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LATTICES WITH A THIRD DISTRIBUTIVE OPERATION

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Preliminaries

Two binary operations $\circ$ and $*$ in a set M are said to be mutually distributive (or the operation $\circ$ is distributive with the operation $*$) if for each $a, b, c \in M$, $a \circ (b * c) = (a \circ b) * (a \circ c)$, $a * (b \circ c) = (a * b) \circ (a * c)$.

B. H. Arnold [1] investigated distributive lattices $(L; \wedge, \vee)$ with an operation $*$ such that $(L; *)$ is a semilattice and the operation $*$ is distributive with $\wedge$ and $\vee$. In [4] there were investigated pairs of distributive lattices $(L; \wedge, \vee)$, $(L; \cap, \cup)$ such that each of the operations $\wedge$, $\vee$ is distributive with each of the operations $\cap$, $\cup$. In this note we shall show that the results of [1, Th. 16] and [4] are valid also without assuming the distributivity of the mentioned lattices.

In the lattices $(L; \wedge, \vee)$ the order will be denoted by $\leqq$, that in the semilattice $(L; \cap)$ by $\subseteq$ (i.e. $x \subseteq y$ iff $x \cap y = x$). Lattice operations in the lattice of equivalence relations in a set M will be denoted by $\wedge$ and $\vee$. ω will denote the least equivalence relations (equality), ι the greatest one. $\Theta . \Phi$ will denote the product of equivalence relations Θ, Φ in the usual sense.

1. Results

Theorem 1. *Let $L = (L; \wedge, \vee)$ be a lattice. There is a $1-1$ correspondence between semilattice operations $\cap$ in L such that $\cap$ is distributive with $\wedge$ and $\vee$, and pairs of congruence relations Θ_1, Θ_2 in L such that $\Theta_1 \wedge \Theta_2 = \omega$, $(a \wedge b) \vee c \Theta_i (a \vee c) \wedge (b \vee c)$ $(i = 1, 2)$ for each $a, b, c \in L$, and $a < b$ implies $a \Theta_1 \Theta_2 b$.*

The congruence relations Θ_i corresponding to $\cap$ are given as follows. $a \Theta_1 b$ iff $a \cap b = a \vee b$, $a \Theta_2 b$ iff $a \cap b = a \wedge b$. Conversely, given Θ_1 and Θ_2, $a \cap b$ is the uniquely determined element c for which $a \wedge b \; \Theta_1 c \Theta_2 a \vee b$.

If the desired operation $\cap$ exists, then L is distributive.

Theorem 2. *Let $L=(L;\ \wedge,\ \vee)$ be a lattice. There is a $1-1$ correspondence between the operations $\cap$ as in Theorem 1 and representations of L as a subdirect product of distributive lattices A, B such that if (a, b), (a', b') are elements of the subdirect product and $(a, b)\leq(a', b')$, then (a, b') belongs to this subdirect product. The subdirect representation belonging to an operation $\cap$ is that given by congruence relations Θ_1, Θ_2 from Theorem 1. The operation $\cap$ corresponding to a subdirect representation $\varphi: L\to A\times B$ is given as follows. If $\varphi(x)=(a, b)$, $\varphi(y)=(a', b')$, then $x\cap y=\varphi^{-1}(a\wedge a', b\vee b')$.*

Theorem 3. *a) The semilattice $(L;\ \cap)$ of Th. 1 turns out to be a lattice*[1]*) iff the corresponding congruence relations Θ_1, Θ_2 commute.*

b) The semilattice $(L;\ \cap)$ of Th. 2 turns out to be a lattice if the subdirect factorization is a direct one.

In both cases the lattice $(L;\ \cap,\ \cup)$ is distributive and the operation $\cup$ is distributive with $\wedge$ and $\vee$, too.

Theorem 4. *a) If for two lattices, $(L;\ \wedge,\ \vee)$ and $(L;\ \cap,\ \cup)$, the operation $\cap$ is distributive with $\wedge$ and $\vee$, then the operation $\cup$ is distributive with these operations too and both lattices are distributive.*

b) Let $L_1=(L;\ \wedge,\ \vee)$ and $L_2=(L;\ \cap,\ \cup)$ be lattices. The operation $\cap$ is distributive with $\wedge$ and $\vee$ iff there are distributive lattices $A=(A;\ \wedge,\ \vee)$, $B=(B;\ \wedge,\ \vee)$ and a map $\varphi: L\to A\times B$ such that φ is an isomorphism of L_1 onto the direct product $A\times B$ and an isomorphism of L_2 onto the direct product $A\times\tilde{B}$ ($\tilde{B}$ being the dual of B).

Remark 1. In Theorem 1 four distributive laws are postulated:

$$x\cap(y\wedge z)=(x\cap y)\wedge(x\cap z),\ x\wedge(y\cap z)=(x\wedge y)\cap(x\wedge z),$$

$x\cap(y\vee z)=(x\cap y)\vee(x\cap z)$, and $x\vee(y\cap z)=(x\vee y)\cap(x\vee z)$. None of these laws can be omitted as the following example shows. Let L_1, L_2 be lattices on the set $\{a, b, c\}$ given by the chains $L_1: a<b<c$, $L_2: a\subset c\subset b$. There hold the first three identities but the last does not.

This example shows also that in Theorem 4 it would not be sufficient to suppose only that one of the operations of L_1 is distributive with one operation of L_2.

Remark 2. From theorems [4, Th. 3.4] and [5, Th. 3.6] there immediately follows the following weakening of Theorem 4b). If each operation of L_1 is distributive with each operation of L_2, then there is an isomorphism of L_1 onto a direct product of two lattices A and B which is also an isomorphism of the lattices L_2 and $A\times\tilde{B}$.

2. Some lemmas

2.0. Lemma. *Congruence relations Θ, Φ of a lattice $(L;\ \wedge,\ \vee)$ commute iff for each $a, b\in L$, $a\leq b$, $a\Theta\Phi b$ is equivalent with $a\Phi\Theta b$.*

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[1]) i.e. there is an operation $\cup$ on L such that $(L;\ \cap,\ \cup)$ is a lattice

Proof. The condition is obviously necessary. Suppose it is satisfied and let x, $y \in L$, $x\Theta z$ and $z\Phi y$. Then $x \wedge y \wedge z \Phi x \wedge z \Theta x$, hence $t \in L$ exists with $x \wedge y \wedge z \Theta t \Phi x$, so that $y \Theta y \vee t$. Further, $x \wedge y \wedge z \Theta y \wedge z \Phi y$, hence $y \wedge z \Phi \Theta y \vee t$, $y \wedge z \Theta \Phi y \vee t$ and $t \Theta y \wedge z$ so that $t \Theta \Phi y \vee t$, $x \Phi t \Phi \Theta y \vee t \Theta y$, hence $x \Phi \Theta y$. This shows that $\Theta\Phi \leqslant \Phi\Theta$, which implies $\Theta\Phi = \Phi\Theta$.

In the paragraphs 2.1—2.5.9 we suppose that $(L; \wedge, \vee)$ is a lattice (with the ordering relation $\leqslant$), $(L; \cap)$ is a semilattice (with the ordering relation $\subseteq$) and that the operation $\cap$ is distributive with both operations $\wedge$ and $\vee$.

2.1. *From the distributivity of $\cap$ with the operations $\wedge$, $\vee$ it follows immediately (see [1])* $x \wedge y \leqslant x \cap y \leqslant x \vee y$, $x \cap y \subseteq x \wedge y$, $x \cap y \subseteq x \vee y$, $x \cap (x \wedge y) = x \wedge (x \cap y)$, $x \cap (x \vee y) = x \vee (x \cap y)$. These relations will be used freely in what follows.

2.2. *$a \leqslant x \leqslant b$ and $a \subseteq b$ imply $a \subseteq x \subseteq b$.*

Proof. $a \cap x = a \cap (x \wedge b) = (a \cap x) \wedge (a \cap b) = (a \cap x) \wedge a = (a \wedge x) \cap a = a \cap a = a$, $b \cap x = (b \vee x) \cap (a \vee x) = (b \cap a) \vee x = a \vee x = x$.

2.3. *$u \leqslant x$, $u \leqslant y$, $u \subseteq x$ and $u \subseteq y$ imply $x \wedge y = x \cap y$.*

Proof. $u \leqslant x \wedge y \leqslant x$ and $u \subseteq x$ yield, by 2.2, $u \subseteq x \wedge y \subseteq x$. Similarly, $x \wedge y \subseteq y$. It follows that $x \wedge y \subseteq x \cap y$, hence $x \wedge y = x \cap y$ (using 2.1).

2.4. *Let the semilattice $(L; \cap)$ form a lattice $(L; \cap, \cup)$ (see the footnote[1])). Then $a \leqslant a \cup b \leqslant b$ holds for any $a \leqslant b$.*

Proof. $[a \vee (a \cup b)] \cap [b \wedge (a \cup b)] = ([a \vee (a \cup b)] \cap b) \wedge ([a \vee (a \cup b)] \cap (a \cup b)) = [(a \cap b) \vee b] \wedge [a \vee (a \cup b)] = b \wedge [a \vee (a \cup b)]$, hence

$$(1) \qquad b \wedge [a \vee (a \cup b)] \subseteq a \vee (a \cup b), \quad b \wedge [a \vee (a \cup b)] \subseteq b \wedge (a \cup b).$$

Further, $a \cap (b \wedge [a \vee (a \cup b)]) = (a \cap b) \wedge a = a$, hence

$$(2) \qquad a \subseteq b \wedge [a \vee (a \cup b)].$$

Using 2.1 we get $b \cap (b \wedge [a \vee (a \cup b)]) \supseteq b \cap (b \cap [a \vee (a \cup b)]) = (b \cap a) \vee b = b$, hence

$$(3) \qquad b \subseteq b \wedge [a \vee (a \cup b)].$$

Further, $[a \vee (a \cup b)] \cap (a \cup b) = a \vee (a \cup b)$, $[b \wedge (a \cup b)] \cap (a \cup b) = b \wedge (a \cup b)$, hence

$$(4) \qquad a \vee (a \cup b) \subseteq a \cup b, \quad b \wedge (a \cup b) \subseteq a \cup b.$$

From (2) and (3) it follows that $a \cup b \subseteq b \wedge [a \vee (a \cup b)]$, which combined with (1) and (4) yields $a \vee (a \cup b) = a \cup b = b \wedge (a \cup b)$, which proves the assertion.

2.5. Define the relations Θ_1, Θ_2 in L as follows. $a\Theta_1 b$ iff $a \cap b = a \vee b$, $a \Theta_2 b$ iff $a \cap b = a \wedge b$.

2.5.1. *Θ_2 is an equivalence relation in L.*

Proof. Reflexivity and symmetry are obvious. Let $a\Theta_2 b$, $b \Theta_2 c$. Then

$$a \cap b = a \wedge b, \; b \cap c = b \wedge c,$$
$$a \wedge b \wedge c = (a \wedge b) \wedge (b \wedge c) = (a \cap b) \wedge (b \cap c) = (a \wedge b) \cap (a \wedge c) \cap b \cap (b \wedge c) =$$
$= (a \cap b) \cap (a \wedge c) \cap b \cap (b \cap c) = a \cap b \cap c \cap (a \wedge c) = a \cap b \cap c$, since $a \cap c \subseteq a \wedge c$. From the relations $a \wedge b \wedge c \leq a, c$; $a \wedge b \wedge c = a \cap b \cap c \subseteq a, c$ it follows by 2.3 that $a \cap c = a \wedge c$, hence $a \Theta_2 c$.

2.5.2. *Θ_2 is a congruence relation in the lattice $(L; \wedge, \vee)$.*

Proof. Let $a \Theta_2 b$, i.e., $a \cap b = a \wedge b$. Then $(a \wedge c) \cap (b \wedge c) = (a \cap b) \wedge c = a \wedge b \wedge c = (a \wedge c) \wedge (b \wedge c)$, hence $a \wedge c \Theta_2 b \wedge c$. Further, $(a \vee c) \wedge (b \vee c) \leq (a \vee c) \cap (b \vee c) = (a \cap b) \vee c = (a \wedge b) \vee c \leq (a \vee c) \wedge (b \vee c)$, which yields $a \vee c \Theta_2 b \vee c$.

2.5.3. *Θ_1 is a congruence relation in the lattice $(L; \wedge, \vee)$.*

Proof. It suffices to consider the semilattice $(L; \cap)$ and the lattice dual to $(L; \wedge, \vee)$, and to use 2.5.2.

2.5.4. *Θ_1, Θ_2 are congruence relations in the semilattice $(L; \cap)$.*

Proof. $a \Theta_1 b$ implies $(a \cap c) \cap (b \cap c) = (a \cap b) \cap c = (a \vee b) \cap c = (a \cap c) \vee (b \cap c)$, hence $a \cap c \Theta_1 b \cap c$. The proof for Θ_2 is similar.

2.5.5. *$\Theta_1 \wedge \Theta_2 = \omega$.*

The assertion follows immediately from the definition 2.5.

2.5.6. *$a \wedge b \Theta_1 a \cap b$, $a \cap b \Theta_2 a \vee b$.*

The assertion follows immediately from 2.5 and 2.1.

2.5.7. *$a \leq b$ implies $a \Theta_1 \Theta_2 b$.*

The assertion follows from 2.5.6.

2.5.8. *If the semilattice $(L; \cap)$ forms a lattice (see the footnote[1])), then $a \Theta_2 \Theta_1 b$ for each $a \leq b$.*

Proof. Using 2.4 we get $a \Theta_2 a \cup b \Theta_1 b$.

2.5.9. *The lattice $(L; \wedge, \vee)$ is distributive.*

Proof. Using 2.5.6 we get for arbitrary $x, y, z \in L$: $(x \vee y) \wedge z \Theta_2 (x \cap y) \wedge z = (x \wedge z) \cap (y \wedge z) \Theta_2 (x \wedge z) \vee (y \wedge z)$. On the other hand, $(x \vee y) \wedge z \Theta_1 (x \vee y) \cap z = (x \cap z) \vee (y \cap z) \Theta_1 (x \wedge z) \vee (y \wedge z)$. This gives $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ by 2.5.5.

3. Proofs of the Theorems

Proof of Th. 1. The existence of the congruence relations Θ_1, Θ_2 for a given semilattice $(L; \cap)$ and the distributivity of the lattice $(L; \wedge, \vee)$ are consequences of 2.5.2, 2.5.3, 2.5.5, 2.5.7 and 2.5.9.

Conversely, let Θ_1, Θ_2 be congruence relations in L satisfying the given conditions. These conditions ensure the existence of the operation $\cap$. Obviously $\cap$ is idempotent and commutative. The elements $d_1 = (a \cap b) \cap c$, $d_2 = a \cap (b \cap c)$ satisfy $d_i \Theta_1 a \wedge b \wedge c$, $d_i \Theta_2 a \vee b \vee c$ $(i = 1, 2)$, hence $d_1 \Theta_1 \wedge \Theta_2 d_2$, which yields $d_1 = d_2$.

To prove the distributivity of the operation $\cap$ with $\wedge$ and $\vee$ we use the definition of $\cap$ (i.e. 2.5.6) and the supposed distributivity of quotient lattices L/Θ_i $(i=1, 2)$. The elements $u_1=(a\wedge b)\cap c$, $u_2=(a\cap c)\wedge(b\cap c)$ satisfy $a\wedge b\wedge c\Theta_1 u_i\ \Theta_2(a\wedge b)\vee c$ $(i=1, 2)$, hence $u_1=u_2$. Similarly we get $(a\cap b)\wedge c=(a\wedge c)\cap(b\wedge c)$ and the distributivity of the operations $\cap$ and $\vee$.

One can easily verify that if Θ_1, Θ_2 are congruence relations corresponding to a given operation $\cap$, then the semilattice operation corresponding to Θ_1, Θ_2, coincides with $\cap$. Similarly, if we start with Θ_1, Θ_2, construct $\cap$ and then the corresponding congruence relations, we get Θ_1, Θ_2. This yields the correspondence stated in the theorem.

Proof of Theorem 2. Let $(L; \cap)$ be a semilattice with the property stated in the theorem and Θ_1, Θ_2 the congruence relations from Th. 1. Then the lattice L is isomorphic to a subdirect product of lattices L/Θ_1, L/Θ_2 under the mapping $\varphi: x\to([x]\Theta_1, [x]\Theta_2)$ ($[x]\Theta_i$ is the class of the congruence relation Θ_i, containing x) (see e.g. [3, § 20]). By Th. 1 the lattices L/Θ_i are distributive. Let (a, b), (a', b') have the same meaning as in the theorem. Then elements $u, v\in L$ exist with $\varphi(u)=(a, b)$, $\varphi(v)=(a', b')$, $u\leq v$. By Th. 1 there is $t\in L$ with $u\Theta_1 t\Theta_2 v$. Then $\varphi(t)=(a, b')$.

Conversely, let $\varphi: L\to A\times B$ be an isomorphism of the lattice L to a subdirect product of lattices A, B having the properties stated in the theorem. Let Θ_1, Θ_2 be the corresponding congruence relations in L [3, § 20]. Then $\Theta_1\wedge\Theta_2=\omega$ and L/Θ_i are isomorphic to A and B, respectively, hence they are distributive. If $a, b\in L$, $a\leq b$, $\varphi(a)=(a, a')$, $\varphi(b)=(b, b')$, let t be the element of L with $\varphi(t)=(a, b')$. Then $a\Theta_1 t\Theta_2 b$, hence the congruence relations Θ_1, Θ_2 have the properties of Theorem 1 so that there is a semilattice operation $\cap$ in L which is distributive with the operations $\wedge$, $\vee$. The relations $x\wedge y\Theta_1 x\cap y\Theta_2 x\vee y$ yield the last assertion of Th. 2 concerning the operation $\cap$.

Proof of Th 3 a) If the lattice $(L; \cap, \cup)$ exists, then $\Theta_1 . \Theta_2=\Theta_2 . \Theta_1$ by 2.0, 2.5.7 and 2.5.8. Conversely, let $\Theta_1 . \Theta_2=\Theta_2 . \Theta_1$. Then for $a\leq b$ we get by Th. 1 $a\Theta_1\Theta_2 b$, hencc $a\Theta_2\Theta_1 b$, too. By Th. 1 there is a semilattice operation $\cup$ in L, which is distributive with the operations $\wedge$, $\vee$, satisfying $a\wedge b\Theta_2 a\cup b\Theta_1 a\vee b$. Hence $(a\cup b)\cap a\Theta_2(a\wedge b)\cap a\Theta_2 a$, $(a\cup b)\cap a\Theta_1(a\vee b)\cap a\Theta_1 a$, which yields $(a\cup b)\cap a\Theta_1\wedge\Theta_2 a$, i.e., $(a\cup b)\cap a=a$. Similarly we get $(a\cap b)\cup a=a$ using $a\wedge b\Theta_1 a\cap b\Theta_2 a\vee b$. Hence $(L; \cap, \cup)$ is a lattice. The distributivity of this lattice follows by Th. 1 (or by 2.5.9) from the distributivity of the operation $\wedge$ with the operations $\cap$, $\cup$.

b) Since $a\Theta_1\Theta_2 b$ for $a\leq b$, we get $\Theta_1\vee\Theta_2=\iota$. Hence the subdirect product is a direct product iff $\Theta_1 . \Theta_2=\Theta_2 . \Theta_1$. By a), this is equivalent to the condition that $(L; \cap)$ forms a lattice.

Proof of Th. 4. The assertion a) follows from Theorems 1 and 3. The assertion b) follows from Theorems 2 and 3.

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СТРУКТУРЫ С ТРЕТЬЕЙ ДИСТРИБУТИВНОЙ ОПЕРАЦИЕЙ

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Резюме

Пусть $(L; \wedge, \vee)$ — структура. В этой статье исследуется бинарная операция $\cap$ на множестве L обладающая следующими свойствами: (а) $(L; \cap)$ является полуструктурой; (б) операция $\cap$ будет дистрибутивной относительно каждой из операций $\wedge$ и $\vee$. Доказано обобщение одного результата Арнольда.