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ON CLIQUISH FUNCTIONS ON PRODUCT SPACES

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The functions considered in this paper are cliquish ones (Bledsoe [1], Marcus [3], Neubrunnová [7]) and the functions which are almost continuous in the sense of Husain (see [2]). In the case when the domain of a function is a product space, we introduce a new class of functions, i.e., functions which are cliquish with respect to one of the variables.

Let M be a metric space with a metric d and let X be a topological space. A function $f: X \rightarrow M$ is called *cliquish at a point* $x \in X$ if for any $\varepsilon > 0$ and for any neighbourhood U of x there exists a non-empty open set $G \subset U$ such that for any $x_1, x_2 \in G$ we have $d(f(x_1), f(x_2)) < \delta$. A function f is called *cliquish* if f is cliquish at each point $x \in X$.

Let X and Y be topological spaces. A function f from X to Y is called *almost continuous* (in the sense of Husain) *at a point* $x \in X$ if for any neighbourhood V of $f(x)$ the set $\operatorname{Int}(\operatorname{cl}(f^{-1}(V)))$ is a neighbourhood of x. A function f is called *almost continuous* if it is almost continuous at each point $x \in X$.

The restriction of a cliquish function to any open (or dense) subspace is a cliquish one. The restriction of a cliquish function to a closed subspace need not be cliquish (see [4], Example 1).

Recall that a function $f: X \rightarrow Y$ is called *almost everywhere continuous* if the set D_f of points at which f is not continuous is of the first category.

We start with the following

Theorem 1. *Let X be a Baire space, Y be a metric one and let $f: X \rightarrow Y$ be an arbitrary function. The function f is cliquish if and only if f is almost everywhere continuous.*

Proof. The "only if" part of the proof is due to Neubrunnová [7, Theorem 8, p. 114]. The proof of the "if" part:

Suppose that f is not cliquish at a point $x \in X$. Then there exists $\varepsilon > 0$ and an open, non-empty subset U of X such that for each open non-empty subset $G \subset U$ we have $\operatorname{diam}(f(G)) \geq \varepsilon$. Hence f is not cliquish at each point of U and therefore f is not continuous at each point of U. Since U is the open, non-empty subset of a Baire space, U is of the second category. Thus U is contained in D_f. But D_f is of the first category and this contradicts the assumption that the function f is almost everywhere continuous.

The next theorem characterizes the continuity of the function f.

Theorem 2. *Let X be a topological space, Y be a metric one. A function $f\colon X\to Y$ is continuous if and only if f is almost continuous and f is cliquish.*

Proof. The necessity is obvious. To see the sufficiency suppose f not to be continuous. Choose a point x_0 of discontinuity. Therefore there exists $\varepsilon>0$ such that for each open neighbourhood U of x_0 there exists $y\in U$ such that $d(f(x_0), f(y))\geqslant\varepsilon$.

Let $B(f(x_0), \varepsilon/3)$ be the open ball in the space Y centred at $f(x_0)$ with the radius $\varepsilon/3$. Since f is almost continuous, there exists a neighbourhood U of x_0 such that the set $f^{-1}(B(f(x_0), \varepsilon/3))$ is dense in U.

Since f is not continuous at x_0, there exists $y\in U$ such that $d(f(x_0), f(y))\geqslant\varepsilon$. Let $B(f(y), \varepsilon/3)$ be a neighbourhood of $f(y)$. Since f is almost continuous, there exists a non-empty, open neighbourhood V_y of y such that $f^{-1}(B(f(y), \varepsilon/3))$ is a dense subset of V_y. From the cliquishness of f it follows that there exists an open subset U_2 of V_y such that for each $x_1, x_2\in U_2$ we have $d(f(x_1), f(x_2))<\varepsilon/3$.

Since $f^{-1}(B(f(y), \varepsilon/3))$ is dense in V_y, it follows that $f^{-1}(B(f(y), \varepsilon/3))\cap U_2\neq\emptyset$. Since $f^{-1}(B(f(x_0), \varepsilon/3))$ is dense in U and thereby dense in V_y, it follows that $f^{-1}(B(f(x_0), \varepsilon/3))\cap U_2\neq\emptyset$. Let $x_1\in U_2\cap f^{-1}(B(f(x_0), \varepsilon/3))$ and $x_2\in U_2\cap f^{-1}(B(f(y), \varepsilon/3))$. Therefore $d(f(x_0), f(y)) \leqslant d(f(x_0), f(x_1)) + d(f(x_1), f(x_2)) + d(f(x_2), f(y)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3=\varepsilon$, a contradiction.

In the following we present some results on generalized continuity on product spaces. Similar problems were investigated by Martin [4], Neubrunn [5], [6] and Piotrowski [8].

Let $f\colon X\times Y\to M$ be a function, where X and Y are topological spaces and M is a metric one. For each $x\in X$ let f_x denotes the function $f_x\colon Y\to M$ such that $f(x, y)=f_x(y)$. The function f^y is defined similarly. We call f_x and f^y the section of f.

Recall that a function $f\colon X\to Y$ is called *quasi-continuous at a point x* provided that for each open neighbourhood U of x and for each open neighbourhood V of $f(x)$ there exists a non-empty open set $G\subset U$ such that $f(G)\subset V$ (see [4, p. 39]).

We prove the following

Theorem 3. *Let X be a Baire space, Y be a space such that for each point $y\in Y$ there exists an open neighbourhood which satisfies the second countability axiom and let M be a metric space with a metric d. Further let $f\colon X\times Y\to M$ be a function such that for each $x\in X$ the section f_x is cliquish and for each $y\in Y$ the section f_y is quasi-continuous. Then f is cliquish.*

Proof. Let $(p, q)\in U\times V\subset X\times Y$, where U and V are open neighbourhoods of p and q, respectively. Let $\varepsilon>0$ be fixed. Without loss of generality we may assume that V has a countable base $\{G_n\}_{n=1}^{\infty}$.

For each $n\in N$, let H_n be the set of all $x\in U$ for which there exists an open set $V_x\subset V$ such that $G_n\subset V_x$ and that for each $y_1, y_2\in V_x$ we have $d(f(x, y_1), f(x, y_2))<\varepsilon/8$.

Observe that $U = \bigcup_n H_n$. In fact, if $x \in U$, then from the cliquishness of f_x there exists a non-empty open set $V_x \subset V$ such that for each $y_1, y_2 \in V_x$ we have $d(f(x, y_1), f(x, y_2)) < \varepsilon/8$. Since $\{G_n\}_{n=1}^{\infty}$ is a base for V, there exists a number n such that $G_n \subset V_x$ and therefore $x \in H_n$. On the other hand if $x \in \bigcup_n H_n$, there exists $n \in N$ such that $x \in H_n$, and from the definition of H_n we have $x \in U$.

Since X is Baire, the set U is not of the first category. Thus there exists a number $n \in N$ and a non-empty open set $U' \subset U$ such that $H_n \cap U'$ is dense in U'. Let $(a, b) \in U' \times G_n$. Since f^b is quasi-continuous, it is cliquish. Hence there exists a non-empty open subset $U'' \subset U'$ such that for each $x_1, x_2 \in U''$ we have $d(f(x_1, b), f(x_2, b)) < \varepsilon/8$.

Consider the set $T = (U'' \times \{b\}) \cup ((H_n \cap U'') \times G_n)$. It is easy to see that $\operatorname{Int}(\operatorname{cl}(T)) = \emptyset$.

Let $(x, y) \in U'' \times G_n$ and $(s, t) \in T$. Since f^y is quasicontinuous, then there exists an open set $U_y \subset U''$ such that for each $x_1 \in U_y$ we have $d(f(x_1, y), f(x, y)) < \varepsilon/8$. Since $H_n \cap U''$ is dense in U'', there exists $x_0 \in U_y \cap U'' \cap H_n$. This implies that $(x_0, y) \in T$. Therefore we have $d(f(x, y), f(s, t)) \leqslant d(f(x_0, y), f(x, y)) + d(f(x_0, y), f(x_0, b)) + d(f(x_0, b), f(s, b)) + d(f(s, b), f(s, t)) < \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 = \varepsilon/2$.

Hence it follows that for each $(x', y'), (x, y) \in U'' \times G_n$ we have $d(f(x', y'), f(x, y)) < \varepsilon$. Since $U'' \times G_n$ is an open, nonempty subset of $U \times V$ we have proved that f is cliquish at (p, q).

Now, we will prove the essentiality of some assumptions of Theorem 3. The following example shows that the assumption of quasi-continuity of the section f^y for each $y \in Y$ cannot be omitted.

Example 4. We construct a function $f: I \times I \to I$, where $I = [0, 1]$ such that for each $x \in I$ the section f_x is cliquish and for each $y \in I$ the section f^y is cliquish and f is not cliquish.

Let A be the set of all points $(x, y) \in I \times I$ such that $x = k/2^n$ and $y = m/2^n$ where k and m are odd numbers less than 2^n and $n = 1, 2, \ldots$.

The set A is dense in $I \times I$, and for each $x \in I$ the sets $(\{x\} \times I) \cap A$ and $(I \times \{x\}) \cap A$ are finite. We define

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in A \\ 0 & \text{if } (x, y) \notin A. \end{cases}$$

The following example shows that the assumption that Y is a Baire space cannot be omitted in Theorem 3.

Example 5. In this construction, we make use of what we call "the cross operation". Let a square S be given in the plane. A cross C means the difference between the union of two closed straight line segments connecting the centres of the opossite sides of the square S and the mentioned centres. In other words, C is the union of two segments without end points, parallel to the sides of S and intersecting at its centre. We distinguish two colours of cross, red and black.

Consider the unit square I^2. We put the red cross first. In the resulting figure, composed of four subsquares, we put red crosses in the upper left subsquare and in the lower right one, and we put black crosses in the rest, i.e., in the upper right subsquares and in the lower left one. Each of the previous subsquares splits into four subsquares, each of which we fill according to the general algoritm: we put red crosses in the upper left and the lower right subsquares and we put black crosses in the upper right and in the lower left subsquares. We continue this procedure countable many times. Let us take the union K of all the crosses (i.e. red and black ones) in the unit square I^2. It is easy to see that $Q \times Q$ is contained in K, where Q is the set of rationals of the form $k/2^n$ with k, $n \in N$ and $k<2^n$.

Now, we define a function f: $Q \times Q \to I$ as follows:

$$f(x, y)=\begin{cases} 0 \text{ for } (x, y) \text{ belonging to a black cross,} \\ 1 \text{ for } (x, y) \text{ belonging to a red cross.} \end{cases}$$

Standard arguments show that all sections f_x and f^y are quasi-continuous, while f is not cliquish. The just presented example shows also the essentiality of the assumption that Y is Baire in Martin [4, Theorem 1, p. 39].

From Theorem 8 in [7, p. 114] of Neubrunnová and Theorem 3 there follows

Corollary 6. *Let X be a Baire space, Y be a second countable Baire space and let M be a metric one. If for a function f: $X \times Y \to M$ the section f_x is cliquish for each $x \in X$ and the section f^y is quasi-continuous for each $y \in Y$, then the set of points of continuity of f is a residual subset of $X \times Y$.*

Example 7. A slight modification of Martin's Example 1 in [4] shows that cliquishness of f: $X \times Y \to M$ does not imply that all sections f_x or all sections f^y are cliquish.

Now we will define the notion of a function cliquish with respect to x, which is similar to the notion of the function quasi-continuous with respect to x, introduced by Martin in [4, p. 41].

Definition 8. *Let X and Y be topological spaces and M be a metric space with a metric d. A function f: $X \times Y \to M$ is said to be cliquish with respect to x at a point $(p, q) \in X \times Y$ if for any $\varepsilon > 0$ and any neighbourhood U' of p included in U, and there exists an open, non-empty subset V' of V such that for each (x, y), $(x', y') \in U' \times V'$ we have $d(f(x, y), f(x', y')) < \varepsilon$. A function f: $X \times Y \to M$ is called cliquish with respect to x if it is cliquish with respect to x at each point of $X \times Y$.*

Obviously each function which is cliquish with respect to x is cliquish.

The following theorem holds

Theorem 9. *Let X and Y be topological spaces such that Y is Baire and M is metric. If a function $f\colon X\times Y\to M$ is cliquish with respect to x, then for all $x\in X$ the set of points of continuity of f is a dense G_δ-subset of $\{x\}\times I$.*

Proof. The same as the proof of Piotrowski [8, Theorem, p. 114].

Let a function $f\colon X\times Y\to M$ be given, where X and Y are topological spaces and M is metric. One can ask whether the conditions that for each $x\in X$ the section f_x is cliquish and for each $y\in Y$ the section f^y is continuous imply that f is cliquish with respect to x. Recall that the corresponding result is true when we put a quasi-continuous function instead of a cliquish one (see Martin [4, Theorem 3, p. 41]). However, the following example shows a negative answer to the above question.

Example 10. Let $n=1, 2, \ldots$ and p be an odd number such that $0<p<2^n$. Define $f\colon I\times I\to I$ putting

$$f(x, y)=\begin{array}{l} 0 \text{ if } x=1/2^n \text{ and } y=p/2^n,\\ 0 \text{ if } x\in I\setminus[1/2^{n+1}, 1/2^{n-1}] \text{ and } y=p/2^n,\\ 2^{n+1}(x-1/2^{n+1}) \text{ if } x\in[1/2^{n+1}, 1/2^n] \text{ and } y=p/2^n,\\ 2^n(1/2^{n-1}-x) \text{ if } x\in[1/2^n, 1/2^{n-1}] \text{ and } y=p/2^n,\\ 0 \text{ for all remaining points } (x, y).\end{array}$$

Of course, for each $y\in Y$ the section f^y is continuous. For each $x\in I$ the section f_x is cliquish, because f_x is discontinuous at a finite number of points. The function f is not cliquish with respect to x at the point $(0, y)$ where $y\in I$, because each neighbourhood of $(0, y)$ contains points at which f takes the value 1 and 0.

Example 11. Now we construct a function $f\colon[-1, 1]\times[-1, 1]\to[0, 1]$ such that for each $x\in X$ the section f_x is quasi-continuous and for each $y\in Y$ the section f^y is quasi-continuous and f is not cliquish with respect to x.

Let $f_1\colon(0, 1]\times(0, 1]\to[0, 1]$ be a function defined by

$$f_1(x, y)=\begin{cases} 1 \text{ if } x\in(1/2^{n+1}, 1/2^n] \text{ and } y\in(1/2^{n+1}, 1/2^n] \text{ for } n=0, 1, \ldots\\ 0 \text{ if } x\in(1/2^{n+1}, 1/2^n] \text{ and } y\in I\setminus(1/2^{n+1}, 1/2^n] \text{ for } n=0, 1, \ldots\end{cases}$$

Let us define $f_2\colon[-1, 0)\times(0, 1]\to[0, 1]$ by $f_2(x, y)=-f_1(-x, y)+1$, $f_3\colon[-1, 0)\times[-1, 0)\to[0, 1]$ by $f_3(x, y)=f_1(-x, -y)$, and $f_4\colon(0, 1]\times[-1, 0)\to[0, 1]$ by $f_4(x, y)=f_2(-x, -y)$.

Now, we define f as follows

$$f(x, y)=\begin{array}{l} 0 \text{ if } x=0 \text{ or } y=0,\\ f_1(x, y) \text{ if } x>0 \text{ and } y>0,\\ f_2(x, y) \text{ if } x<0 \text{ and } y>0,\\ f_3(x, y) \text{ if } x<0 \text{ and } y<0,\\ f_4(x, y) \text{ if } x>0 \text{ and } y<0.\end{array}$$

This example also shows that a quasi-continuous function need not be cliquish with respect to x or to y.

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О ФУНКЦИЯХ CLIQUISH НА ПРЯМОМ ПРОИЗВЕДЕНИИ ПРОСТРАНСТВ

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Резюме

Результаты, касающиеся функции cliquish на топологическом пространстве, получены в [1], [3], [7]. В статье представлены некоторые новые результаты в том же круге задач (теоремы 1–2). Кроме этого получено некоторую характеризацию функции cliquish на прямых произведениях топологических пространств (теорема 3). В работе обращается внимание на существенность вводимых предпосылок, иллюстрируя это примерами.