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SOME SUFFICIENT CONDITIONS FOR FINDING A SECOND SOLUTION OF THE QUADRATIC EQUATION IN A BANACH SPACE

IOANNIS K. ARGYROS

Introduction. Consider the quadratic equation

$$x = y + B(x, x) \tag{1}$$

in a Banach space X, where B is a bounded symmetric bilinear operator on X and $y \in X$ is fixed. If $\bar{x} \in X$ is a solution of (1), then any other solution is given by

$$x = \bar{x} + h,$$

where h is a nonzero solution of the equation

$$(I - 2B(\bar{x}))(h) = B(h, h). \tag{2}$$

If the linear operator $I - 2B(\bar{x})$ is invertible, then equation (2) is equivalent to

$$h = \bar{B}(h, h),$$

where

$$\bar{B} = (I - 2B(\bar{x}))^{-1} B. \tag{3}$$

Here we introduce the iteration

$$h_{n+1} = (\bar{B}(h_n))^{-1}(h_n) \tag{4}$$

for some $h_0 \in X$ to find nonzero solutions of (3). Iteration (4) has the property that if $\|h_0\| \geq d$ for some d such that $0 < d \leq \frac{1}{\|\bar{B}\|}$ with $\|\bar{B}\| \neq 0$, then $\|h_n\| \geq d$, $n = 0, 1, 2, \ldots$, therefore if iteration (4) converges to some $h \in X$, then $h \neq 0$ and $x = \bar{x} + h$ is another solution of (1).

Sufficient conditions for a solution $\bar{x}$ of (1) can be found in [2], [6], [7], [9]. The results in this paper can obviously be extended to include multilinear equations of the form

$$x = y + \underbrace{M_k(x, x, \ldots, x)}_{\text{-k times-}}$$

where M_k is a k-linear operator on X, $k = 2, 3, \ldots$.

Proposition 1. *Assume that iteration* (4) *is well defined for all* $n = 0, 1, 2, \ldots$ *for some* $h_0 \in X$ *such that* $\|h_0\| \geq d$ *and for some* d *such that* $0 < d \leq \frac{1}{\|\bar{B}\|}$ *with* $\|\bar{B}\| \neq 0$, *then* $\|h_n\| \geq d$, $n = 0, 1, 2, \ldots$.

Proof. We proceed by induction. We assume that $\|h_k\| \geq d$, $k = 0, 1, 2, \ldots, n$, then by iteration (4)

$$\bar{B}(h_n, h_{n+1}) = h_n$$

and

$$\|\bar{B}\| \cdot \|h_n\| \cdot \|h_{n+1}\| \geq \|\bar{B}(h_n, h_{n+1})\| = \|h_n\|$$

or

$$\|h_{n+1}\| \geq \frac{1}{\|\bar{B}\|}.$$

To complete the induction it suffices to show that

$$\frac{1}{\|\bar{B}\|} \geq d,$$

which is true by hypothesis.

We now state the following lemma. The proof can be found in [10].

Lemma 1. *Let* L_1 *and* L_2 *be bounded linear operators in a Banach space* X, *where* L_1 *is invertible, and* $\|L_1^{-1}\| \cdot \|L_2\| < 1$. *Then* $(L_1 + L_2)^{-1}$ *exists, and*

$$\|(L_1 + L_2)^{-1}\| \leq \frac{\|L_1^{-1}\|}{1 - \|L_2\| \cdot \|L_1^{-1}\|}.$$

Lemma 2. *Let* $z \neq 0$ *be fixed in* X. *Assume that the linear operator* $\bar{B}(z)$ *is invertible, then* $\bar{B}(x)$ *is also invertible for all* $x \in U(z, r) = \{x \in X \mid \|x - z\| < r\}$, *where* $r \in (0, r_0)$ and $r_0 = [\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|]^{-1}$.

Proof. We have

$$\begin{aligned}\|\bar{B}(x - z)\| \cdot \|\bar{B}(z)^{-1}\| &\leq \|\bar{B}\| \cdot \|x - z\| \cdot \|\bar{B}(z)^{-1}\| \\ &\leq \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r \\ &< 1\end{aligned}$$

for $r \in (0, r_0)$. The result now follows from Lemma 1 for $L_1 = \bar{B}(z)$, $L_2 = \bar{B}(x - z)$ and $x \in U(z, r)$.

Definition 1. *Let* $z \neq 0$ *be fixed in* X. *Assume that the linear operator* $\bar{B}(z)$ *is invertible.*

Define the operators P, T *on* $U(z, r)$ *by*

$$P(x) = \bar{B}(x, x) - x, \quad T(x) = (\bar{B}(x))^{-1}(x)$$

and the real polynomials $f(r)$, $g(r)$ *on* R *by*

$$f(r) = a'r^2 + b'r + c', \quad g(r) = ar^2 + br + c,$$
$$a' = (\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|)^2,$$
$$b' = -2\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|,$$
$$c' = 1 - \|\bar{B}(z)^{-1}\| - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 \cdot \|z\|,$$
$$a = \|\bar{B}\| \, \|\bar{B}(z)^{-1}\|,$$
$$b = \|\bar{B}(z)^{-1}(I - \bar{B}(z))\| - 1, \quad \textit{and}$$
$$c = \|\bar{B}(z)^{-1}P(z)\|.$$

Theorem 1. *Let* $z \in X$ *be such that* $\bar{B}(z)$ *is invertible and that the following are true*:

a) $c' > 0$;

b) $b < 0$, $b^2 - 4ac > 0$; *and*

c) *there exists* $r > 0$ *such that* $f(r) > 0$ *and* $g(r) \leq 0$. *Then the iteration*

$$h_{n+1} = \bar{B}(h_n)^{-1}(h_n), \quad n = 0, 1, 2, \ldots$$

is well defined and it converges to a unique solution h *of* (3) *for any* $h_0 \in \bar{U}(z, r)$. *Moreover, if* $\|h_0\| \geq d$ *for some* d *such that* $0 < d \leq \dfrac{1}{\|\bar{B}\|}$, *then* $\|h\| \geq d$.

Proof. T is well defined by Lemma 2.

claim 1. T *maps* $\bar{U}(z, r)$ *into* $\bar{U}(z, r)$.

If $x \in \bar{U}(z, r)$, then

$$\begin{aligned} T(x) - z &= \bar{B}(x)^{-1}(x) - z \\ &= \bar{B}(x)^{-1}[(I - \bar{B}(z))(x - z) - P(z)], \end{aligned}$$

so

$$\|T(x) - z\| \leq r$$

if

$$\frac{1}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| r}[\|\bar{B}(z)^{-1}(I - \bar{B}(z))\| r + \|\bar{B}(z)^{-1}P(z)\|] \leq r$$

(using Lemma 1 for $L_1 = \bar{B}(z)$ and $L_2 = \bar{B}(x - z)$) or $g(r) \leq 0$, which is true by hypothesis.

claim 2. T *is a contraction operator on* $\bar{U}(z, r)$.

If $w, v \in \bar{U}(z, r)$ then

$$\begin{aligned} &\|T(w) - T(v)\| \\ &= \|\bar{B}(w)^{-1}(w) - \bar{B}(v)^{-1}(v)\| \\ &= \|\bar{B}(w)^{-1}[I - \bar{B}(\bar{B}(v)^{-1}(v))](w - v)\| \end{aligned}$$

$$= \| \bar{B}(w)^{-1}[I - \bar{B}(\bar{B}(v)^{-1}(v - z)) + \bar{B}(\bar{B}(v)^{-1}(z))](w - v)\|$$

$$\leq \frac{1}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| r} \left[\|\bar{B}(z)^{-1}\| + \frac{\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 r + \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 \|z\|}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r} \right] \cdot$$

$$\cdot \|w - v\| = q \cdot \|w - v\|.$$

So T is a contraction on $\bar{U}(z, r)$ if $0 < q < 1$, where

$$q = \frac{1}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r} \left[\|\bar{B}(z)^{-1}\| + \frac{\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 r + \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 \|z\|}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r} \right],$$

which is true since $f(r) > 0$.

Iteration (4) can be written as

$$h_{n+1} = h_n - \bar{B}(h_n)^{-1}(P(h_n)), \quad n = 0, 1, 2, \ldots. \tag{5}$$

The corresponding Newton-Kantorovich method can be written as

$$z_{n+1} = z_n - (2\bar{B}(z_n) - I)^{-1} P(z_n), \quad n = 0, 1, 2, \ldots. \tag{6}$$

Iteration (6) is faster and easier to use most of the time, but if we choose an h_0 such that $\|h_0\| \geq d$, then (6) does not guarantee that the limit $w = \lim_{n \to \infty} z_n$ is such that $w \neq 0$. This is exactly the advantage of iteration (5) when compared with (6).

The basic defect of (5) is that each step involves the solution of an equation with a different invertible operator $\bar{B}(h_n)$. For this reason we can study the following modified method

$$h_{n+1} = h_n - \bar{B}(z)^{-1} P(h_n), \quad n = 0, 1, 2, \ldots. \tag{7}$$

Iteration (7), however, does not necessarily satisfy the conclusion of Proposition 1.

The proof of the following theorem is omitted as similar to that of Theorem 1.

Theorem 2. *Let $z \in X$, assume that the operator $\bar{B}(z)$ is invertible and that the following are true:*

(a) $\|\bar{B}(z)^{-1}(I - \bar{B}(z))\| < 1$,

(b) $D = (\|\bar{B}(z)^{-1}(I - \bar{B}(z))\| - 1)^2 - 4\|\bar{B}(z)^{-1}\| \, \|\bar{B}\| \, \|\bar{B}(z)^{-1} P(z)\| > 0$,

then the iteration (7) *is well defined and it converges to a unique solution x of* (1) *for any $x_0 \in \bar{U}(z, r)$, where r is such that*

$$c_1 \leq r < c_2$$

with

$$c_1 = \frac{1 - \|\bar{B}(z)^{-1}(I - \bar{B}(z))\| - \sqrt{D}}{2\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|},$$

$$c_2 = \frac{1 - \|\bar{B}(z)^{-1}(I - B(z))\|}{2\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|}.$$

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НЕКОТОРЫЕ ДОСТАТОЧНЫЕ УСЛОВИЯ ДЛЯ НАХОЖДЕНИЯ ВТОРОГО РЕШЕНИЯ КВАДРАТНОГО УРАВНЕНИЯ В БАНАХОВОМ ПРОСТРАНСТВЕ

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Резюме

Рассматривается квадратное уравнение

$$x = y + B(x, x) \tag{1}$$

в банаховом пространстве X, где B — ограниченный симетрический билинейный оператор на X, а $y \in X$ фиксированное. Если $\bar{x} \in X$ — решение (1) и линейный оператор $I - 2B(\bar{x})$ обратимый, то введем итерацию

$$h_{n+1} = (\bar{B}(h_n))^{-1}(h_n),$$

где

$$\bar{B} = (I - 2B(\bar{x}))^{-1} B$$

для некоторого $h_0 \in X$. В работе нейдены достаточные условия сходимости вышеприверенной итерации к ненулевому $h \in X$. Если такое $h \in X$ может быть найдено, то

$$x = \bar{x} + h$$

является вторым решением у равнения (1).

Условия существования решения (малого) $\bar{x}$ уравнения (1) уже известны из работ Л. Б. Ралла и ево учеников.