = I$$

in accordance to Proposition 2.11. For $J \in \mathrm{Id}\,\gamma(A)$ it is easy to see that $\gamma\gamma^{-1}(J) = J$. □

**COROLLARY 10.** *$K\,\mathrm{Spec}\, A$ and $\mathrm{Spec}\,\gamma(A)$ are homeomorphic.*

Consider an $f$-ring $R$ and $D_2(R)$ the lattice constructed in [9; p. 210]. A construction of $D_2(R)$ can also be done in Belluce's style [2]. Consider the equivalence relation on $R^+$: $x \sim y \iff d(x) = d(y)$ for $x, y \in R^+$ (here $d(x) = \{P \in K\,\mathrm{Spec}\, A : \ x \notin P\}$). Denote $D_2(R) = R^+/{\sim}$ and $D_2(x)$ the class of $x \in R^+$. We define the operations of $D_2(A)$: $D_2(x) \vee D_2(y) = D_2(x \vee y)$ and $D_2(x) \wedge D_2(y) = D_2(x \star y)$ for $x, y \in R^+$. Thus $D_2(R)$ is a bounded distributive lattice.

**PROPOSITION 7.1.** *If $A$ is an $f$-algebra, then the lattices $\gamma(A)$ and $D_2(D(A))$ are isomorphic.*

Proof. For any $x, y \in \mathrm{Rad}\, A$ we have:

$$\begin{aligned}\gamma(x) = \gamma(y) &\iff d_A(x) = d_A(y)\\ &\iff d_{D(A)}\big([x,0]\big) = d_{D(A)}\big([y,0]\big)\\ &\iff D_2\big([x,0]\big) = D_2\big([y,0]\big).\end{aligned}$$

Thus one can prove that $\gamma(x) \mapsto D_2\big([x,0]\big)$ is a lattice isomorphism. □

**COROLLARY 11.** *$\gamma(A)$ is a normal lattice.*

Proof. By [9; p. 213] and Proposition 7.1. □

Recall that a $\star$-identity element is an element $e \in \mathrm{Rad}\, A$ such that $e \star x = x \star e = x$ for $x \in \mathrm{Rad}\, A$. The $\star$-identity element is unique. It is clear that $e$ is the $\star$-identity element of $A$ if and only if $[e, 0]$ is the identity element of $D(A)$.

**PROPOSITION 7.2.** *For an $f$-algebra $A$ with the $\star$-identity $e$ the following are equivalent:*

(1) *For any $a \in \mathrm{Rad}\, A$ there exist $b, c \in \mathrm{Rad}\, A$, $(a \star b) \vee c \geq e$ and $(a \star b) \wedge c = 0$.*
(2) *For any $a \in \mathrm{Rad}\, A$, $\langle a\rangle \vee a^{\perp} = \mathrm{Rad}\, A$.*
(3) *$\gamma(A)$ is a Boolean algebra.*
(4) *$K\,\mathrm{Spec}\, A$ is a Boolean space.*
(5) *Any irreducible $\star$-ideal is a maximal $\star$-ideal.*

Proof. We shall prove that (1) is equivalent to

(i) for $u \in D(A)^+$ there exist $v, w \in D(A)^+$ such that

$$(u \cdot v) \vee w \geq [e, 0] \qquad \text{and} \qquad (u \cdot v) \wedge w = [0, 0].$$

(1) $\Longrightarrow$ (i):
Consider $u \in D(A)^+$, so $u = [a, 0]$, $a \in \operatorname{Rad} A$, hence $(a \star b) \vee c \geq e$ and $(a \star b) \wedge c = 0$ for some $b, c \in \operatorname{Rad} A$. Thus for $v = [b, 0]$, $w = [c, 0]$ we obtain the relations in (i).

(i) $\Longrightarrow$ (1):
Similarly.

The condition (2) is equivalent to

(ii) for $u \in D(A)^+$ we have $\langle u \rangle \vee u^{\perp} = D(A)$.

This follows by $D(\langle a \rangle) = \langle [a, 0] \rangle$, $D(a^{\perp}) = [a, 0]^{\perp}$ for $a \in \operatorname{Rad} A$ and the lattice isomorphism $I \mapsto D(I)$ between $\mathcal{I}\mathrm{d}\, A$ and $\mathcal{I}\mathrm{d}\, D(A)$.

Thus our proposition follows from [9; p. 217, Proposition 4.10] and Proposition 7.1. □

**Lemma 7.3.** *If $I$ is a $\star$-ideal in $A$ and $J$ an ideal of the lattice $\gamma(A)$, then $(\gamma(I))^{\perp} = \gamma(I^{\perp})$ and $(\gamma^{-1}(J))^{\perp} = \gamma^{-1}(J)$.*

Proof. Straightforward. □

An $f$-algebra is *locally stonian* (resp. *locally strongly stonian*) if $x^{\perp} \vee x^{\perp\perp} = \operatorname{Rad} A$ (resp. $I^{\perp} \vee I^{\perp\perp} = \operatorname{Rad} A$) for each $x \in \operatorname{Rad} A$ (resp. $I \in \mathcal{I}\mathrm{d}\, A$).

**Proposition 7.3.** *For an $f$-algebra $A$ the following are equivalent:*

(1) *$A$ is locally stonian (resp. locally strongly stonian);*
(2) *$\gamma(A)$ is a stonian (resp. strongly stonian) lattice.*

Proof. By Lemmas 7.2 and 7.3. □

Now we shall define the second reticulation of an $f$-algebra. Denote by $E(A)$ the set of $\star$-ideals having the form:

$$K = \bigvee_{i=1}^{n} \langle x_{i_1} \rangle \star \cdots \star \langle x_{i_{n(i)}} \rangle, \qquad x_{ij} \in \operatorname{Rad} A.$$

Consider the following equivalence relation on $E(A)$:

$$K_1 \equiv K_2 \iff \sqrt{K_1} = \sqrt{K_2}.$$

Denote $\delta(K)$ the class of $K \in E(A)$ and define

$$\delta(K_1) \vee \delta(K_2) = \delta(K_1 \vee K_2) \qquad \text{and} \qquad \delta(K_1) \wedge \delta(K_2) = \delta(K_1 \star K_2)$$

for $K_1, K_2 \in E(A)$. Thus $\delta(A) = E(A)/\equiv$ has a structure of distributive lattice.

For $a \in \operatorname{Rad} A$ denote $\delta(a) = \delta(\langle a \rangle)$. Thus

$$\delta(a+b) = \delta(a \vee b) = \delta(a) \vee \delta(b) \qquad \text{and} \qquad \delta(a \star b) \leq \delta(a) \wedge \delta(b)$$

because $\langle a+b \rangle = \langle a \vee b \rangle = \langle a \rangle \vee \langle b \rangle$ and $\langle a \star b \rangle \subseteq \langle a \rangle \star \langle b \rangle$.

For $I \in \mathcal{I}\mathrm{d}\, A$, $I^\star = \{\delta(K) : K \in E(A),\ K \subseteq I\}$ is an ideal of $\delta(A)$.

For an ideal $J$ of $\delta(A)$, $J_\star = \{a \in \operatorname{Rad} A : \delta(a) \in J\}$ is a $\star$-ideal in $A$.

The maps $I \mapsto I^\star$, $J \mapsto J_\star$ are order-preserving and $I \subseteq (I^\star)_\star$, $J \subseteq (J_\star)^\star$.

The following result can be proved as in [2].

**PROPOSITION 7.4.** *$P \in \operatorname{Spec} A \implies P = (P^\star)_\star$ and $P^\star \in \operatorname{Spec} \delta(A)$.*

**PROPOSITION 7.5.** *The following are equivalent:*

(1) *For any ideal $J$ of $\delta(A)$, $J = (J_\star)^\star$.*
(2) *$J \in \operatorname{Spec} \delta(A) \implies J_\star \in \operatorname{Spec} A$.*

**PROPOSITION 7.6.** *If the equivalent conditions of Proposition 7.5 are fulfilled, then $\operatorname{Spec} A$ and $\operatorname{Spec} \delta(A)$ are homeomorphic.*

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