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BOUNDED OPERATORS ON NON-ARCHIMEDIAN ORTHOMODULAR SPACES[1]

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ABSTRACT. The single most important fact underlying the theory of infinite dimensional Hilbert spaces is embodied in the projection theorem: every orthogonally closed subspace is an orthogonal summand. A hermitian space which enjoys this property is said to be orthomodular. Besides the obvious Hilbert spaces, there do exist other infinite dimensional orthomodular spaces, examples of which have so far only been constructed over complete fields with a non-archimedian valuation. In this article, we study bounded linear operators on such spaces, many features of which are found to diverge sharply from those of bounded operators in the classical Hilbert space setting. In particular, we construct an operator algebra of von Neumann type that contains no orthogonal projections at all. For this algebra a representation theorem is derived, which implies that it is commutative.

Introduction

Orthomodular spaces are generalizations of Hilbert spaces. More precisely, let E be a vector space over an arbitrary skew field endowed with a hermitian form Φ. Then (E, Φ) is called orthomodular if

$$U \subseteq E,\ U^{\perp\perp} = U \implies E = U \oplus U^{\perp} \tag{P}$$

holds true. If $\dim E < \infty$, then the "Projection Theorem" (P) is simply equivalent to anisotropy of the form Φ and is therefore not interesting. In the infinite dimensional case, in which (P) is a very strong requirement, the classical examples are the real, complex and quaternionic Hilbert spaces. Although for a long time no other examples were known, in 1979 new kinds of infinite dimensional orthomodular spaces were discovered. All such spaces are constructed over certain complete fields with a non-archimedean valuation and are endowed with a natural non-archimedean norm with respect to which they are complete. Since they

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were first introduced in [Ke], many aspects of these spaces, such as, for example, Clifford algebras, measures and orthogonal groups, have been investigated.

The purpose of this paper is to initiate a study of bounded linear operators on non-archimedean orthomodular spaces. In spite of numerous analogies between such operators and their classical counterparts, some striking differences surface. These deviations from the classical theory arise principally because the base field is never algebraically closed, and the underlying geometry diverges sharply from that of Hilbert spaces. We will make manifest some of the salient new features by closely examining a particular example.

In §1, we summarize preliminary material on valuations and non-archimedianly normed spaces. In §2, we outline the construction of the orthomodular space E under consideration. After establishing a few general results in §3, we introduce in §4 a particularly interesting bounded self adjoint linear operator $A\colon E \to E$. It turns out that, although the spectrum of A consists of the single point 1, this number is not an eigenvalue of A, and we shall prove that A, in fact, possesses no invariant closed subspace at all. Even though these properties of A mean that no satisfactory spectral theory for such operators is to be expected, the basic idea of spectral decompositions, namely the expression of complex operators in terms of a family of simpler ones, is not to be abandoned. Indeed, we shall give such a spectral-type representation for the operators in the von Neumann algebra $\mathcal{A} = \{A\}' = \{B \mid\ B \circ A = A \circ B\}$, which is studied in §5; this representation will allow us to derive the surprising fact that $\mathcal{A}$ is commutative and, in fact, an integral domain.

1. Preliminaries

1.1. Valuation.

Let K be a field and Γ a totally ordered abelian group, written additively. A map

$$v\colon K \to \Gamma \cup \{\infty\}$$

is called a *Krull valuation* provided that, for all $\xi, \eta \in K$,

(i) $v(\xi) = \infty \iff \xi = 0$,
(ii) $v(\xi\eta) = v(\xi) + v(\eta)$,
(iii) $v(\xi + \eta) \geq \min\{v(\xi), v(\eta)\}$.

Here we adopted the usual conventions that $\gamma < \infty$ and $\gamma + \infty = \infty$ for all $\gamma \in \Gamma$. The valuation v gives rise to a field topology on K defined by taking $\{U_\varepsilon \mid\ \varepsilon \in \Gamma\}$, where $U_\varepsilon := \{\xi \in K \mid\ v(\xi) > \varepsilon\}$, as a neighbourhood base of the point $0 \in K$. Notice that a sequence $(\xi_i)_{i\in\mathbb{N}}$ in K converges to 0 in the valuation topology if and only if $v(\xi_i) \to \infty$.

1.2. Non-Archimedian norms.

Let E be a vector space over a field K and $v\colon K \to \Gamma \cup \{\infty\}$ a valuation. A map

$$\|\,.\,\|\colon E \to \Gamma \cup \{\infty\}$$

is called a (*non-archimedian*) *norm* provided that, for all $x, y \in E$ and all $\xi \in K$

(i) $\|x\| = \infty \iff x = 0$,
(ii) $\|\xi x\| = 2v(\xi) + \|x\|$,
(iii) $\|x + y\| \geq \min\{\|x\|, \|y\|\}$.

The corresponding norm topology on E is obtained by taking the sets $V_\varepsilon := \{x \in E \mid \|x\| > \varepsilon\}$ $(\varepsilon \in \Gamma)$ as a neighbourhood base of $0 \in E$.

1.3. Definite spaces.

Consider a vector space E over a valued field (K, v), $\operatorname{char} K \neq 2$, endowed with a non-degenerate, symmetric bilinear form $\Phi\colon E \times E \to K$. For $x \in E$ we let $N(x) := v\big(\Phi(x, x)\big)$ and we ask whether N is a norm. Conditions (i) and (ii) in 1.2 are certainly satisfied, however, (iii) may fail.

Lemma 1.1. ([Kü]) *Assume that $v(2) = 0$. The following conditions are equivalent:*

(i) $\forall x, y \in E:\ N(x + y) \geq \min\{N(x), N(y)\}$ (*"triangle inequality"*).
(ii) $\forall x, y \in E:\ x \perp y \implies N(x + y) = \min\{N(x), N(y)\}$ (*"Pythagoras"*).
(iii) $\forall x, y \in E:\ 2v\big(\Phi(x, y)\big) \geq N(x) + N(y)$ (*"Cauchy-Schwarz"*).

If one (and hence all) of the conditions (i), (ii), (iii) in Lemma 1.1 is satisfied, we say that (E, Φ) is a *definite space over* (K, v). Notice that a definite space is always anisotropic. For, if E contains an isotropic vector $x \neq 0$, then, by non-degeneracy, there exists an isotropic $y \neq 0$ with $\Phi(x, y) = 1$ in which case (iii) certainly fails.

Thus, if (E, Φ) is definite, then $N(x) = v\big(\Phi(x, x)\big)$ is a norm on E (and conversely), and we shall often write "$\|x\|$" instead of "$N(x)$".

2. Construction of the orthomodular space (E, Φ)

In this section, we summarize the construction of our non-classical orthomodular space. For detailed proofs, we refer to [Ke] and [G-Kü]. The original reasonings in [Ke] are in the framework of ordered fields, in [G-Kü] these arguments are transferred and generalized to valued fields.

We let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$.

2.1. The value group.

For $j \in \mathbb{N}$ let Γ_j be an isomorphic copy of the additive group $\mathbb{Z}$ of integers and let Γ be the direct sum

$$\Gamma := \bigoplus_{j=1}^{\infty} \Gamma_j = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \oplus \cdots .$$

Thus Γ consists of all sequences $\gamma = (q_j)_{j\in\mathbb{N}}$ with $q_j \in \Gamma_j$ for which $\{j \in \mathbb{N} \mid q_j \neq 0\}$ is finite. We order Γ antilexicographically, that means, if $0 \neq \gamma = (q_j)_{j\in\mathbb{N}} \in \Gamma$ and $k := \max\{j \in \mathbb{N} \mid q_j \neq 0\}$, then

$$\gamma > 0 \text{ in } \Gamma \iff q_k > 0 \text{ in } \Gamma_k .$$

2.2. The base field.

We start with the field $K_0 := \mathbb{R}(X_i)_{i\in\mathbb{N}}$ of all rational functions in the variables $X_1, X_2, \ldots, X_n, \ldots$ with real coefficients. We define a valuation $v_0\colon K_0 \to \Gamma \cup \{\infty\}$ as follows.

(a) v_0 is trivial on $\mathbb{R}$, i.e., $v_0(\xi) = 0$ for $0 \neq \xi \in \mathbb{R}$; $v_0(0) = \infty$.

(b) Suppose that $\mu \in K_0$ is a monomial with coefficient 1, say

$$\mu = X_1^{s_1} X_2^{s_2} \ldots X_m^{s_m}, \qquad \text{where} \quad s_1 \geq 0, \ldots, s_m \geq 0 .$$

Then we let

$$v_0(\mu) := (-s_1, -s_2, \ldots, -s_m, 0, \ldots) .$$

Notice that, if two such monomials μ and μ' are different, then $v_0(\mu) \neq v_0(\mu')$.

(c) If $\rho \in K_0$ is a polynomial, say

$$\rho = \alpha_0\mu_0 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s ,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_s \in \mathbb{R}$ and $\mu_0, \mu_1, \ldots, \mu_s$ are monomials, then we put

$$v_0(\rho) := \min\{v_0(\mu_0), v_0(\mu_1), \ldots, v_0(\mu_s)\} .$$

The value $v_0(\rho)$ reflects the "highest" monomial involved in ρ.

(d) Finally, every $\xi \in K_0$ is the quotient of two polynomials, say $\xi = \rho/\rho'$. We put

$$v_0(\xi) = v_0(\rho) - v_0(\rho') .$$

It is readily verified that $v_0\colon K_0 \to \Gamma \cup \{\infty\}$ is indeed a Krull valuation.

We finish the construction of the field by passing to the completion

$$K := \widetilde{K}_0$$

of the valued field (K_0, v_0) (for details see [Ri]). v_0 extends uniquely to a valuation v on K with the same value group Γ.

For reference later on we need the following elementary fact which indicates the "gaps" in the square function $K \to K^2$, $\xi \mapsto \xi^2$.

In order to unify formulas, we put down the convention that $X_0 := 1$.

LEMMA 2.1. *Let $\xi \in K$ and let $n \geq 1$.*

(i) *If $v(\xi^2 X_n) \geq 0$, then $v(\xi^2 X_n) > v(X_{n-1}^r)$ for all $r \in \mathbb{Z}$.*
(ii) *If $v(\xi^2 / X_n) \geq 0$, then $v(\xi^2 / X_n) > v(X_{n-1}^r)$ for all $r \in \mathbb{Z}$.*

Proof.

(i) In case $\xi = 0$, the claim is trivial, so assume $\xi \neq 0$ and write $v(\xi) = (q_1, q_2, \dots) \in \Gamma$. Then

$$v(\xi^2 X_n) = 2v(\xi) + v(X_n) = (2q_1, \dots, 2q_{n-1}, 2q_n - 1, 2q_{n+1}, \dots).$$

Clearly, $v(\xi) \neq (0, 0, \dots)$. Put $k := \max\{i \in \mathbb{N} \mid q_i \neq 0\}$. The hypothesis $v(\xi^2 X_n) \geq 0$ implies that $k \geq n$. Moreover, if $k > n$, then $2q_k \geq 0$, and, if $k = n$, then $2q_k - 1 \geq 0$, hence $q_k \geq 1$ in both cases. If $n > 1$, then $v(X_{n-1}^r) = (0, \dots, 0, -r, 0, \dots)$, where $-r$ is in the place $(n-1) < k$; if $n = 1$, then $v(X_{n-1}^r) = (0, 0, 0, \dots)$ and the claim follows.

(ii) Apply (i) with ξ / X_n in place of ξ. □

2.3. The space.

Recall the convention that $X_0 = 1$. Consider

$$E := \Big\{ (\xi_i)_{i \in \mathbb{N}_0} \in K^{\mathbb{N}_0} \mid \sum_{i=0}^{\infty} \xi_i^2 X_i \text{ converges in the valuation topology} \Big\}.$$

Notice that a series $\sum_{i=0}^{\infty} \xi_i^2 X_i$ converges if and only if $v(\xi_i^2 X_i) \to \infty$ for $i \to \infty$. Let $x := (\xi_i)_i$, $y := (\eta_i)_i \in E$. Since $v(\xi_i \eta_i X_i) \geq \min\{v(\xi_i^2 X_i), v(\eta_i^2 X_i)\}$ for every i, it follows that both the series $\sum_{i=0}^{\infty} \xi_i \eta_i X_i$ and $\sum_{i=0}^{\infty} (\xi_i + \eta_i)^2 X_i$ are convergent. Hence E is a vector space over K under componentwise operations. Moreover, we can define a symmetric, bilinear form $\Phi \colon E \times E \to K$ by

$$\Phi(x, y) := \sum_{i=0}^{\infty} \xi_i \eta_i X_i \qquad \text{for} \quad x = (\xi_i)_{i \in \mathbb{N}_0}, \ y = (\eta_i)_{i \in \mathbb{N}_0} \in E.$$

This completes the construction of the quadratic space (E, Φ).

2.4. Basic properties of (E, Φ).

The most fundamental properties of the space constructed above are:

THEOREM 2.2. *E is an orthomodular space.*

THEOREM 2.3. *(E, Φ) is a definite space.*

Thus the assignment $x \to \|x\| = v\big(\Phi(x, x)\big) \in \Gamma \cup \{\infty\}$ is a norm on E, and, if $x \perp y$, then $\|x + y\| = \min\{\|x\|, \|y\|\}$.

THEOREM 2.4.

(i) *E is complete in the norm topology, i.e., a Banach space.*

(ii) *A linear subspace of E is topologically closed if and only if it is orthogonally closed.*

For proofs we refer to [Ke].

2.5. Types.

In the theory of non-classical orthomodular spaces, a device called types turned out to be crucial. The concept of types is developed in full generality in [G-Kü]; here we only explain it for the present space. Recall that the value group is $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \dots$. To each $\gamma = (q_j)_{j \in \mathbb{N}} \in \Gamma$ we assign a type, denoted by $T(\gamma)$, as follows. If $\gamma \in 2\Gamma$, then all the numbers q_j are even and we put $T(\gamma) := 0$; if $\gamma \notin 2\Gamma$, then finitely many of the numbers q_j are odd and we put

$$T(\gamma) := \max\{j \in \mathbb{N} \mid \ q_j \text{ is odd}\}.$$

Evidently, $T(\gamma) = T(\gamma + 2\gamma')$ for all $\gamma' \in \Gamma$.

Next we define the type of a non-zero scalar $0 \neq \xi \in K$ by

$$T(\xi) := T\big(v(\xi)\big).$$

Then $T(\lambda^2 \xi) = T(\xi)$ for all $0 \neq \lambda \in K$, i.e., the type function $T \colon K^* \to \mathbb{N}_0$ is constant on square classes. Observe that $T(X_i) = i$ for all $i = 0, 1, \dots$.

Since Φ is anisotropic, we can assign a type to every non-zero vector $0 \neq x \in E$ by

$$T(x) := T\big(\Phi(x,x)\big) = T\big(\|x\|\big).$$

Clearly, $T(\lambda x) = T(x)$ for all $0 \neq \lambda \in K$, thus there is a type attached to each straight line G in E, denoted also by $T(G)$.

For reference later on we need

LEMMA 2.5. *Let $\gamma, \delta \in \Gamma$. Then $T(\gamma+\delta) \leq \max\{T(\gamma), T(\delta)\}$; if $T(\gamma) \neq T(\delta)$, then $T(\gamma+\delta) = \max\{T(\gamma), T(\delta)\}$.*

The proof is straightforward and will be omitted.

The next theorem expresses one of the most outstanding geometric properties of non-classical orthomodular spaces, the so called invariance of types.

THEOREM 2.6. ([Ke])

(i) *Let x, y be two non-zero vectors in E. If $x \perp y$, then $T(x) \neq T(y)$.*

(ii) *Let U be a topologically closed subspace of E. Then, in any two maximal orthogonal families in U, there occur the same types.*

It follows that, in particular, E cannot be isomorphic to a proper (closed) subspace.

2.6. The standard basis.

For every $i \in \mathbb{N}_0$ we let

$$e_i := (0, \ldots, 0, 0, 1, 0, \ldots) \in E, \qquad \text{where } 1 \text{ is in place } i+1.$$

Then $e_i \perp e_j$ for $i \neq j$. Clearly, $\{e_i \mid i \in \mathbb{N}_0\}$ is a maximal orthogonal family in E. Notice that $\Phi(e_i, e_i) = X_i$; in particular, $T(e_i) = i$ for all $i \in \mathbb{N}_0$.

Consider an arbitrary vector $x = (\xi_i)_{i \in \mathbb{N}_0} \in E$. The norm of x is given by

$$\|x\| = v\left(\sum_{i=0}^{\infty} \xi_i^2 X_i\right).$$

For $i \neq j$ we have $v(\xi_i^2 X_i) \neq v(\xi_j^2 X_j)$ as types are different. Hence

$$\|x\| = \min\{v(\xi_i^2 X_i) \mid i \in \mathbb{N}_0\}.$$

For $n \in \mathbb{N}_0$ we put $x_n := \sum_{i=0}^{n} \xi_i e_i$. Then, by the above formula,

$$\|x - x_n\| = \min\{v(\xi_i^2 X_i) \mid i \geq n+1\},$$

which shows that $\|x - x_n\| \to \infty$, hence $x_n \to x$ when $n \to \infty$. This means that x can be expressed as

$$x = \sum_{i=0}^{\infty} \xi_i e_i.$$

Convergence of the series is, of course, in the norm topology on E. Notice that

$$\|x\| = \min\{\|\xi_i e_i\| \mid i \in \mathbb{N}_0\}.$$

We shall refer to $\{e_i \mid i \in \mathbb{N}_0\}$ as the standard base of E.

2.7. Residual spaces.

A proper subgroup Δ of the value group Γ is called *isolated* (*or convex*) if

$$\delta \in \Delta, \ \gamma \in \Gamma, \ 0 \leq \gamma \leq \delta \implies \gamma \in \Delta.$$

The *isolated subgroups* of Γ are easily described. For $n \in \mathbb{N}_0$ we let

$$\Delta_n := \{(q_j)_{j \in \mathbb{N}} \in \Gamma \mid q_j = 0 \text{ for } j > n\} = \Gamma_1 \oplus \cdots \oplus \Gamma_n \oplus \{0\} \oplus \{0\} \oplus \ldots.$$

Every Δ_n is an isolated subgroup of Γ and there are no others. To each Δ_n there belongs, by general valuation theory, a valuation ring

$$R_n := \{\xi \in K \mid v(\xi) \geq \delta \text{ for some } \delta \in \Delta_n\}.$$

R_n is a local ring, its unique maximal ideal is

$$J_n := \{\xi \in K \mid v(\xi) > \delta \text{ for all } \delta \in \Delta_n\}.$$

We let $\widehat{K}_n := R_n/J_n$ be the residue field and $\Theta_n\colon R_n \to \widehat{K}_n$ the canonical epimorphism. It is easy to check that

$$\widehat{K}_n \cong \mathbb{R}(X_1, X_2, \ldots, X_n).$$

We have seen that the space (E,Φ) has the basic property of being definite. This entails that the above reduction of K to $\widehat{K}_n$ carries over to a corresponding reduction of (E,Φ). As a counterpart of the residue fields $\widehat{K}_n$, we shall obtain the residual spaces $(\widehat{E}_n, \widehat{\Phi}_n)$.

It follows from the triangle inequality that

$$M_n := \{x \in E \mid \ \Phi(x,x) \in R_n\} = \{x \in E \mid \ \|x\| \geq \delta \text{ for some } \delta \in \Delta_n\}$$

is a module over the ring R_n, and

$$S_n := \{x \in E \mid \ \Phi(x,x) \in J_n\}$$

is a submodule. Let $\widehat{E}_n := M_n/S_n$ and let $\pi_n\colon M_n \to \widehat{E}_n$ be the canonical epimorphism. Since $J_n \cdot M_n \subseteq S_n$, the quotient $\widehat{E}_n$ is turned into a vector space over the residue field $\widehat{K}_n$ by putting

$$\Theta_n(\xi)\cdot\pi_n(x) := \pi_n(\xi\cdot x) \qquad (\text{where } \xi \in R_n\,,\ x \in M_n\,).$$

Moreover, $\widehat{E}_n$ inherits a bilinear form $\widehat{\Phi}_n$ from (E,Φ) by

$$\widehat{\Phi}_n\big(\pi_n(x),\pi_n(y)\big) := \Theta_n\big(\Phi(x,y)\big) \qquad (\text{where } x,y \in M_n\,).$$

$(\widehat{E}_n, \widehat{\Phi}_n)$ is called the *residual space* of (E,Φ) belonging to the isolated subgroup Δ_n.

Clearly, every subspace $U \subseteq E$ is reduced under π_n to a subspace

$$\widehat{U}_n := \{\pi_n(x) \mid \ x \in U \cap M_n\}.$$

By abuse of language we sometimes write "$\pi_n(U)$" instead of "$\widehat{U}_n$".

LEMMA 2.7. *If the subspaces $U, W \subset E$ are orthogonal, $U \perp W$, then $\pi_n(U) \perp \pi_n(W)$ and $\pi_n(U \oplus W) = \pi_n(U) \oplus \pi_n(W)$.*

Proof. Clearly, $\pi_n(U) \perp \pi_n(W)$. For the second assertion we have to show that $(U \oplus W) \cap M_n = (U \cap M_n) \oplus (W \cap M_n)$. Let $x \in (U \oplus W) \cap M_n$ and decompose $x = u + w$ with $u \in U$, $w \in W$. Then $\|x\| = \min\{\|u\|, \|w\|\}$ since E is definite. So $x \in M_n$ implies that $u, w \in M_n$, and consequently, $x = u + w \in (U \cap M_n) \oplus (W \cap M_n)$. The inverse inclusion is trivial. □

Lemma 2.8. *Let G be a one-dimensional subspace of E. Then $\widehat{G}_n = \{0\}$ if and only if the type of G is greater than n, $T(G) > n$.*

Proof. Suppose first that $i := T(G) > n$. Let $0 \neq x \in G$. Then $\|x\| = (q_1, q_2, \ldots, q_i, \ldots) \in \Gamma$ with q_i odd. Hence $k := \max\{j \in \mathbb{N} \mid q_j \neq 0\} \geq i > n$. If $q_k < 0$, then $x \notin M_n$; if $q_k > 0$, then $x \in S_n$, and therefore $\pi_n(x) = 0$. Thus $\widehat{G}_n = \pi_n(G \cap M_n) = \{0\}$. On the other hand, suppose that $i = T(G) \leq n$. It is easy to find a multiple $x' = \lambda x \in G$ such that $\|x'\| = (q_1, \ldots, q_{i-1}, -1, 0, 0, \ldots)$. In that case, we have $x' \in M_n$, $x' \notin S_n$, and therefore $\pi_n(x') \neq 0$. □

From the above results we deduce

Theorem 2.9. *The residual space $\widehat{E}_n$ has dimension $n+1$. The vectors $\widehat{e}_i := \pi_n(e_i)$ $(i = 0, \ldots, n)$ form an orthogonal basis for $(\widehat{E}_n \widehat{\Phi}_n)$, and $\widehat{\Phi}_n \sim$ $\operatorname{diag}(1, X_1, X_2, \ldots, X_n)$.*

Proof. Let $U := \operatorname{span}\{e_0, e_1, \ldots, e_n\}$. Then $E = U \oplus U^{\perp}$ by orthomodularity, so $\widehat{E}_n = \pi_n(E) = \pi_n(U) \oplus \pi_n(U^{\perp})$. We have $T(e_i) = i$ for all i. In view of Theorem 2.6.(i), $U^{\perp}$ contains only vectors of types $i > n$, and therefore, by Lemma 2.8, $\pi_n(U^{\perp}) = \{0\}$. Thus $\widehat{E}_n = \pi_n(U)$ and the assertion follows by Lemma 2.7. □

3. Bounded linear operators

A linear operator $B \colon E \to E$ is called *bounded* if there exists a $\gamma \in \Gamma$ such that

$$\|B(x)\| - \|x\| \geq \gamma \qquad \text{for all} \quad 0 \neq x \in E.$$

The set $\mathcal{B}(E)$ of all bounded linear operators on E is closed under the usual addition and composition, i.e., $\mathcal{B}(E)$ is an algebra. A bounded linear operator B is determined by the images $B(e_i)$ of the vectors e_i of the standard basis, so B can be represented by a countably infinite matrix.

Remark 1. Clearly, a bounded linear operator B is continuous in the norm topology, however, as shown in [Fä], continuous operators need not be bounded.

Remark 2. In general, a bounded linear operator cannot be assigned a norm in the usual way because norms are in the ordered group Γ where bounded subsets may fail to possess an infimum.

There are two naturally arising topologies on $\mathcal{B}(E)$. Each of them turns $\mathcal{B}(E)$ into a topological algebra. The norm topology on $\mathcal{B}(E)$ has all sets

$$\mathcal{U}_\varepsilon(B_0) := \bigl\{B \in \mathcal{B}(E) \mid \ \|(B - B_0)(x)\| - \|x\| \geq \varepsilon \text{ for all } 0 \neq x \in E\bigr\},$$

where ε ranges over Γ, as a neighbourhood base of $B_0 \in \mathcal{B}(E)$. The topology of pointwise convergence is defined by taking all sets

$$\mathcal{U}_{\varepsilon,x}(B_0) := \big\{B \in \mathcal{B}(E) \mid \ \|(B - B_0)(x)\| - \|x\| \geq \varepsilon\big\}\,,$$

where ε ranges over Γ and x ranges over $E \setminus \{0\}$ as a neighbourhood subbase of B_0.

The norm topology is finer than the topology of pointwise convergence. As a matter of fact we mention that $\mathcal{B}(E)$ is complete in the norm topology, thus $\mathcal{B}(E)$ is a Banach algebra.

Remark. Given a linear map $B\colon E \to E$, the question whether B is bounded, or whether B has an inverse in $\mathcal{B}(E)$, can be settled by examining the set $\big\{\|B(e_i)\| - \|e_i\| \mid \ i \in \mathbb{N}_0\big\}$ as for upper and lower bounds.

Lemma 3.1. *A map $B_0\colon \{e_i \mid \ i \in \mathbb{N}_0\} \to E$ extends to a bounded linear operator $B\colon E \to E$ if (and only if) the set $\big\{\|B_0(e_i)\| - \|e_i\| \mid \ i \in \mathbb{N}_0\big\} \subset \Gamma$ is bounded from below.*

Proof. Assume that there exists an element $\gamma \in \Gamma$ such that $\|B_0(e_i)\| - \|e_i\| \geq \gamma$ for all $i \in \mathbb{N}_0$. B_0 extends to a linear map $B_0^*\colon E_0 \to E$, where $E_0 := \operatorname{span}\{e_i \mid \ i \in \mathbb{N}_0\}$. We first show that B_0^* is bounded by γ. Consider a non-zero vector $x = \sum\limits_{i=0}^{n} \xi_i e_i \in E_0$. Let $k \in \{0, \dots, n\}$ be such that $\|B_0^*(\xi_k e_k)\| = \min\big\{\|B_0^*(\xi_i e_i)\| \mid \ i = 0, \dots, n\big\}$. Then

$$\|B_0^*(x)\| \geq \|B_0^*(\xi_k e_k)\| = 2v(\xi_k) + \|B_0(e_k)\|\,.$$

We have $\|x\| = \min\big\{\|\xi_i e_i\| \mid \ i = 0, \dots, n\big\}$, hence

$$\|x\| \leq \|\xi_k e_k\| = 2v(\xi_k) + \|e_k\|\,.$$

Subtracting these inequalities we get $\|B_0^*(x)\| - \|x\| \geq \|B_0(e_k)\| - \|e_k\|$, thus $\|B_0^*(x)\| - \|x\| \geq \gamma$ as claimed. Now E_0 is dense in E in the norm topology, so the assertion follows by standard arguments. □

Remark. The above proof shows that any lower bound γ of $\big\{\|B(e_i)\| - \|e_i\| \mid \ i \in \mathbb{N}_0\big\}$ is a lower bound for the operator B. In particular, if $\big\{\|B(e_i)\| - \|e_i\| \mid \ i \in \mathbb{N}_0\big\}$ has an infimum γ_0, then γ_0 is a norm for B in the usual sense.

Lemma 3.2. *Let U be a subspace of E and let $B\colon U \to E$ be a linear map. Suppose that there is a closed subspace $V \subseteq U$ with $\dim U/V < \infty$ such that the restriction of B to V is bounded. Then B is bounded on all of U.*

Proof. It is sufficient to consider the case where $\dim U/V = 1$; the claim then follows by induction. So suppose that $U = V \oplus K(w)$. If B were not bounded on U, then there would exist a sequence $(v_i)_{i\in\mathbb{N}}$ in V such that

$$\|B(v_i) + B(w)\| - \|v_i + w\| \to -\infty \qquad \text{for} \quad i \to \infty\,.$$

Since V is closed and $w \notin V$, the set $\{\|v_i + w\| \mid i \in \mathbb{N}\}$ is bounded from above. It follows that $\|B(v_i) + B(w)\| \to -\infty$. This can happen only when $\|B(v_i)\| \to -\infty$. Since B is bounded on V it follows that $\|v_i\| \to -\infty$. Consequently, for all sufficiently large $i \in \mathbb{N}$ we have $\|B(v_i) + B(w)\| = \|B(v_i)\|$ and $\|v_i + w\| = \|v_i\|$. But then $\{\|B(v_i) + B(w)\| - \|v_i + w\| \mid i \in \mathbb{N}\}$ is bounded from below, which is a contradiction. $\square$

LEMMA 3.3. *Let $B\colon U \to U'$ be a linear, bijective map between two subspaces U, U' of E. Suppose that there exists a closed subspace $V \subseteq U$ with $U/V < \infty$ such that*

$$\{\|B(v)\| - \|v\| \mid v \in V\}$$

has an upper and a lower bound in Γ. Then also the set

$$\{\|B(u)\| - \|u\| \mid u \in U\}$$

is bounded from below and from above, in other words, both B and $B^{-1}\colon U' \to U$ are bounded linear operators.

Proof. Put $V' := B(V)$. V is closed, hence complete. By hypothesis, the restriction $B|_V\colon V \to V'$ is a topological isomorphism. Hence V' is complete and therefore closed. The claim now follows by applying Lemma 3.2 to B and to B^{-1}. $\square$

THEOREM 3.4. *Let $B\colon E \to E$ be an injective, bounded linear operator on E. If the set*

$$\{\|B(e_i)\| - \|e_i\| \mid i \in \mathbb{N}_0\}$$

has an upper bound in Γ, then B maps E onto E and the algebraic inverse $B^{-1}\colon E \to E$ is bounded, that is, B^{-1} belongs to $\mathcal{B}(E)$.

Proof. Let $\delta \in \Gamma$ be such that

$$-\delta \le \|B(e_i)\| - \|e_i\| \le \delta \qquad \text{for all } i \in \mathbb{N}_0 \,. \tag{1}$$

We have $\delta \in \Delta_n$ for some $n \in \mathbb{N}_0$, where Δ_n is the isolated subgroup of Γ belonging to n. Then $\|B(e_i)\| - \|e_i\| \in \Delta_n$ for all i. In particular, for all $i \in \mathbb{N}_0$, the type of $\|B(e_i)\| - \|e_i\|$ is one of $0, 1, \dots, n$. Recall that $T(\|e_i\|) = i$. Hence, if $i > n$, then $T(\|B(e_i)\| - \|e_i\|) < T(\|e_i\|)$. By Lemma 2.5 (applied with $\gamma := \|B(e_i)\|$, $\delta := \|e_i\|$), this implies that

$$T(\|B(e_i)\|) = T(\|e_i\|) = i \qquad \text{for all } \; i > n \,. \tag{2}$$

Let $U_0 := \operatorname{span}\{e_i \mid i > n\}$, $W := \operatorname{span}\{e_i \mid i \le n\}$ and let $U := \bar{U}_0$ be the closure of U_0. By orthomodularity, $E = U \overset{\perp}{\oplus} W$ and, in particular, $\dim E/U = \dim W = n + 1 < \infty$.

Consider any vector $x \in U$ and write $x = \sum_{i=n+1}^{\infty} \xi_i e_i$. Then

$$\|x\| = \min\{\|\xi_i e_i\| \mid i > n\} = \min\{2v(\xi_i) + \|e_i\| \mid i > n\}. \tag{3}$$

We have seen that, if $i > n$, then $T(B(e_i)) = i$, thus $T(B(\xi_i e_i)) = i$. In particular, if $i > j > n$, then $| B(\xi_i e_i)\| \neq \|B(\xi_j e_j)\|$. It follows that

$$\begin{aligned} \|B(x)\| = \Big\| \sum_{i=n+1}^{\infty} B(\xi_i e_i) \Big\| &= \min\{\|B(\xi_i e_i)\| \mid i > n\} \\ &= \min\{2v(\xi_i) + \|B(e_i)\| \mid i > n\}. \end{aligned} \tag{4}$$

From (1), (3), (4), we deduce that

$$-\delta \leq \|B(x)\| - \|x\| \leq \delta \qquad \text{for all} \quad x \in U.$$

This means that the restriction of B to U is a topological isomorphism between U and $U' := B(U)$, and consequently, U' is complete, so U' is closed. Hence $E = U' \overset{\perp}{\oplus} U'^{\perp}$.

If $i > n$, then $T(B(e_i)) = i$ and $B(e_i) \in U'$, thus U' contains vectors of each of the types $i > n$. By Theorem 2.6.(i), $U'^{\perp}$ can only contain vectors of types $\leq n$. It follows, again by that theorem, that $\dim U'^{\perp} \leq n + 1$. Hence $\dim E/U' \leq n + 1$. On the other hand, since B is one-to-one, we have

$$\dim E/U' = \dim E/B(U) \geq \dim B(E)/B(U) = \dim E/U = \dim W = n + 1.$$

We conclude that $\dim E/U' = \dim B(E)/U' = n + 1$, hence $B(E) = E$, i.e., B maps E onto E. The remaining assertion follows by Lemma 3.3. $\square$

4. The operator A

Let $u := \sum_{i=0}^{\infty} \frac{1}{X_i} \cdot e_i \in E$ and define the map $A_0\colon \{e_i \mid i \in \mathbb{N}_0\} \to E$ by

$$A_0(e_i) := u + \Big(1 - \frac{1}{X_i}\Big) \cdot e_i.$$

It is readily verified that $\|A_0(e_0)\| - \|e_0\| = v\Big(\frac{1}{X_1}\Big) > 0$, $\|A_0(e_i)\| - \|e_i\| = 0$ for $i > 0$. By Lemma 3.1, A_0 extends to a linear operator $A\colon E \to E$ with

$$\|A(x)\| - \|x\| \geq 0 \qquad \text{for all} \quad x \in E. \tag{6}$$

For $i \neq j$ we have $\Phi(A(e_i), e_j) = \Phi(u, e_j) = 1 = (e_i, A(e_j))$, thus A is selfadjoint.

The matrix of A with respect to the standard basis $\{e_i \mid i \in \mathbb{N}_0\}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \frac{1}{X_1} & 1 & \frac{1}{X_1} & \frac{1}{X_1} & \dots \\ \frac{1}{X_2} & \frac{1}{X_2} & 1 & \frac{1}{X_2} & \dots \\ \frac{1}{X_3} & \frac{1}{X_3} & \frac{1}{X_3} & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

We shall first prove that the operator A has no eigenvalues. Our reasonings rely on residual spaces and on types.

It follows from (6) that A maps each valuation ring R_n and each ideal J_n into itself. Consequently, A induces an operator

$$\widehat{A}_n \colon \widehat{E}_n \to \widehat{E}_n \quad \text{given by} \quad \pi_n(x) \mapsto \pi_n\big(A(x)\big) \qquad (x \in R_n)\,.$$

Recall that the base field of $\widehat{E}_n$ is $F_n := \widehat{K}_n \cong \mathbb{R}(X_1, X_1, \dots, X_n)$.

Lemma 4.1. *If $n \geq 1$, then the equation*

$$\sum_{i=0}^{n} \frac{1}{1-\rho X_i} = 1 \tag{7}$$

in the indeterminate ρ has no solution in F_n.

Proof. Suppose, indirectly, that (7) had some solution $\rho = \rho_0 \in F_n$. Clearly $\rho_0 \neq 0$. Write $\rho_0 = \dfrac{\varphi(X_n)}{\tau(X_n)}$, were $\varphi(X_n)$, $\tau(X_n)$ are relatively prime polynomials in $F_{n-1}[X_n]$. Then

$$\sum_{i=0}^{n} \frac{\tau(X_n)}{\tau(X_n) - \varphi(X_n) \cdot X_i} = 1\,. \tag{8}$$

We interpret (8) as an equality in the field $\bar{F}_{n-1}(X_n)$, where $\bar{F}_{n-1}$ is the algebraic closure of F_{n-1}. If $\deg \varphi(X_n) > 0$, then there exists a $\xi \in \bar{F}_{n-1}$ such that $\varphi(\xi) = 0$ and $\tau(\xi) \neq 0$. Substitution $X_n = \xi$ into (8) yields $n+1 = 1$, a contradiction. If $\deg \tau(X_n) > 0$, then we can find a $\xi \in \bar{F}_{n-1}$ such that $\varphi(\xi) \neq 0$ and $\tau(\xi) = 0$. Substituting into (8) we get $0 = 1$, a contradiction. Thus there only remains the case where both $\varphi(X_n)$ and $\tau(X_n)$ are constant, i.e., elements of F_{n-1}. But then (8) would entail that $X_n \in F_{n-1}$, which is impossible. □

Lemma 4.2. *If $n \geq 1$, then the operator $\widehat{A}_n \colon \widehat{E}_n \to \widehat{E}_n$ has no eigenvectors.*

Proof. For simplicity, we write $\widehat{A} = \widehat{A}_n$, $\widehat{E} = \widehat{E}_n$. $\widehat{E}$ has the base $\{\widehat{e}_0, \dots, \widehat{e}_n\}$, and $\widehat{A}$ is given by

$$\widehat{A}(\widehat{e}_k) = \sum_{j=0}^{n} \frac{1}{X_j} \widehat{e}_j + \Big(1 - \frac{1}{X_k}\Big) \widehat{e}_k\,.$$

Suppose that $\widehat{A}$ admits an eigenvalue λ and let $\widehat{x} = \sum\limits_{k=0}^{n} \xi_k \widehat{e}_k \in \widehat{E}$ be a corresponding eigenvector. Then

$$\lambda \cdot \left(\sum_{k=0}^{n} \xi_k \widehat{e}_k \right) = \widehat{A} \left(\sum_{k=0}^{n} \xi_k \widehat{e}_k \right) = \sum_{k=0}^{n} \xi_k \left(\widehat{e}_k + \sum_{i \neq k} \frac{1}{X_i} \widehat{e}_i \right).$$

Comparing the coefficients of $\widehat{e}_k$ we get

$$\lambda \cdot \xi_k = \xi_k + \frac{1}{X_k} \sum_{i \neq k} \xi_i ,$$

from which we obtain

$$\big[1 + (\lambda - 1) \cdot X_k\big] \cdot \xi_k = \sum_{i=0}^{n} \xi_i \qquad \text{for all} \quad k = 0, 1, \dots, n. \tag{9}$$

Put $\eta := \sum\limits_{i=0}^{n} \xi_i$. If $\eta = 0$, then we easily deduce from (9) that all ξ_k are 0, which is impossible. So $\eta \neq 0$ and $\big[1 + (\lambda - 1) \cdot X_k\big] \cdot \xi_k \neq 0$ for all $k = 0, 1, \dots, n$. Then

$$\xi_k = \frac{\eta}{1 + (\lambda - 1) \cdot X_k} \qquad \text{for} \quad k = 0, 1, \dots, n.$$

Summing up these equations and dividing by η we obtain

$$1 = \sum_{k=0}^{n} \frac{1}{1 + (\lambda - 1) \cdot X_k}.$$

But this is excluded by Lemma 4.1. □

THEOREM 4.3. *The operator A admits no closed invariant subspaces (except the trivial ones).*

P r o o f. Suppose, indirectly, that there is a closed subspace $U \neq \{0\}$ in E that is invariant under A. Then $E = U \oplus U^{\perp}$ by orthomodularity. $U^{\perp}$ is also invariant, for A is selfadjoint. We now look at the types of vectors in U and $U^{\perp}$. By Theorem 2.6.(i), no type can occur in both U and $U^{\perp}$. Hence there exists an integer $n \geq 1$ such that U contains vectors of each of types $0, \dots, n-1$ and $U^{\perp}$ contains a vector of type n. We examine the reduced operator $\widehat{A}_n$ on the residual space

$$\widehat{E}_n = \pi_n(E) = \pi_n(U) \overset{\perp}{\oplus} \pi_n(U^{\perp}) = \widehat{U}_n \oplus \widehat{U}_n^{\perp} ;$$

$\widehat{U}_n$ and $\widehat{U}_n^{\perp}$ are invariant under $\widehat{A}_n$. Let G be a one-dimensional subspace of $U^{\perp}$ spanned by a vector of type n. Then $U^{\perp} = G \overset{\perp}{\oplus} (U^{\perp} \cap G^{\perp})$ and $\pi_n(U^{\perp}) =$

$\pi_n(G)\overset{\perp}{\oplus}\pi_n(U^\perp \cap G^\perp)$. By the choice of n and by Theorem 2.6.(i), $U^\perp \cap G^\perp$ contains only vectors of types $i > n$, and therefore, by Lemma 2.8, $\pi_n(U^\perp \cap G^\perp) = \{0\}$. Hence $\pi_n(U^\perp) = \pi_n(G)$. Now $\pi_n(G)$ is one-dimensional (by Lemma 2.8) and is invariant under $\widehat{A}_n$. In other words, $\widehat{A}_n$ has an eigenvalue. But this is impossible by Lemma 4.2. □

COROLLARY 4.4. *The operator A has no eigenvalues.*

We proceed to determine the *spectrum* of A which is defined as usual by

$$\operatorname{spec}(A) := \bigl\{\lambda \in K \mid (A - \lambda I) \text{ has no inverse in } \mathcal{B}(E)\bigr\},$$

where I is the identity operator. The main result is

THEOREM 4.5. *$spec(A) = \{1\}$.*

Proof. By Corollary 4.4, each operator $A - \lambda I$ is injective. In view of Theorem 3.4, $A - \lambda I$ is invertible in $\mathcal{B}(E)$ if and only if $\bigl\{\|(A-\lambda I)(e_i)\| - \|e_i\| \mid i \in \mathbb{N}_0\bigr\} \subseteq \Gamma$ has an upper bound. From the definition of A, we obtain

$$(A - \lambda I)(e_i) = u + \Bigl(1 - \frac{1}{X_i}\Bigr)e_i - \lambda e_i = \sum_{k \neq i} \frac{1}{X_k} e_k + (1 - \lambda)e_i \,.$$

Put $w_i := \sum\limits_{k\neq i} \dfrac{1}{X_k} e_k$; then $\|w_0\| = -v(X_1)$ and $\|w_i\| = 0$ for $i > 0$.

a) Suppose that $\lambda \neq 1$. Then

$$\|(1 - \lambda)e_i\| = 2v(1 - \lambda) + \|e_i\| = 2v(1 - \lambda) + v(X_i) \to -\infty \qquad \text{for} \quad i \to \infty \,.$$

Hence, for all sufficiently large i we have

$$\begin{aligned} \|(A - \lambda I)(e_i)\| - \|e_i\| &= \|w_i + (1 - \lambda)e_i\| - \|e_i\| \\ &= \|(1 - \lambda)e_i\| - \|e_i\| = 2v(1 - \lambda)\,, \end{aligned}$$

so the set $\bigl\{\|(A - \lambda I)(e_i)\| - \|e_i\| \mid i \in \mathbb{N}_0\bigr\}$ is bounded, as claimed.

b) In case $\lambda = 1$, we have $(A - I)(e_i) = w_i$, thus $\|(A - I)(e_i)\| = 0$ for $i > 0$.

It follows that $\|(A - I)(e_i)\| - \|e_i\| = -\|e_i\| \to \infty$ when $i \to \infty$, so $(A - I)$ is not invertible. □

5. The algebra $\mathcal{A}$

We now turn to von Neumann subalgebras of $\mathcal{B}(E)$. The most straightforward way to single out such algebras is by taking all elements in $\mathcal{B}(E)$ which commute with some specific selfadjoint operator. It turns out that the operator

A considered in Section 4 gives rise to a particularly interesting algebra. We therefore introduce

$$\mathcal{A} := \{B \in \mathcal{B}(E) \mid A \circ B = B \circ A\}.$$

It is readily seen that $\mathcal{A}$ is a *von Neumann* algebra in the sense that

(i) $\mathcal{A}$ is closed under taking adjoints,
(ii) $\mathcal{A}$ coincides with its double centralizer.

Notice that $\mathcal{A}$ is closed in the norm topology on $\mathcal{B}(E)$, hence also in the topology of pointwise convergence.

The fact that A has no invariant closed subspaces entails that every non-zero vector is topologically cyclic which in turn has the rather peculiar consequence that two distinct operators $B, C \in \mathcal{A}$ differ on every non-zero vector in E.

LEMMA 5.1. *For every vector $x \neq 0$ in E the linear subspace $S := \operatorname{span}\{A^n(x) \mid n \in \mathbb{N}_0\}$ is topologically dense in E.*

Proof. Clearly S is invariant under A. Since A is continuous, the topological closure $\overline{S}$ is also invariant, moreover $0 \neq x \in \overline{S}$. By Theorem 4.3, we conclude that $\overline{S} = E$. □

LEMMA 5.2. *If two operators $B, C \in \mathcal{A}$ coincide on some vector $x \neq 0$, then $B = C$.*

Proof. $B(x) = C(x)$ implies that $B(A^n(x)) = C(A^n(x))$ for all $n \geq 0$. Thus B and C are equal on $S = \operatorname{span}\{A^n(x) \mid n \in \mathbb{N}_0\}$. By continuity, B and C are equal on $\overline{S} = E$. □

COROLLARY 5.3. *Every non-zero operator $B \in \mathcal{A}$ is injective.*

The above Lemma 5.2 states that the operators $B \in \mathcal{A}$ are determined by their action on some fixed vector $x \neq 0$ and can therefore be represented by their image vectors $B(x)$. For computational reasons it is convenient to take $x := e_0$. Thus we introduce the linear, injective map

$$\Psi \colon \mathcal{A} \to E \quad \text{defined by} \quad B \mapsto \Psi(B) := B(e_0).$$

We are going to determine the range of Ψ, $W := \{B(e_0) \mid B \in \mathcal{A}\}$. We first prove that all base vectors e_m belong to W, and then we establish the main result in this section, which provides a representation of the operators $B \in \mathcal{A}$ in terms of the matches $C_m := \Psi^{-1}(e_m)$.

LEMMA 5.4. *For $m \in \mathbb{N}_0$ let $\rho_m := 1 - 1/X_m$, and put $C_m := I + \rho_m(A - \rho_m I)^{-1}$. Then*

(i) $C_m(e_0) = e_m$,

(ii) $C_m(e_k) = \dfrac{X_m(X_k-1)}{X_k - X_m} e_k - \dfrac{X_k(X_m-1)}{X_k - X_m} e_m$ *for* $k \neq m$,

(iii) $C_m(e_m) = X_m \cdot \Big(1 + \sum\limits_{k \neq m} \dfrac{X_m - 1}{X_k - X_m}\Big) \cdot e_m - (X_m - 1) \cdot \sum\limits_{k \neq m} \dfrac{X_m}{X_k - X_m} e_k$.

In particular, $e_m \in W$ *and* $C_m = \Psi^{-1}(e_m)$.

Proof. Notice that $\rho_m \notin \operatorname{spec}(A)$, so $D_m := (A - \rho_m I)$ has an inverse in $\mathcal{B}(E)$. Clearly, D_m^{-1} commutes with A, thus $D_m^{-1} \in \mathcal{A}$.

If $m = 0$, then $\rho_0 = 0$, $C_0 = I$, and our formulas are trivially true. Assume $m > 0$. Recall that $A(e_k) = u + \rho_k e_k$, where $u = \sum\limits_{i=0}^{\infty} \dfrac{1}{X_i} e_i$. Thus

$$D_m(e_k) = (A - \rho_m I)(e_k) = u + (\rho_k - \rho_m) e_k ,$$
$$D_m(e_0) = u - \rho_m e_0 .$$

Subtraction yields $D_m(e_k - e_0) = (\rho_k - \rho_m)e_k + \rho_m e_0$, and applying D_m^{-1} we obtain

$$e_k - e_0 = (\rho_k - \rho_m) D_m^{-1}(e_k) + \rho_m D_m^{-1}(e_0) . \tag{10}$$

In particular, putting $k = m$ in (10) we get

$$\rho_m \cdot D_m^{-1}(e_0) = e_m - e_0 . \tag{11}$$

It follows that $C_m(e_0) = (I + \rho_m D_m^{-1})(e_0) = e_m$, which proves (i).

Next, substituting (11) into (10) yields

$$e_k - e_m = (\rho_k - \rho_m) D_m^{-1}(e_k) .$$

In the case $k \neq m$, it follows that $D_m^{-1}(e_k) = \dfrac{1}{\rho_k - \rho_m} \cdot (e_k - e_m)$, hence $C_m(e_k) = (I + \rho_m D_m^{-1})(e_k) = \Big(1 + \dfrac{\rho_m}{\rho_k - \rho_m}\Big) e_k - \dfrac{\rho_m}{\rho_k - \rho_m} e_m$. After replacing ρ_m, ρ_k by $1 - 1/X_m$ and $1 - 1/X_k$ respectively, we obtain (ii).

Finally, to establish (iii), we start with

$$D_m(e_m) = u + (\rho_m - \rho_m) e_m = u = \frac{1}{X_m} e_m + \sum_{k \neq m} \frac{1}{X_k} e_k ,$$

from which we obtain $e_m = X_m \cdot \Big[D_m(e_m) - \sum\limits_{k \neq m} \dfrac{1}{X_k} e_k\Big]$, hence $D_m^{-1}(e_m) = X_m \Big[e_m - \sum\limits_{k \neq m} \dfrac{1}{X_k'} D_m^{-1}(e_m)\Big]$. It follows that

$$C_m(e_m) = e_m + \rho_m D_m^{-1}(e_m) = e_m + \rho_m X_m \Big[e_m - \sum_{k \neq m} \frac{1}{X_k} D_m^{-1}(e_k)\Big]$$
$$= e_m + \rho_m X_m \Big[e_m - \sum_{k \neq m} \frac{1}{X_k} \cdot \frac{1}{\rho_k - \rho_m} (e_k - e_m)\Big] .$$

Gathering terms on the right hand side and replacing ρ_m and ρ_k by $1-1/X_m$, $1-1/X_k$ we arrive at (iii). The proof is complete. $\square$

Notice that along with A every operator $C_m = I + \rho_m(A-\rho_m I)^{-1}$ is selfadjoint, moreover, C_m commutes with all $B \in \mathcal{A}$.

COROLLARY 5.5. *Every $B \in \mathcal{A}$ is selfadjoint.*

P r o o f. Using Lemma 5.4.(i),(ii) we see that $C_m(e_k) = C_k(e_m)$ for all $m, k \in \mathbb{N}_0$. Consequently, for any $B \in \mathcal{A}$,

$$\begin{aligned}\Phi\big(B(e_m), e_k\big) &= \Phi\big(B\big(C_m(e_0)\big), e_k\big) = \Phi\big(C_m\big(B(e_0)\big), e_k\big)\\ &= \Phi\big(B(e_0), C_m(e_k)\big) = \Phi\big(B(e_0), C_k(e_m)\big)\\ &= \Phi\big(C_k\big(B(e_0)\big), e_m\big) = \Phi\big(B\big(C_k(e_0)\big), e_m\big)\\ &= \Phi\big(B(e_k), e_m\big) = \Phi\big(e_m, B(e_k)\big)\end{aligned}$$

as claimed. $\square$

We shall need the norm of the vectors $C_m(e_k)$.

LEMMA 5.6.

(i) $\|C_m(e_k)\| = 2v(X_k) + v(X_m)$ *for* $k < m$,
(ii) $\|C_m(e_k)\| = 2v(X_m) + v(X_k)$ *for* $k > m$,
(iii) $\|C_m(e_m)\| = 3v(X_m)$ *for* $m \neq 1$,
(iv) $\|C_1(e_1)\| = 2v(X_1)$.

P r o o f. This is established by a straightforward verification using formulae (i), (ii), (iii) in Lemma 5.4 and the definition of the non-archimedian valuation v. $\square$

Recall that a typical vector $x \in E$ has the shape $x = \sum_{i=0}^{\infty} \xi_i e_i$, where $v(\xi_i^2 x_i) \to \infty$ when $i \to \infty$. It is convenient to put $\lambda_i := \xi_i X_i$ and to write x as

$$x = \sum_{i=0}^{\infty} \frac{\lambda_i}{X_i} e_i \,.$$

The numerators λ_i are subject to the condition that $v(\lambda_i^2/X_i) \to \infty$ when $i \to \infty$. This condition does of course not imply that the values $v(\lambda_i) \in \Gamma$ are bounded from below. We now show that for vectors x in $W = \mathrm{Im}(\Psi)$ the values $v(\lambda_i)$ must be bounded.

LEMMA 5.7. *Let $B \in \mathcal{A}$ and write $w := B(e_0) = \sum_{k=0}^{\infty} \frac{\lambda_k}{X_k} e_k$. Then $\big\{v(\lambda_k) \mid k \in \mathbb{N}_0\big\} \subseteq \Gamma$ has a lower bound.*

P r o o f. Replacing B by some scalar multiple ηB, if necessary, we may assume that $\|B(x)\| - \|x\| \geq 0$ for all $x \in E$. In particular, $\|B(e_0)\| - \|e_0\| \geq 0$, so

$$\|w\| = \Big\|\sum_{k=0}^{\infty} \frac{\lambda_k}{X_k} e_k\Big\| = \min\Big\{\Big\|\frac{\lambda_k}{X_k} e_k\Big\| \ \Big|\ k \in \mathbb{N}_0\Big\} \geq 0 .$$

Thus for all $k \in \mathbb{N}_0$ we have $\Big\|\frac{\lambda_k}{X_k} e_k\Big\| = 2v(\lambda_k) - v(X_k) \geq 0$.

Now suppose, indirectly, that $\{v(\lambda_k) \mid \mathbb{N}_0\}$ has no lower bound. Then we can pick an integer $m > 1$ such that

$$v(\lambda_m) < 0 \quad \text{and} \quad v(\lambda_m) < v(\lambda_k) \qquad \text{for all} \quad k < m . \tag{12}$$

We compute the norm of the vector

$$B(e_m) = B\big(C_m(e_0)\big) = C_m\big(B(e_0)\big) = \sum_{k=0}^{\infty} \frac{\lambda_k}{X_k} C_m(e_k) .$$

a) Using Corollary 5.6.(iii) we obtain

$$\Big\|\frac{\lambda_m}{X_m} C_m(e_m)\Big\| = 2v(\lambda_m) - 2v(X_m) + \|C_m(e_m)\| = 2v(\lambda_m) + v(X_m) . \tag{13}$$

b) If $k < m$, then using Corollary 5.6.(i), we find $\Big\|\frac{\lambda_k}{X_k} C_m(e_k)\Big\| = 2v(\lambda_k) + v(X_m)$. Since $v(\lambda_m) < v(\lambda_k)$, it follows that

$$\Big\|\frac{\lambda_m}{X_m} C_m(e_m)\Big\| < \Big\|\frac{\lambda_k}{X_k} C_m(e_k)\Big\| \qquad \text{for all} \quad k < m . \tag{14}$$

c) Similarly, if $k > m$, then we find

$$\Big\|\frac{\lambda_k}{X_k} C_m(e_k)\Big\| = 2v(\lambda_k) + 2v(X_m) - v(X_k) .$$

Here $2v(\lambda_k) - v(X_k) \geq 0$ and $k > m$, from which it follows by Lemma 2.1.(ii) that $2v(\lambda_k) - v(X_k) > -v(X_{k-1}) \geq -v(X_m)$. Hence

$$\Big\|\frac{\lambda_k}{X_k} C_m(e_k)\Big\| > 2v(X_m) - v(X_m) = v(X_m) .$$

Using (12) and (13) we see that

$$\Big\|\frac{\lambda_m}{X_m} C_m(e_m)\Big\| < \Big\|\frac{\lambda_k}{X_k} C_m(e_k)\Big\| \qquad \text{for all} \quad k > m . \tag{15}$$

From (14), (15) we conclude that

$$\|B_m(e_m)\| = \Big\|\sum_{k=0}^{\infty} \frac{\lambda_k}{X_k} C_m(e_k)\Big\| = \Big\|\frac{\lambda_m}{X_m} C_m(e_m)\Big\| = 2v(\lambda_m) + v(X_m) .$$

Therefore $\|B_m(e_m)\| - \|e_m\| = 2v(\lambda_m)$, thus $\|B_m(e_m)\| - \|e_m\| < 0$ by (12). But this contradicts the assumption on B. The proof is complete. □

LEMMA 5.8. *For every vector $x \in E$ we have*

$$\frac{1}{X_m} C_m(x) \to 0 \qquad \textit{when} \quad m \to \infty .$$

Proof. Write $x = \sum_{k=0}^{\infty} \xi_k e_k$. We may assume that $\|x\| \geq 0$, thus $\|\xi_k e_k\| = 2v(\xi_k) + v(X_k) \geq 0$ for all k. Let $m > 1$. We estimate the norm of the vector

$$\frac{1}{X_m} C_m(x) = \sum_{k=0}^{\infty} \frac{1}{X_m} \cdot \xi_k \cdot C_m(e_k) .$$

(i) Suppose that $k < m$. Using Corollary 5.6.(i) we find

$$\begin{aligned}\left\| \frac{1}{X_m} \xi_k C_m(e_k) \right\| &= 2v(\xi_k) + 2v(X_k) - v(X_m) \\ &= \big(2v(\xi_k) + v(X_k)\big) + \big(v(X_k) - v(X_m)\big) \\ &\geq v(X_k) - v(X_m) > -v(X_{m-1}) .\end{aligned}$$

(ii) Suppose that $k \geq m$. Then Corollary 5.6.(ii) or (iii) yields

$$\left\| \frac{1}{X_m} \xi_k C_m(e_k) \right\| = 2v(\xi_k) + v(X_k) .$$

Now $2v(\xi_k) + v(X_k) \geq 0$ by assumption, so by Lemma 2.1.(i), we have $2v(\xi_k) + v(X_k) > -v(X_{k-1}) \geq -v(X_{m-1})$, i.e.,

$$\left\| \frac{1}{X_m} \xi_k C_m(e_k) \right\| > -v(X_{m-1}) .$$

Combining the above inequalities we see that

$$\left\| \frac{1}{X_m} C_m(x) \right\| > -v(X_{m-1}) \qquad \text{for all} \quad m > 1 ,$$

hence $\left\| \frac{1}{X_m} C_m(x) \right\| \to \infty$ when $m \to \infty$, as claimed. □

THEOREM 5.9. *Let $B \in \mathcal{A}$ and write $w = B(e_0) = \sum_{m=0}^{\infty} \frac{\lambda_m}{X_m} e_m$. Then B can be represented as the limit of the series of operators*

$$\sum_{m=0}^{\infty} \frac{\lambda_m}{X_m} C_m$$

in the topology of pointwise convergence.

Proof. By Lemma 5.7, there exists a $\gamma \in \Gamma$ such that $v(\lambda_m) \geq \gamma$ for all $m \in \mathbb{N}_0$. From Lemma 5.8, we deduce that, for every $x \in E$,

$$\frac{\lambda_m}{X_m} C_m(x) \to 0 \qquad \text{when} \quad m \to \infty .$$

Since the norm on E is non-archimedian, it follows that the series $\sum\limits_{m=0}^{\infty} \frac{\lambda_m}{X_m} C_m$ converges pointwise to some operator $D \colon E \to E$. Using Corollary 5.6. it is readily checked that

$$\|D(e_k)\| - \|e_k\| \geq 2\gamma \qquad \text{for all} \quad k \in \mathbb{N}_0 \, .$$

Thus, by Theorem 3.4, D is bounded. Each C_m commutes with A, hence so does D. In other words, $D \in \mathcal{A}$. Now

$$D(e_0) = \sum_{m=0}^{\infty} \frac{\lambda_m}{X_m} C_m(e_0) = \sum_{m=0}^{\infty} \frac{\lambda_m}{X_m} e_m = B(e_0) \, ,$$

and, by Lemma 5.2, we conclude that $B = D$. □

From the above proof and from Lemma 5.7. we deduce

COROLLARY 5.10. *W consists of all vectors $\sum\limits_{m=0}^{\infty} \frac{\lambda_m}{X_m} e_m$ for which $\{v(\lambda_m) \mid m \in \mathbb{N}_0\}$ is bounded from below.*

A rather unexpected consequence of Theorem 5.9. is the following.

COROLLARY 5.11. *The algebra $\mathcal{A}$ is commutative.*

Proof. Any two elements $B, D \in \mathcal{A}$ can be represented as $B = \sum\limits_{m=0}^{\infty} \frac{\lambda_m}{X_m} C_m$, $D = \sum\limits_{m=0}^{\infty} \frac{\mu_m}{X_m} C_m$ (limits are in the topology of pointwise convergence). Now $C_m \circ C_{m'} = C_{m'} \circ C_m$ for all $m, m' \in \mathbb{N}_0$, consequently B commutes with D. □

We have established that $\mathcal{A}$ is a commutative algebra. In view of Corollary 5.3, it follows that $\mathcal{A}$ is in fact an integral domain. It seems now natural to study $\mathcal{A}$ along the line of concepts of ring theory such as factorization, prime ideals and others. This task will be pursued in another article.

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