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A CATEGORICAL CONTRIBUTION TO THE KUMMER THEORY OF IDEAL NUMBERS

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ABSTRACT. This article is partly a brief survey of known results which are going back as far as E. E. Kummer (1847), then to modern algebraic language of Z. I. Borevich and I. R. Shafarevich (1964) introducing the notion of theory of divisors, and to author's results (1973–75) using categorical methods in this area. The presented conception is chosen for better understanding the motivation of the new results and the notions.

The main result of this paper is the description of all maximal δ_1-categories by means of so called α-ultrapseudofilters and ultrastars. A δ_1-category is a subcategory $\mathcal{M}$ of the category $\mathcal{L}$ of all δ_1-semigroups (which are semigroups possessing a divisor theory in the sense of Arnold) with semigroup homomorphisms, having the same objects as $\mathcal{L}$, containing δ^*-homomorphisms (defined by means of v-ideals) as morphisms, and with the divisor theory as a reflection for the reflective subcategory of $\mathcal{M}$ of all semigroups with unique factorization.

It is shown that these maximal δ_1-categories form a set with cardinal number equal to $\exp\exp\aleph_0$, while all the δ_1-categories form a class which is not a set.

1. Introduction

In his monumental work E. E. Kummer introduced the concept of "*ideal complex numbers*" (*ideale komplexe Zahlen*) ([7] (1845), in more detail [8] (1847); cf. [11]) for the ring of integers of the λth cyclotomic field (λ an odd prime) to remove the defect of these rings that the law of unique factorization into irreducible elements fails. In the present algebraic language this concept can be expressed by means of that of the *divisor theory of an integral domain* introduced by Z. I. Borevich and I. R. Shafarevich (cf. [2] (1966), Russian original 1964) which can be formulated as follows (a slight modification):

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DEFINITION 1.1. Let R be an integral domain, D a semigroup with unique factorization, and h a (semigroup) homomorphism from the multiplicative semigroup R^* into D. The homomorphism h (together with D) is called a *theory of divisors for the ring* R if h satisfies the conditions:

(1) An element $\alpha \in R^*$ is divisible by $\beta \in R^*$ in the ring R if and only if $h(\alpha)$ is divisible by $h(\beta)$ in the semigroup D.

(2) Let $\alpha, \beta, \alpha \pm \beta \in R^*$, and $a \in D$. If $h(\alpha)$ and $h(\beta)$ are divisible by a in D, then $h(\alpha \pm \beta)$ are also divisible by a in D.

(3) Let $a, b \in D$. If $\{r \in R^* : a \text{ divides } h(r)\} = \{r \in R^* : b \text{ divides } h(r)\}$, then $a = b$.

Note that the condition (2) can be derived from the conditions (1) and (3) (cf. [12]) and it is not hard to see that the integral domains possessing a theory of divisors are just the Krull domains. Semigroups with unique factorization are just free semigroups and the generators are just the irreducible elements.

To use the categorical methods, it is appropriate to transfer the considered algebraic structures to those of the same kind, it means to pass over the rings to the semigroups. In this paper a semigroup will be considered to be commutative, with an identity element and satisfying the cancellation law. The concept of a theory of divisors (= a divisor theory) for a ring is transferred to a divisor theory for a semigroup as follows:

DEFINITION 1.2. A semigroup S is called a *δ-semigroup* if it possesses a divisor theory, which is a homomorphism ϱ from S into a semigroup D with unique factorization satisfying the following conditions:

(a) $\bigl(s_1 \in S \ \&\ s_2 \in S \ \&\ \varrho(s_1) \underset{D}{/} \varrho(s_2)\bigr) \implies s_1 \underset{S}{/} s_2$,

(b) $d \in D \implies \exists$ a positive integer n and elements $s_1, \dots, s_n \in S$ such that $\gcd_D\{\varrho(s_1), \dots, \varrho(s_n)\} = d$.

Here, the symbols $\underset{D}{/}$ and $\underset{S}{/}$ denote the divisibility relation in the semigroups D and S, respectively, and $\gcd_D\{d_1, \dots, d_n\}$ is the greatest common divisor of the set $\{d_1, \dots, d_n\}$ in D for $d_1, \dots, d_n \in D$.

If instead of (b) the stronger axiom

$$(\mathrm{b}_1) \quad (a, b \in D) \implies (\exists\, c \in D)\bigl(\gcd_D\{b, c\} = 1_D \ \ \&\ \ a \cdot c \in \varrho(S)\bigr)$$

is satisfied, then we call S a *δ_1-semigroup*.

Remark. Clifford studied the δ-semigroups in slightly more general form in his papers [3] (1934) and [4] (1938). Arnold paid attention to the δ_1-semigroups in [1] (1929). The axioms (a) and (b) are equivalent to the axioms (1) and (3) of Borevich and Shafarevich (cf. [12; 2.4]). The multiplicative semigroups of the integral domains possessing a theory of divisors (hence of the Krull rings) are δ_1-semigroups.

From Clifford's result the following assertion follows:

THEOREM 1.1 (Uniqueness of the divisor theory). *Let S be a δ-semigroup with divisor theories $\varrho\colon S \to D$, $\varrho'\colon S \to D'$. Then there exists a unique isomorphism f from D onto D' such that the following diagram commutes.*

$$\begin{tikzcd} & D \arrow[dd, dashed, "f"] \\ S \arrow[ur, "\varrho"] \arrow[dr, "\varrho'"'] & \\ & D' \end{tikzcd}$$

Therefore the divisor theory $\varrho\colon S \to D$ is uniquely determined with the exception of S-isomorphisms f. For a δ-semigroup S we denote by $e_S\colon S \to \mathfrak{c}S$ a divisor theory of S ($\mathfrak{c}S$ is a semigroup with unique factorization). For elements $s_1, s_2 \in S$ we have $e_S(s_1) = e_S(s_2)$ *if and only if* s_1, s_2 *are associates in* S.

In the area of divisibility theory of a semigroup the important role is played by a special kind of ideal, at present called a *v-ideal*, introduced by Arnold [1] (1929), which can be defined as follows:

DEFINITION 1.3. Let S be a semigroup. A non-empty subset I of S ($\emptyset \neq I \subseteq S$) is called a *$v$-ideal of the semigroup S* if

$$I = \Big\{ s \in S :\ \Big((s_1, s_2 \in S)\ \&\ (\forall i \in I)\big(s_1 \underset{S}{/} i s_2\big) \Big) \implies \big(s_1 \underset{S}{/} s s_2\big) \Big\}.$$

The set $(s) = \{sx :\ x \in S\}$ is a v-ideal of S for each $s \in S$, which is called the *principal v-ideal of S generated by s*.

The set of all v-ideals of S will be denoted by $\mathcal{I}(S)$ and for $I, J \in \mathcal{I}(S)$ we denote by $I \circ J$ the v-ideal generated by the set $I \cdot J$, therefore

$$I \circ J = \bigcap_{\substack{K \in \mathcal{I}(S) \\ K \supseteq I \cdot J}} K.$$

Then $\circ$ is an operation on $\mathcal{I}(S)$ and $(\mathcal{I}(S), \circ)$ *is a semigroup* (the cancellation law need not be satisfied in general).

Put $\varrho_S(s) = (s)$ for each $s \in S$. Then $\varrho_S\colon S \to (\mathcal{I}(S), \circ)$ is a homomorphism and according to [4] we have:

THEOREM 1.2. *If a semigroup S possesses a divisor theory, then the homomorphism $\varrho_S\colon S \to (\mathcal{I}(S), \circ)$ is a divisor theory of the semigroup S.*

The significance of Kummer's idea — to supplement the multiplicative semigroup of the ring of the λth cyclotomic field by new elements (ideal complex numbers) to save the uniqueness of decomposition into irreducible factors

— can be found in many abstract constructions in present mathematics. If a mathematical structure does not possess some "good" property, then this defect is removed by supplementing new elements in this way that the new structure has the required "good" property. Such constructions are compactifications of a topological space (the Čech-Stone β-compactification) and completions of an ordered set (the Mac Neile completion).

This fact is expressed in the category theory language by means of the concept of the reflection. In Section 2 we will define special homomorphisms for semigroups (δ-homomorphisms) by means of v-ideals and the category $\mathcal{K}$ of all δ-semigroups, where the morphisms are just the δ-homomorphisms. The full subcategory $\mathcal{D}$ of all unique factorization semigroups is a reflective subcategory of $\mathcal{K}$ and $e_S\colon S \to \mathfrak{c}S$ is a $\mathcal{D}$-reflection for each δ-semigroup S (Theorem 2.1). The same property has the category $\mathcal{K}_1$ of all δ_1-semigroups with δ^*-homomorphisms (Theorem 2.2), where the δ^*-homomorphisms occur in algebraic number theory and are again defined by means of v-ideals.

The question which is investigated in this paper concern the maximality of the choice of these morphisms to preserve the property of reflection. In [14] it was shown that this choice is maximal in the category $\mathcal{L}$ of all δ-semigroups with homomorphisms (Theorem 3.1).

Much more complicated is the situation in the category $\mathcal{L}_1$ of all δ_1-semigroups with homomorphisms. All maximal choices of morphisms of the category $\mathcal{L}_1$ which involve δ^*-homomorphisms and preserve the mentioned property of reflection were described in [14] by means of generalized matrices with integral entries (bundles) forming so called ultrastars which are maximal stars (Definition 3.4).

It is shown in this paper that these maximal choices of morphisms form a set with cardinal number equal to $\exp\exp\aleph_0$ (Theorem 5.4). On the contrary all choices of morphisms with these properties form a class which is not a set (Theorem 6.7).

The tool for the investigation is the Čech-Stone compactification $\beta\mathbf{P}$ of the discrete topological space $\mathbf{P}$ of all primes. Special systems (α-pseudofilters) of special subsets (α-sets) of $\mathbf{P}$ are introduced (Definition 4.2 and 4.3), a one-to-one mapping from $\beta\mathbf{P}$ to the set of all maximal α-pseudofilters (α-ultrapseudofilters) is introduced, and the connections between stars and α-pseudofilters are shown (Propositions 5.1–5.3). Using Pospíšil's formula (cf. [10]): $\operatorname{card}\beta\mathbf{P} = \exp\exp\aleph_0$, cardinal numbers of all ultrastars and all α-ultrapseudofilters are calculated (Theorems 4.5 and 5.4).

In this article we will use only the basic notions of the category theory (e.g., [9] or [6]) and furthermore of the reflective subcategories (e.g., [5]). If $\mathcal{C}$ is a category, we denote by $\mathcal{O}(\mathcal{C})$ the class of all objects of $\mathcal{C}$. If $\mathcal{R}$ is a full subcategory of $\mathcal{C}$, then an *$\mathcal{R}$-reflection for* a $\mathcal{C}$-object X is a morphism $\varrho_X \in$

$\mathrm{Hom}_{\mathcal{C}}(X, R(X))$ $(R(X) \in \mathcal{O}(\mathcal{R}))$ such that for each $\mathcal{R}$-object Y and each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ there exists a unique morphism $\bar{f} \in \mathrm{Hom}_{\mathcal{C}}(R(X), Y)$ such that the following diagram commutes.

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[dr, "\varrho_X"'] & \\ & R \arrow[uu, dashed, "\bar{f}"'] \end{tikzcd}$$

If each $\mathcal{C}$-object possesses an $\mathcal{R}$-reflection, then the subcategory $\mathcal{R}$ is said to be *reflective*.

2. Categorical approach to the divisor theory

DEFINITION 2.1. Let S_1 and S_2 be semigroups and f a homomorphism from S_1 to S_2. Then f is called a *δ-homomorphism* if for each v-ideal J of S_2 the set $f^{-1}(J)$ is empty or it is a v-ideal of S_1 ([13; Definition 2.8]).

Furthermore we denote by

$\mathcal{K}$ the category of all δ-semigroups with δ-homomorphisms,
$\mathcal{D}$ the full subcategory of $\mathcal{K}$ of all unique factorization semigroups,
$\mathcal{L}$ the category of all δ-semigroups with (semigroup) homomorphisms (thus $\mathcal{O}(\mathcal{K}) = \mathcal{O}(\mathcal{L})$).

The divisor theory $\varrho\colon S \to D$ of a δ-semigroup S has the property of "$\mathcal{D}$-reflection", more exactly we have ([13; 5.3]):

THEOREM 2.1. *$\mathcal{D}$ is a reflective subcategory of the category $\mathcal{K}$ and $e_S\colon S \to \mathfrak{c}S$ is a $\mathcal{D}$-reflection for each $\mathcal{K}$-object S.*

The choice of δ-homomorphisms is appropriate to express the divisor theory as $\mathcal{D}$-reflection, but this choice does not involve, e.g., the norm homomorphism N from $\mathfrak{c}R^*$ to $\mathfrak{c}S^*$, where S is an integral domain possessing a theory of divisors with quotient field k and R is the integral closure of S in a finite extension K (see [2; Chap. 3, Sec. 5]).

In the paper [13; Definitions 3.1, 3.7] for a semigroup G a topology $\delta^* = \delta^*_G$ was defined by means of the set of all v-ideals of G as a subbasis for closed sets and then the concept of δ-homomorphism is transferred to that of *δ^*-homomorphism.*

DEFINITION 2.2. A homomorphism f from a semigroup G to a semigroup H is called a *δ^*-homomorphism* if f is a continuous mapping from the topological space (G, δ^*_G) to the topological space (H, δ^*_H).

To preserve the property of $\mathcal{D}$-reflection for δ-semigroups with δ^*-homomorphisms, we must contract the class of δ-semigroups to the class of δ_1-semigroups.

We denote further by

$\mathcal{K}_1$ the category of all δ_1-semigroups with δ^*-homomorphisms,
$\mathcal{D}_1$ the full subcategory of $\mathcal{K}_1$ of all unique factorization semigroups (thus $\mathcal{O}(\mathcal{D}_1) = \mathcal{O}(\mathcal{D})$),
$\mathcal{L}_1$ the category of all δ_1-semigroups with (semigroup) homomorphisms (hence $\mathcal{O}(\mathcal{L}_1) = \mathcal{O}(\mathcal{K}_1)$).

Note that the category $\mathcal{K}_1$ contains above mentioned norm homomorphism as a morphism.

The $\mathcal{K}_1$ analogue of Theorem 2.1 is also true ([13; 5.3]):

THEOREM 2.2. *$\mathcal{D}_1$ is a reflective subcategory of the category $\mathcal{K}_1$ and $e_S \colon S \to \mathfrak{c}S$ is a $\mathcal{D}_1$-reflection for each $\mathcal{K}_1$-object S.*

3. Maximal δ_1-categories

Now there arises the question if we can enlarge the classes of morphism in the categories $\mathcal{K}$ and $\mathcal{K}_1$ remaining in $\mathcal{L}$ and $\mathcal{L}_1$ for Theorems 2.1 and 2.2, respectively, to keep validity. It was shown ([14; Satz 1.3]) that in case of the category $\mathcal{K}$ it is not possible. We have more exactly:

THEOREM 3.1. *Let $\mathcal{M}$ be a subcategory of $\mathcal{L}$ containing $\mathcal{K}$ and let $\bar{\mathcal{D}}$ be the full subcategory of $\mathcal{M}$ with $\mathcal{O}(\bar{\mathcal{D}}) = \mathcal{O}(\mathcal{D})$ fulfilling the following conditions:*

(a) *$\bar{\mathcal{D}}$ is a reflective subcategory of $\mathcal{M}$,*
(b) *$e_S \colon S \to \mathfrak{c}S$ is a $\bar{\mathcal{D}}$-reflection for each $\mathcal{K}$-object S.*

Then $\mathcal{M} = \mathcal{K}$.

In the case of the category $\mathcal{K}_1$ the situation is much more complicated. To size up this question, the following concepts were introduced ([14; Definition 1.4]):

DEFINITION 3.1. Let $\mathcal{M}$ be a subcategory of $\mathcal{L}_1$ containing $\mathcal{K}_1$ and let $\bar{\mathcal{D}}_1$ be the full subcategory of $\mathcal{M}$ with $\mathcal{O}(\bar{\mathcal{D}}_1) = \mathcal{O}(\mathcal{D}_1)$. The category $\mathcal{M}$ is called a *δ_1-category* if $\bar{\mathcal{D}}_1$ is a reflective subcategory of $\mathcal{M}$ and $e_S \colon S \to \mathfrak{c}S$ is a $\bar{\mathcal{D}}_1$-reflection for each $\mathcal{K}_1$-object S. A δ_1-category $\mathcal{M}$ is said to be a *maximal δ_1-category* if for each δ_1-category $\bar{\mathcal{M}}$ containing $\mathcal{M}$ we have $\bar{\mathcal{M}} = \mathcal{M}$. The category $\mathcal{K}_1$ is the least δ_1-category.

The description of the maximal δ_1-categories makes use of the following Definitions 3.2–3.4 ([14; Definitions 2.1, 2.3, 3.2, 4.1]):

DEFINITION 3.2. A *bundle* $A = [a_{ij}]$ $(1 \le i < u+1,\ 1 \le j < v+1)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

of size $u \times v$ (u, v are positive integers or the symbol ∞ and $\infty + 1 = \infty$) is a sequence $\{S_j\}_{j=1}^{v}$, where S_j is a sequence $\{a_{ij}\}_{i=1}^{u}$ and a_{ij} are non-negative integers such that for each integer i $(1 \le i < u+1)$ the set $\{j :\ 1 \le j < v+1 \ \&\ a_{ij} \ne 0\}$ is finite. The sequence S_j will also be called a *column* of A and we also say that S_j is of size u.

If for each $1 \le j < v+1$ the set $\{i :\ 1 \le i < u+1 \ \&\ a_{ij} \ne 0\}$ is finite, then the bundle A is called an *almost zero bundle.*

Two bundles $A = [a_{ij}]$ of size $u \times v$ and $B = [b_{jk}]$ of size $v \times w$ can be multiplied in the usual way; the product $A \cdot B$ is the bundle $C = [c_{ik}]$ of size $u \times w$ and

$$c_{ik} = \sum_{1 \le j < v+1} a_{ij} b_{jk}$$

for each $1 \le i < u+1$, $1 \le k < w+1$.

DEFINITION 3.3. The sequences $\xi = \{x_i\}_{i=1}^{u}$, $\eta = \{y_j\}_{j=1}^{v}$ of non-negative integers (u, v are positive integers or the symbol ∞) are called *parallel* if for each positive integer A there exists a positive integer B such that the g.c.d. (the greatest common divisor) of the set

$$\{x_i :\ n \le i < u+1\} \cup \{y_j :\ n \le j < v+1\}$$

is greater than A or equal to zero for each integer $n > B$. (The g.c.d. of the empty set is zero.) Then we shall write $\xi \parallel \eta$.

DEFINITION 3.4. A system $\mathfrak{S}$ of bundles containing all almost zero bundles is called a *star* if we have

(a) $A, B \in \mathfrak{S}$, A has size $u \times v$, B has size $v \times w$ $\Longrightarrow$ $A \cdot B \in \mathfrak{S}$.

(b) If $A = [a_{ij}] \in \mathfrak{S}$ has size $u \times v$ and $B = [b_{k\ell}] \in \mathfrak{S}$ has size $r \times s$, then for each $1 \le j < v+1$, $1 \le \ell < s+1$ the sequences $\{a_{ij}\}_{i=1}^{u}$, $\{b_{k\ell}\}_{k=1}^{r}$ are parallel.

A maximal element in the system of all stars ordered by inclusion $\subseteq$ is called an *ultrastar*. Clearly, the set of all almost zero bundles forms the least star, which will be denoted by $\mathfrak{S}_0$.

In [14; Sec. 4] a natural one-to-one correspondence from the set of all ultrastars onto the class (which is therefore a set) of all maximal δ_1-categories was constructed and it was shown (Satz 4.11):

PROPOSITION 3.2. *For each δ_1-category $\mathcal{M}$ there exists a maximal δ_1-category $\bar{\mathcal{M}}$ such that $\mathcal{M}$ is a subcategory of $\bar{\mathcal{M}}$.*

Since the set of all bundles has cardinal $\exp \aleph_0$, we get:

PROPOSITION 3.3. *The set of all ultrastars and the set of all maximal δ_1-categories have the same cardinal $\leq \exp\exp \aleph_0$.*

4. α-Pseudofilters

For more detailed description of ultrastars and for the proof of converse inequality in Proposition 3.3 we will introduce and investigate the concept of *α-pseudofilter*.

We denote by $\mathbf{P}$ the set of all primes with discrete topology and by $\beta\mathbf{P}$ the Čech-Stone compactification of $\mathbf{P}$. For $M \subseteq \beta\mathbf{P}$ the closure of M in $\beta\mathbf{P}$ will be denoted by $\mathrm{cl}_{\beta\mathbf{P}} M$.

Remind that Čech-Stone compactification $\beta\mathbf{P}$ of the space $\mathbf{P}$ consist of all ultrafilters of the set $\mathbf{P}$, and each $p \in \mathbf{P}$ is identified with the fixed ultrafilter of the set $\mathbf{P}$ generated by p. The system of all sets of the form $\{\mathbf{u} \in \beta\mathbf{P} : U \in \mathbf{u}\}$, where $U \subseteq \mathbf{P}$ forms an open base of the topological space $\beta\mathbf{P}$. This system is also the system of all clopen (open-and-closed) sets of $\beta\mathbf{P}$ (see, e.g., [15; 1.19, 1.37]).

DEFINITION 4.1. Let $\xi = \{x_i\}_{i=1}^{u}$ (u is a positive integer or ∞) be a sequence of non-negative integers x_i. Put

$$\pi_1(\xi) = \{p \in \mathbf{P} : (\exists k \in \mathbb{N})(\forall i \in \mathbb{N})(k < i < u+1 \implies p/x_i)\},$$

$$\pi_2(\xi) = \{p \in \mathbf{P} : (\forall N \in \mathbb{N})(\exists k_N \in \mathbb{N})(\forall i \in \mathbb{N})(k_N < i < u+1 \implies p^N/x_i)\},$$

($/$ is the divisibility relation among integers,

$\mathbb{N}$ is the set of all positive integers),

$$\alpha(\xi) = \left(\mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\xi) - \mathbf{P}\right) \cup \pi_2(\xi).$$

PROPOSITION 4.1. *Sequences ξ, η of non-negative integers are parallel if and only if $\alpha(\xi) \cap \alpha(\eta) \neq \emptyset$.*

Proof. Let $\xi = \{x_i\}_{i=1}^{u}$, $\eta = \{y_i\}_{i=1}^{v}$ (u, v are positive integers or the symbol ∞) be sequences of non-negative integers.

I. Assume $\xi \parallel \eta$. If the set $\pi_1(\xi) \cap \pi_1(\eta)$ is infinite, then there exists $\mathbf{u} \in \mathrm{cl}_{\beta\mathbf{P}}(\pi_1(\xi) \cap \pi_2(\eta)) - \mathbf{P}$ and clearly $\mathbf{u} \in \alpha(\xi) \cap \alpha(\eta)$.

Let the set $\pi_1(\xi) \cap \pi_1(\eta)$ be finite and let $q \in \mathbf{P} - \pi_1(\xi) \cap \pi_1(\eta)$. Then q does not divide the g.c.d. of the set $\{x_i : n \leq i < u+1\} \cup \{y_j : n \leq j < v+1\}$ for each positive integer n. Therefore there exists $p \in \pi_2(\xi) \cap \pi_2(\eta) \subseteq \alpha(\xi) \cap \alpha(\eta)$.

II. Let $\mathbf{u} \in \alpha(\xi) \cap \alpha(\eta)$. If $\mathbf{u} \in \beta\mathbf{P} - \mathbf{P}$, then $\mathbf{u} \in \mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\xi) \cap \mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\eta)$ and the set $\pi_1(\xi) \cap \pi_1(\eta)$ is infinite and $\xi \parallel \eta$.

If $\mathbf{u} = p \in \mathbf{P}$, then $p \in \pi_2(\xi) \cap \pi_2(\eta)$, which follows that $\xi \parallel \eta$. □

DEFINITION 4.2. A subset A of $\beta\mathbf{P}$ is called an *α-set* if there exists a sequence ξ of non-negative integers such that $A = \alpha(\xi)$.

This concept of an α-set can also be characterized only by purely topological tools as follows:

PROPOSITION 4.2. *Let $A \subseteq \beta\mathbf{P}$. Then the following statements are equivalent:*

(a) *A is an α-set.*
(b) *There exist a continuous mapping f from $\beta\mathbf{P}$ to $\mathbb{R}$ (the space of real numbers) and a neighbourhood U of the point 0 in $\mathbb{R}$ with the following properties:*
   (i) *if $r \in \mathbb{R}$ is a cluster point of the set $f(\beta\mathbf{P}) \cap U$, then $r = 0$,*
   (ii) *for $s \in U - \{0\}$ the set of $f^{-1}(s)$ is finite,*
   (iii) *$A = f^{-1}(0)$.*
(c) *There exists a clopen set G of $\beta\mathbf{P}$ such that $G \supseteq A$ and $G \cap (\beta\mathbf{P} - \mathbf{P}) = A \cap (\beta\mathbf{P} - \mathbf{P})$.*

Proof.

I. Suppose that the statement (a) is valid and ξ is a sequence of non-negative integers with $\alpha(\xi) = A$. If $\pi_1(\xi)$ is finite, then A is finite, $A \subseteq \mathbf{P}$, and clearly the condition (b) is satisfied. Assume that $\pi_1(\xi) = \{p_1, p_2, \dots\}$ is infinite and the primes p_i are mutually different. Put $U = \{x \in \mathbb{R} : \ -1 < x < 1\}$ and

$$f(t) = \begin{cases} 0 & \text{if } t \in A, \\ \frac{1}{n} & \text{if } t = p_n \text{ and } t \notin A, \\ 1 & \text{if } t \in \beta\mathbf{P} - A \cup \pi_1(\xi). \end{cases}$$

Then f is a continuous mapping from $\beta\mathbf{P}$ to $\mathbf{P}$ fulfilling (i)–(iii) from (b).

II. Assume that the statement (b) is true. We can suppose that U is an open set of $\mathbb{R}$. Setting $G = f^{-1}(U)$ we get a clopen set G of $\beta\mathbf{P}$ with $G \supseteq A$.

Let $\mathbf{u} \in G \cap (\beta\mathbf{P} - \mathbf{P})$. If $f(\mathbf{u}) \neq 0$, then there exists a neighbourhood V of $f(\mathbf{u})$ in $\mathbb{R}$ such that $V \subseteq U$ and $f(\beta\mathbf{P}) \cap V = f(\mathbf{u})$. Hence the set $f^{-1}(V)$ is finite and therefore $f^{-1}(V)$ is not a neighbourhood of $\mathbf{u}$ in $\beta\mathbf{P}$, which is a contradiction. Consequently, $G \cap (\beta\mathbf{P} - \mathbf{P}) = A \cap (\beta\mathbf{P} - \mathbf{P})$.

III. (c) $\implies$ (a): Let there exist a clopen set G of $\beta\mathbf{P}$ such that $G \supseteq A$ and $G \cap (\beta\mathbf{P} - \mathbf{P}) = A \cap (\beta\mathbf{P} - \mathbf{P})$. If $G \cap \mathbf{P}$ is finite, we can suppose $G = A$.

For a positive integer n put

$$b_n = \begin{cases} 1 & \text{if } G \cap \mathbf{P} \text{ is finite,} \\ p_1 \dots p_n & \text{if } G \cap \mathbf{P} = \{p_1, p_2, \dots\} \text{ is infinite,} \end{cases}$$

where $p_1, p_2, \dots$ are mutually different,

$$c_n = \begin{cases} q_1^n \cdot q_2^n \dots q_n^n & \text{if } A \cap \mathbf{P} = \{q_1, q_2, \dots\}, \\ 1 & \text{if } A \cap \mathbf{P} = \emptyset \text{ and } A \neq \emptyset, \end{cases}$$

$$\left.\begin{aligned} c_{2n} &= 2^n \\ c_{2n-1} &= 3^n \end{aligned}\right\} \text{ if } A = \emptyset.$$

Consider the sequence $\xi = \{b_n c_n\}_{n=1}^{\infty}$. Then $\pi_1(\xi) = G \cap \mathbf{P}$, $\pi_2(\xi) = A \cap \mathbf{P}$, therefore $\alpha(\xi) = A$. It follows that A is an α-set. □

COROLLARY 4.3. *Each α-set is a zero set and therefore closed in $\beta\mathbf{P}$. Each clopen set of $\beta\mathbf{P}$ is an α-set.*

DEFINITION 4.3. A non-empty system $\mathfrak{a}$ of α-sets is said to be an *α-pseudofilter* if it satisfies the following conditions:

(a) $A \in \mathfrak{a}$, $B \in \mathfrak{a} \implies A \cap B \neq \emptyset$,

(b) C is an α-set, $A \in \mathfrak{a}$, $A \subseteq C \implies C \in \mathfrak{a}$.

A maximal element of the system of all α-pseudofilters ordered by inclusion $\subseteq$ is called an *α-ultrapseudofilter*. If $\mathbf{u} \in \beta\mathbf{P}$, we denote by $\chi(\mathbf{u})$ the system of all α-sets containing $\mathbf{u}$.

Using Corollary 4.3 we get:

PROPOSITION 4.4. *If $\mathbf{u} \in \beta\mathbf{P}$, then $\chi(\mathbf{u})$ is an α-ultrapseudofilter. The mapping $\mathbf{u} \mapsto \chi(\mathbf{u})$ from $\beta\mathbf{P}$ to the set of all α-ultrapseudofilters is one-to-one.*

According to Pospíšil's Theorem ([10] or [15; 3.2]) $\operatorname{card} \beta\mathbf{P} = \exp\exp \aleph_0$ and since the set of all α-sets has cardinal $\exp \aleph_0$, we get from Proposition 4.4:

THEOREM 4.5. *The set of all α-ultrapseudofilters has cardinal* $\exp\exp \aleph_0$.

DEFINITION 4.4. Let $A = [a_{ij}]$ $(1 \leq i < u+1,\ 1 \leq j < v+1)$ be a bundle of size $u \times v$ and $\mathfrak{h}$ be an α-pseudofilter. We put $A \to \mathfrak{h}$ if the following conditions are valid:

(1) $(\forall n \in \mathbb{N})\Big(1 \leq n < v+1 \implies \alpha\big(\{a_{in}\}_{i=1}^{u}\big) \in \mathfrak{h}\Big)$,

(2) if ξ is a sequence of non-negative integers of size v such that $\alpha(\xi) \in \mathfrak{h}$, then $\alpha(A \cdot \xi) \in \mathfrak{h}$.

PROPOSITION 4.6. *Let $\mathfrak{h}$ be an α-pseudofilter.*

(a) *For a sequence η of non-negative integers (η is a bundle of size $u \times 1$) we have $\eta \to \mathfrak{h}$ if and only if $\alpha(\eta) \in \mathfrak{h}$.*

(b) *If A, B are bundles of sizes $u \times v$, $v \times w$, respectively, then*

$$(A \to \mathfrak{h} \ \& \ B \to \mathfrak{h}) \implies A \cdot B \to \mathfrak{h}.$$

Proof.

I. Let η be a sequence of non-negative integers. From Definition 4.4 we get immediately that if $\eta \to \mathfrak{h}$, then $\alpha(\eta) \in \mathfrak{h}$.

Let ξ be a sequence of size 1 (hence a bundle of size 1×1) with the property $\alpha(\xi) \in \mathfrak{h}$. Then $\xi = [0]$ and $\alpha(\eta \cdot \xi) = \beta \mathbf{P} \in \mathfrak{h}$. Thus $\eta \to \mathfrak{h}$.

II. Let $A = [a_{ij}]$, $B = [b_{ij}]$ be bundles of sizes $u \times v$, $v \times w$, respectively, and $A \to \mathfrak{h}$, $B \to \mathfrak{h}$. Then $C = A \cdot B = [c_{ij}]$ is a bundle of size $u \times w$ and

$$c_{ij} = \sum_{k=1}^{v} a_{ik} b_{kj} \qquad \text{for each} \quad 1 \le i < u+1, \quad 1 \le j < w+1.$$

For each integer n ($1 \le n < w+1$) we get $\{c_{in}\}_{i=1}^{u} = A \cdot \{b_{kn}\}_{k=1}^{v}$, therefore $\alpha(\{c_{in}\}_{i=1}^{u}) \in \mathfrak{h}$.

Let ξ be a sequence of non-negative integers of size w such that $\alpha(\xi) \in \mathfrak{h}$. Then $\alpha(B \cdot \xi) \in \mathfrak{h}$ and $\alpha(C \cdot \xi) = \alpha(A \cdot (B \cdot \xi)) \in \mathfrak{h}$. Therefore $A \cdot B \to \mathfrak{h}$. □

For further proofs we will need the following assertion:

LEMMA 4.7.

(a) *Let ξ be a sequence of non-negative integers of size v and A an almost zero bundle of size $u \times v$. Then*

$$\pi_1(\xi) \subseteq \pi_1(A \cdot \xi), \qquad \pi_2(\xi) \subseteq \pi_2(A \cdot \xi), \qquad \alpha(\xi) \subseteq \alpha(A \cdot \xi).$$

(b) *Let M_1, M_2 be α-sets and $M_1 \subseteq M_2$. Let $M_1 = \alpha(\xi)$, where ξ is a sequence of non-negative integers of size v. Then there exists an almost zero bundle A of size $\infty \times v$ such that $M_2 = \alpha(A \cdot \xi)$.*

Proof. The part (a) of Lemma 4.7 is readily shown.

Let γ be a sequence of non-negative integers with $\alpha(\gamma) = M_2$ and let $\xi = \{x_i\}_{i=1}^{v}$. Let $q_1, q_2, \dots$ be integers with the following property:

If $\pi_1(\gamma)$ is infinite, then $q_1, q_2, \dots$ are mutually different and $\pi_1(\gamma) = \{q_1, q_2, \dots\}$. If $\pi_1(\gamma)$ contains only k elements (k a non-negative integer), then $q_\nu = 1$ for $\nu \ge k+1$ and $\pi_1(\gamma) = \{q_\nu : 1 \le \nu \le k\}$. For positive integers i, j ($j < v+1$) set

$$y_{ij} = \begin{cases} 1 & \text{if } q_i \notin \pi_2(\gamma), \\ ij & \text{if } q_i \in \pi_2(\gamma), \end{cases} \qquad a_{ij} = \begin{cases} 0 & \text{if } i \ne j \\ q_1^{y_{1j}} q_2^{y_{2j}} \dots q_j^{y_{jj}} & \text{if } i = j. \end{cases}$$

Then $A = [a_{ij}]$ $(1 \leq i < \infty,\ 1 \leq j < v+1)$ is an almost zero bundle of size $\infty \times v$. Let $\eta = A \cdot \xi = \{y_i\}_{i=1}^{\infty}$. Since $y_i = a_{ii}x_i$ for each positive integer i, we have $\pi_2(\eta) = \pi_2(\gamma)$ and $\pi_1(\eta) = \pi_1(\xi) \cup \pi_1(\gamma)$. Consequently

$$\begin{aligned}
\mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\eta) - \mathbf{P} &= \left(\mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\xi) \cup \mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\gamma)\right) - \mathbf{P} \\
&= \left(\mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\xi) - \mathbf{P}\right) \cup \left(\mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\gamma) - \mathbf{P}\right) \\
&= \mathrm{cl}_{\beta\mathbf{P}}\, \pi_1(\gamma) - \mathbf{P},
\end{aligned}$$

therefore $\alpha(A \cdot \xi) = \alpha(\eta) = \alpha(\gamma) = M_2$, which is what we wanted to prove. □

5. Relationship between stars and α-pseudofilters

DEFINITION 5.1. For an α-pseudofilter $\mathfrak{h}$ we denote by $\mathcal{A}(\mathfrak{h})$ the *system of all bundles A with $A \to \mathfrak{h}$*.

For a star $\mathfrak{S}$ put

$$\begin{aligned}
\mathcal{B}(\mathfrak{S}) &= \{\alpha(\xi) :\ (\exists A \in \mathfrak{S})(\exists n \in \mathbb{N})\big(A = [a_{ij}] \text{ of size } u \times v\ \& \\
&\qquad\qquad\qquad 1 \leq n < v+1\ \&\ \xi = \{a_{in}\}_{i=1}^{u}\big)\} \\
&= \{\alpha(\xi) :\ \xi \text{ is a sequence of non-negative integers with } \xi \in \mathfrak{S}\}.
\end{aligned}$$

PROPOSITION 5.1. *If $\mathfrak{h}$ is an α-pseudofilter, then $\mathcal{A}(\mathfrak{h})$ is a star. If $\mathfrak{h}$ is an α-ultrapseudofilter, then $\mathcal{A}(\mathfrak{h})$ is an ultrastar.*

Proof.

I. If $A = [a_{ij}]$ is an almost zero bundle of size $u \times v$, then for each integer n $(1 \leq n < v+1)$ we have $\alpha(\{a_{in}\}_{i=1}^{u}) = \beta\mathbf{P} \in \mathfrak{h}$ and for each sequence ξ of non-negative integers of size v with $\alpha(\xi) \in \mathfrak{h}$ the set $\alpha(A \cdot \xi)$ belongs to $\mathfrak{h}$ by Lemma 4.7(a). Therefore $A \to \mathfrak{h}$ and $A \in \mathcal{A}(\mathfrak{h})$.

II. Let $A, B \in \mathcal{A}(\mathfrak{h})$. If A, B have sizes $u \times v$, $v \times w$, respectively, then according to Proposition 4.6, $A \cdot B \to \mathfrak{h}$, thus $A \cdot B \in \mathcal{A}(\mathfrak{h})$.

If A, B have sizes $u \times v$, $r \times s$ and ξ, η are columns of A, B, respectively, then $\alpha(\xi) \in \mathfrak{h}$, $\alpha(\eta) \in \mathfrak{h}$ (by Definition 4.4), hence $\alpha(\xi) \cap \alpha(\eta) \neq \emptyset$. Using Proposition 4.1 we get $\xi \parallel \eta$, which implies that $\mathcal{A}(\mathfrak{h})$ is a star.

III. Let $\mathfrak{h}$ be an α-ultrapseudofilter, $\mathfrak{S}$ a star, $\mathfrak{S} \supseteq \mathcal{A}(\mathfrak{h})$, and let $C = [c_{ij}] \in \mathfrak{S}$ be a bundle of size $u \times v$.

Suppose that n is an integer $(1 \leq n < v+1)$ and put $\gamma = \{c_{in}\}_{i=1}^{u}$. Let η be a sequence of non-negative integers with $\alpha(\eta) \in \mathfrak{h}$. Using Proposition 4.6(a) we get $\eta \to \mathfrak{h}$, therefore $\eta \in \mathcal{A}(\mathfrak{h})$ and $\eta \in \mathfrak{S}$. Consequently $\eta \parallel \gamma$ and by Proposition 4.1, $\alpha(\eta) \cap \alpha(\gamma) \neq \emptyset$, which follows that $\alpha(\gamma) \in \mathfrak{h}$.

If ξ is a sequence of non-negative integers of size v with $\alpha(\xi) \in \mathfrak{h}$, then $\xi \to \mathfrak{h}$ (Proposition 4.6(a)) and thus $\xi \in \mathfrak{S}$. Therefore $C \cdot \xi \in \mathfrak{S}$ and $\eta \parallel C \cdot \xi$. Consequently (Proposition 4.1) $\alpha(\eta) \cap \alpha(C \cdot \xi) \neq \emptyset$, which gives $\alpha(C \cdot \xi) \in \mathfrak{h}$.

Hence $C \to \mathfrak{h}$ and $C \in \mathcal{A}(\mathfrak{h})$, thus $\mathcal{A}(\mathfrak{h})$ is an ultrastar. This proves the result. □

PROPOSITION 5.2. *If $\mathfrak{S}$ is a star, then $\mathcal{B}(\mathfrak{S})$ is an α-pseudofilter. For stars $\mathfrak{S}_1$, $\mathfrak{S}_2$ with $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ we have $\mathcal{B}(\mathfrak{S}_1) \subseteq \mathcal{B}(\mathfrak{S}_2)$.*

Proof. Let $\mathfrak{S}$ be a star. Since the zero sequence $\omega = \{o_i\}_{i=1}^{\infty}$ ($o_i = 0$ for all $i = 1, 2 \dots$) as an almost zero bundle belongs to $\mathfrak{S}$, we have $\beta\mathbf{P} = \alpha(\omega) \in \mathcal{B}(\mathfrak{S})$, thus $\mathcal{B}(\mathfrak{S}) \neq \emptyset$.

Using Proposition 4.1 we get $X \cap Y \neq \emptyset$ for $X, Y \in \mathcal{B}(\mathfrak{S})$. Suppose that $B \subseteq C \subseteq \beta\mathbf{P}$, $B \in \mathcal{B}(\mathfrak{S})$, C is an α-set and $\alpha(\xi) = B$, where ξ is a sequence of non-negative integers of size v with $\xi \in \mathfrak{S}$. According to Lemma 4.7(b) there exists an almost zero bundle A of size $\infty \times v$ such that $C = \alpha(A \cdot \xi)$. Then the relation $C \in \mathcal{B}(\mathfrak{S})$ follows from $A \cdot \xi \in \mathfrak{S}$. Therefore $\mathcal{B}(\mathfrak{S})$ is an α-pseudofilter.

For stars $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ the inclusion $\mathcal{B}(\mathfrak{S}_1) \subseteq \mathcal{B}(\mathfrak{S}_2)$ is clear. □

PROPOSITION 5.3. *If $\mathfrak{h}$ is an α-pseudofilter, then $\mathcal{B}\mathcal{A}(\mathfrak{h}) = \mathfrak{h}$.*

Proof. Suppose $X \in \mathfrak{h}$ and $X = \alpha(\xi)$, where ξ is a sequence of non-negative integers. According to Proposition 4.6(a), $\xi \to \mathfrak{h}$, hence $\xi \in \mathcal{A}(\mathfrak{h})$ and $X = \alpha(\xi) \in \mathcal{B}\mathcal{A}(\mathfrak{h})$.

Conversely, assume that $X \in \mathcal{B}\mathcal{A}(\mathfrak{h})$. Then there exists a sequence ξ of non-negative integers with $X = \alpha(\xi)$ and $\xi \in \mathcal{A}(\mathfrak{h})$, therefore $\xi \to \mathfrak{h}$ and $X = \alpha(\xi) \in \mathfrak{h}$ (Proposition 4.6(a)). The result follows. □

From Propositions 5.1–5.3 we can derive easily that $\mathcal{A}$, $\mathcal{B}$ are mappings from the set $\mathcal{P}$ of all α-pseudofilters to the set $\mathcal{S}$ of all stars, from $\mathcal{S}$ to $\mathcal{P}$, respectively, $\mathcal{A}$ is one-to-one (injection) and $\mathcal{B}$ is onto (surjection). Furthermore the restriction of $\mathcal{A}$ to the set $\mathcal{P}_u$ of all α-ultrapseudofilters is a one-to-one mapping from $\mathcal{P}_u$ to the set $\mathcal{S}_u$ of all ultrastars. Hence $\operatorname{card}\mathcal{P}_u \leq \operatorname{card}\mathcal{S}_u$ and using Propositions 4.5 and 3.3 we get:

THEOREM 5.4. *The set of all ultrastars and the set of all maximal δ_1-categories have the same cardinal equal to* $\exp\exp\aleph_0$.

Note that the question whether for each ultrastar $\mathfrak{S}$ the α-pseudofilter $\mathcal{B}(\mathfrak{S})$ is an α-ultrapseudofilter remains open.

6. Some examples

The aim of this last section is to show that the mapping $\mathcal{B}\colon \mathcal{S} \to \mathcal{P}$ is not one-to-one (Proposition 6.2) and that the class of all δ_1-categories does not form a set (Theorem 6.7).

LEMMA 6.1. *Let $\mathfrak{A}$ be the set of all almost zero bundles and ξ a sequence of non-negative integers of size v with $\alpha(\xi) \neq \emptyset$. Then the set*

$$\mathfrak{S} = \mathfrak{S}(\xi) = \{A \cdot \xi \cdot Z : \ A \in \mathfrak{A} \text{ has size } u \times v \text{ and } Z \text{ is a bundle of size } 1 \times w\} \cup \mathfrak{A}$$

is a star.

Proof. Evidently $\mathfrak{A} \subseteq \mathfrak{S}$ and the condition (a) of Definition 3.4 is satisfied. If σ is a column of the bundle $A \cdot \xi \cdot Z$, where $A \in \mathfrak{A}$ has size $u \times v$ and Z is a bundle of size $1 \times w$, then according to Lemma 4.7(a), $\alpha(\xi) \subseteq \alpha(A \cdot \xi) \subseteq \alpha(\sigma)$. □

EXAMPLE. Let p be a prime, $\xi_1 = \{p^i\}_{i=1}^{\infty}$, $\xi_2 = \{p^{i-1}\}_{i=1}^{\infty}$. If $\xi_2 \in \mathfrak{S}(\xi_1)$, then there exist $A = [a_{ij}] \in \mathfrak{A}$ of size $u \times \infty$ and a bundle $Z = [z]$ of size 1×1 such that $\xi_2 = A \cdot \xi_1 \cdot Z$, therefore $1 = \sum\limits_{j=1}^{\infty} a_{1j} p^j z$, which is a contradiction.

The stars $\mathfrak{S}(\xi_1)$ and $\mathfrak{S}(\xi_2)$ are different and $\mathcal{B}(\mathfrak{S}(\xi_1)) = \mathcal{B}(\mathfrak{S}(\xi_2)) = \{X : X \text{ is an } \alpha\text{-set with } p \in X\} = \chi(p)$, using notation of Definition 4.3. Thus we can state:

PROPOSITION 6.2. *The mapping $\mathcal{B}\colon \mathcal{S} \to \mathcal{P}$ is not one-to-one.*

DEFINITION 6.1. ([14; 4.9]) Let D and D' be semigroups with unique factorization and let f be a homomorphism from D to D'. An irreducible element $\mathfrak{q}$ of the semigroup D' will be called an *α-element of f* if the set $\{\mathfrak{p} : \mathfrak{p} \text{ is an irreducible element of } D \text{ such that } f(\mathfrak{p}) \text{ is divisible by } \mathfrak{q}\}$ is infinite.

If $\mathfrak{m}$ is an infinite cardinal number and the set of all α-elements of f has cardinal equal to or less than $\mathfrak{m}$, we call f an *$\alpha(\mathfrak{m})$-homomorphism.*

PROPOSITION 6.3. *Suppose that $\mathfrak{m}$ is an infinite cardinal number, D_1, D_2, D_3 are semigroups with unique factorization and $f\colon D_1 \to D_2$, $g\colon D_2 \to D_3$ are $\alpha(\mathfrak{m})$-homomorphisms. Then $g \circ f\colon D_1 \to D_3$ is an $\alpha(\mathfrak{m})$-homomorphism.*

The proof is almost word for word the same as the proof of [14; 4.9.1], where this proposition was shown for $\mathfrak{m} = \aleph_0$.

DEFINITION 6.2. ([14; Sec. 4]) Let p be a prime and D_1, D_2 be semigroups with unique factorization. Denote by $(D_1, D_2)_{\mathcal{M}(p)}$ the *set of all homomorphisms from D_1 to D_2* with the following property:

Let $\mathfrak{q}$ be an irreducible element of the semigroup D_2 and let $\{\mathfrak{p}_i\}_{i=1}^{u}$ be a sequence of size u (u is a positive integer or ∞) of mutually different irreducible elements of the semigroup D_1. Let a_i be the order of $f(\mathfrak{p}_i)$ at $\mathfrak{q}$ for $1 \le i < u+1$. Then $p \in \pi_2(\{a_i\}_{i=1}^{u})$.

For δ_1-semigroups G, H put

$$(G,H)_{\mathcal{M}(p)} = \{g : g \text{ is a homomorphism from } G \text{ to } H \text{ such that there exists } f \in (\mathfrak{c}G, \mathfrak{c}H)_{\mathcal{M}(p)} \text{ with } f \circ e_G = e_H \circ g\}.$$

We define the category $\mathcal{M}(p)$ whose objects are δ_1-semigroups, $\mathrm{Hom}_{\mathcal{M}(p)}(G,H) = (G,H)_{\mathcal{M}(p)}$ for δ_1-semigroups G, H, and the operation of morphisms equals the composition of mappings. Using the results of [14; Sec. 4] we get:

PROPOSITION 6.4. *The category $\mathcal{M}(p)$ is a maximal δ_1-category for each prime p.*

DEFINITION 6.3. Let p be a prime and $\mathfrak{m}$ be an infinite cardinal number. For δ_1-semigroups G, H put

$$(G,H)_{\mathcal{M}(p,\mathfrak{m})} = \{g : g \text{ is a homomorphism from } G \text{ to } H \text{ such that there exists } f \in (\mathfrak{c}G, \mathfrak{c}H)_{\mathcal{M}(p)} \text{ with } f \circ e_G = e_H \circ g \text{ and } f \text{ is an } \alpha(\mathfrak{m})\text{-homomorphisms}\}.$$

We define a subcategory $\mathcal{M}(p,\mathfrak{m})$ of the category $\mathcal{M}(p)$ whose objects are δ_1-semigroups and $\mathrm{Hom}_{\mathcal{M}(p,\mathfrak{m})}(G,H) = (G,H)_{\mathcal{M}(p,\mathfrak{m})}$ for δ_1-semigroups G, H (see Proposition 6.3). Since for the semigroups D_1, D_2 with unique factorization and for each δ^*-homomorphism f from D_1 to D_2 the set of all α-elements of f is empty ([13; Korollar 3.10]), the category $\mathcal{K}_1$ is a subcategory of the category $\mathcal{M}(p,\mathfrak{m})$ and therefore:

PROPOSITION 6.5. *The category $\mathcal{M}(p,\mathfrak{m})$ is a δ_1-category for each prime p and each infinite cardinal number $\mathfrak{m}$.*

PROPOSITION 6.6. *Let p be a prime and $\mathfrak{m}$, $\mathfrak{n}$ be infinite cardinal numbers with $\mathfrak{m} < \mathfrak{n}$. Then the category $\mathcal{M}(p,\mathfrak{m})$ is a proper subcategory of the category $\mathcal{M}(p,\mathfrak{n})$.*

Proof. Clearly, $\mathcal{M}(p,\mathfrak{m})$ is a subcategory of $\mathcal{M}(p,\mathfrak{n})$. We will show that $\mathcal{M}(p,\mathfrak{m}) \neq \mathcal{M}(p,\mathfrak{n})$. Let Q be a set with $\operatorname{card} Q = \mathfrak{n}$ and let $\mathfrak{p}_{i\mathfrak{q}}$ be mutually different symbols for $\mathfrak{q} \in Q$ and i being a positive integer. There exist semigroups D_1, D_2 with unique factorization such that $\{\mathfrak{p}_{i\mathfrak{q}} : \mathfrak{q} \in Q,\ i \in \mathbb{N}\}$ and Q are the sets of all irreducible elements of D_1 and D_2, respectively.

For a positive integer i and $\mathfrak{q} \in Q$ set

$$f(\mathfrak{p}_{i\mathfrak{q}}) = \mathfrak{q}^{p^i}$$

and extend f to a homomorphism from D_1 to D_2. Then $f \in (D_1, D_2)_{\mathcal{M}(p)}$ and Q is the set of all α-elements of f. Thus $f \in \operatorname{Hom}_{\mathcal{M}(p,\mathfrak{n})}(D_1, D_2) - \operatorname{Hom}_{\mathcal{M}(p,\mathfrak{m})}(D_1, D_2)$. □

Using Propositions 6.5 and 6.6 we obtain:

THEOREM 6.7. *The class of all δ_1-categories does not form a set.*

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