Mathematica Slovaca

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Mathematica Slovaca, Vol. 52 (2002), No. 2, 195--206

Persistent URL: `http://dml.cz/dmlcz/128952`

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METRIC DISTRIBUTION RESULTS FOR SEQUENCES $(\{q_n\vec{\alpha}\})$

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. In this paper a recent result of Philipp and Tichy (2000) on the well-distribution measure of certain binary pseudorandom sequences in the unit interval is generalized. Furthermore the average value of the L^2-discrepancy of sequences $(\{q_n\vec{\alpha}\})_{n\geq 1}$ is calculated, where $(q_n)_{n\geq 1}$ is a given sequence of positive integers and $\vec{\alpha} \in [0,1]^d$.

1. Introduction

Let $\{x\} := x - [x]$ denote the fractional part of a real number x and for any set M let c_M be the characteristic function of M. In this paper we study sequences of the type $\omega = (\{q_n\vec{\alpha}\})_{n\geq 1}$, where $\vec{\alpha} = (\alpha_1,\dots,\alpha_d)$ is a vector in the d-dimensional unit cube $U^d = [0,1]^d$ and $(q_n)_{n\geq 1}$ is a sequence of positive integers. Here $\{q_n\vec{\alpha}\}$ stands for the vector $(\{q_n\alpha_1\},\{q_n\alpha_2\},\dots,\{q_n\alpha_d\})$. In the special case $q_n = n$ we have the so-called Kronecker sequence $(n\vec{\alpha})$, which is uniformly distributed mod 1 if and only if $1,\alpha_1,\dots,\alpha_d$ are linearly independent over $\mathbb{Z}$ (cf. [1]).

For such a sequence $\omega = (\{q_n\vec{\alpha}\})_{n\geq 1}$, the standard discrepancy with arbitrary weights $k_i \geq 0$ $(i = 1,\dots,N)$, where $\sum\limits_{i=1}^{N} k_i = 1$, is defined by

$$D_N(\omega) = \sup_{[\vec{x},\vec{y})\in J^d} \left| \sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}(\{q_n\vec{\alpha}\}) - \lambda_d([\vec{x},\vec{y}))\right|, \tag{1.1}$$

where J^d is the set of all intervals of the form $[\vec{x},\vec{y}) := [x_1,y_1) \times [x_2,y_2) \times \dots \cdots \times [x_d,y_d)$ with $0 \leq x_i \leq y_i \leq 1$, $i = 1,\dots,d$, and λ_d denotes the

2000 Mathematics Subject Classification: Primary 11K55, 11K31.
Keywords: sequence, distribution, discrepancy, well-distribution.

Supported by the Austrian Science Foundation Project S8308-MAT.

d-dimensional Lebesgue measure. Furthermore the L^p-discrepancy of ω is defined by

$$D_N^{(p)}(\omega) := \left(\int\limits_{J^d} \left| \sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}(\{q_n \vec{\alpha}\}) - \lambda_d([\vec{x},\vec{y})) \right|^p \mathrm{d}\vec{x}\,\mathrm{d}\vec{y} \right)^{\frac{1}{p}}. \tag{1.2}$$

The L^p-discrepancy $D_N^{(p)}(\omega)$ for $p = 2$ and $d = 1$ is known as a diaphony which has been introduced by Zinterhof [12], see also Strauch [11].

For a survey on discrepancies and other important concepts in the theory of uniform distribution, we refer to the textbooks of Drmota and Tichy [1] and Kuipers and Niederreiter [5].

Let now $E_N = E_N(\omega) = \{e_1, \dots, e_N\}$ with

$$e_n := \begin{cases} +1 & \text{for } \{q_n \vec{\alpha}\} \in \left[0, \frac{1}{2^{1/d}}\right)^d, \\ -1 & \text{for } \{q_n \vec{\alpha}\} \notin \left[0, \frac{1}{2^{1/d}}\right)^d, \end{cases} \qquad 1 \le n \le N. \tag{1.3}$$

An important measure of the pseudorandomness of such a binary sequence E_N is its well-distribution measure defined as

$$W(E_N) := \max_{\substack{a \in \mathbb{Z}, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} \left| \sum_{j \le t} e_{a+bj} \right|, \qquad N \ge 1.$$

Clearly, $W(E_N)$ can be bounded by the discrepancy of the defining sequence $\left(\{q_n \vec{\alpha}\}\right)_{n \ge 1}$ in the form

$$\begin{aligned} W(E_N) &= \max_{\substack{a \in \mathbb{Z}, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} \left| 2 \sum_{n=1}^{t} c_{\left[0, \frac{1}{2^{1/d}}\right)^d}(\{q_n \vec{\alpha}\}) - t \right| \\ &\le 2 \max_{\substack{a \in \mathbb{Z}, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} t D_t\left(\{q_{a+bj} \vec{\alpha}\},\ j \le t\right), \end{aligned} \tag{1.4}$$

where D_t is the discrepancy defined in (1.1) with equal weights $k_n = 1/N$, $n = 1, \dots, N$.

For $q_n = n^k$ ($n = 1, \dots, N$, $k \in \mathbb{N}$) and $d = 1$, sequences of type (1.3) were considered by Mauduit and Sárközy [6], who, among other things, proved metric results on asymptotic upper bounds for the right hand side of (1.4). Recently these bounds were improved by Philipp and Tichy [8] and at the same time generalized to arbitrary increasing sequences of positive integers $(q_n)_{n \ge 1}$.

In this paper we will derive a metric result on the asymptotic upper bound of (1.4) for arbitrary sequences of distinct positive integers $(q_n)_{n \ge 1}$ and arbitrary dimension $d \ge 1$.

Finally, in Section 3 we calculate the L^2-norm of $D_N^{(2)}(\{q_n\vec{\alpha}\})$ and $D_N^{*(2)}(\{q_n\vec{\alpha}\})$ for arbitrary weights $k_n \geq 0$, $n = 1,\ldots,N$, $\sum_{n=1}^{N} k_n = 1$ and arbitrary dimension $d \geq 1$.

2. A metric theorem for bounding $W(E_N)$

Theorem 2.1. *Let $(q_n,\ n \geq 1)$ be a sequence of distinct positive integers and let $\vec{\alpha} = (\alpha_1,\ldots,\alpha_d) \in [0,1]^d$ for arbitrary $d \geq 1$. Then for almost all $\vec{\alpha}$ and arbitrary $\varepsilon > 0$ we have*

$$\begin{aligned}\max\Big(tD_t(\{q_{a+bj}\vec{\alpha}\},\ j \leq t) : a \in \mathbb{Z},\ b,t \in \mathbb{N},\ 1 \leq a + bt \leq N\Big)& \\ \ll N^{2/3}(\log N)^{1+2d/3+\varepsilon}\,.&\end{aligned} \tag{2.1}$$

Remark. This result is an extension of Theorem 1 of Philipp and Tichy [8], where (2.1) has been established for increasing sequences (q_n) for the one-dimensional case $d = 1$ with the sharper estimate $N^{2/3}(\log N)^{1+\varepsilon}$ on the right hand side.

Proof. The proof is based on a technique developed in [8]. Since the discrepancy $D_t \leq 1$, we only have to consider

$$N^{2/3}(\log N)^{1+2d/3+\varepsilon} \leq t \leq N \tag{2.2}$$

and for the number b in the maximum we thus have without loss of generality

$$b \leq N/(t-1) \leq N^{1/3}(\log N)^{-1-2d/3-\varepsilon}\,. \tag{2.3}$$

Furthermore, by application of the triangle equality, it is easy to see that we can assume

$$|a| \leq b \tag{2.4}$$

without loss of generality.

Now, for $\vec{h} = (h_1,\ldots,h_d) \in \mathbb{Z}^d$ set

$$r(\vec{h}) = \prod_{j \leq d} \max(1, |h_j|) \qquad \text{and} \qquad \|\vec{h}\|_\infty = \max_{j=1,\ldots,d} |h_j|\,.$$

Then for fixed a, b, t and $\vec{\alpha}$ the Erdős-Turán inequality yields

$$\begin{aligned}&tD_t(\{q_{a+bj}\vec{\alpha}\},\ j \leq t)\\ &\quad\leq \left(\frac{3}{2}\right)^d \left(\frac{2t}{H+1} + \sum_{0<\|\vec{h}\|_\infty \leq H} \frac{1}{r(\vec{h})} \left|\sum_{j \leq t} e\Big(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle\Big)\right|\right),\end{aligned} \tag{2.5}$$

where $e(x) = \exp(2\pi \mathrm{i}\, x)$, $\langle \cdot, \cdot \rangle$ denotes the dot product for d-dimensional vectors and H is an arbitrary positive integer (see for example Drmota and Tichy [1]). From (2.5) we obtain

$$t^2 D_t^2\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big) \le \left(\frac{3}{2}\right)^{2d} \left(8(t/H)^2 + 2\left(\sum_{\|\vec{h}\|_\infty \le H} \frac{1}{r(\vec{h})} \left| \sum_{j \le t} e\Big(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle\Big)\right|\right)^2\right). \tag{2.6}$$

For each $\vec{h}$ with $\|\vec{h}\|_\infty \le H$ we have for all $1 \le j_1 \le j_2 \le N$

$$\begin{aligned}
\mathbb{E}\left| \sum_{j_1 \le j \le j_2} e\Big(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle\Big)\right|^2 &= \mathbb{E}\left| \sum_{j_1 \le j \le j_2} \mathrm{e}^{2\pi \mathrm{i}(h_1\alpha_1 + \cdots + h_d\alpha_d) q_{a+bj}}\right|^2 \\
&= \mathbb{E}\left(\sum_{j_1 \le l_1, l_2 \le j_2} \mathrm{e}^{2\pi \mathrm{i}(h_1\alpha_1 + \cdots + h_d\alpha_d)(q_{a+bl_1} - q_{a+bl_2})}\right) \\
&= j_2 - j_1 + 1,
\end{aligned}$$

since $(q_n,\ n \ge 1)$ is a sequence of distinct positive integers.

But now we can apply Lemma A.1 (see appendix) with $\gamma = 2$ and the superadditive function $g(i,j) := j - i + 1$ (that $g(i,j)$ is indeed superadditive, can be checked easily). Thereby we obtain

$$\mathbb{E} \max_{t \le N/b} \left| \sum_{j \le t} e\Big(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle\Big)\right|^2 \le C_1 \frac{N}{b} (\log N)^2 \tag{2.7}$$

for some constant $C_1 \ge 1$. By choosing $H = \lfloor (N/b)^{\frac{1}{2}} \rfloor + 1$ we obtain for fixed a and b from (2.6), (2.7) and Minkowski's inequality

$$\mathbb{E} \max_{t \le N/b} t^2 D_t^2\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big) \le \left(\frac{3}{2}\right)^{2d} \left(C_2 \frac{N}{b} + C_3 \frac{N}{b} \cdot (\log N)^2 (\log N)^{2d}\right) \tag{2.8}$$

for constants $C_2, C_3 \ge 1$ and thus

$$\mathbb{E} \max_{t \le N/b} t^2 D_t^2\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big) \ll \frac{N}{b} \cdot (\log N)^{2d+2}. \tag{2.9}$$

Now we can apply Markov's inequality and together with (2.3) and (2.4) we obtain

$$\begin{aligned}
&\mathbb{P}\left(\max_{\substack{t < N/b \\ |a| < b \le N^{1/3}(\log N)^{-1-2d/3-\varepsilon}}} t D_t\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big) \ge N^{2/3} (\log N)^{1+2d\ 3+} \right) \\
&\ll N^{\ \frac{4}{3}} (\log N)^{-1d/3\ \ 2} \max_{b < N^{1\ 3}(\log N)^{1\ d/3}} N/b \cdot (\log N)^{\ } \cdot b^2 \\
&\ll (\log N)^{1\ 3}.
\end{aligned}$$

Hence we have for fixed $r \ge 1$

$$\mathbb{P}\left(\max_{\substack{t \le 2^r/b \\ |a| \le b \le 2^{r/3} r^{-1-2d/3-\varepsilon}}} tD_t\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big) \ge 2^{2r/3} r^{1+2d/3+\varepsilon} \right) \ll r^{-1-3\varepsilon}$$

from which it finally follows by the Borel-Cantelli lemma that with probability 1

$$\max_{\substack{t \le 2^r/b \\ |a| \le b \le 2^{r/3} r^{-1-2d/3-\varepsilon}}} \Big(tD_t\big(\{q_{a+bj}\vec{\alpha}\},\ j \le t\big)\Big) \ll 2^{2r/3} r^{1+2d/3+\varepsilon},$$

which completes the proof of (2.1). □

3. The mean of the L^2-discrepancy of $(\{q_n\vec{\alpha}\})$

We now allow for arbitrary sequences of positive integers $(q_n)_{n\ge1}$ and define for $[\vec{x},\vec{y}) \in J^d$ the remainder function

$$R_N(\vec{x},\vec{y},\vec{\alpha}) = \sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}\big(\{q_n\vec{\alpha}\}\big) - \prod_{i=1}^{d}(y_i - x_i), \tag{3.1}$$

where $k_n \ge 0$, $n = 1,\dots,N$ and $\sum\limits_{n=1}^{N} k_n = 1$. This can be considered as a weighted local discrepancy function of the sequence $\big(\{q_n\vec{\alpha}\}\big)$.

Koksma [4] was the first to investigate the integral $\int\limits_{[0,1]^d} R_N^2(\vec{x},\vec{y},\vec{\alpha})\,\mathrm{d}\vec{\alpha}$ for $d = 1$ and equal weights $k_n = \frac{1}{N}$, $(n = 1,\dots,N)$, and Strauch [10] obtained an explicit expression for this case. The following proposition generalizes [10; Theorem 1] in that it allows for arbitrary weights k_n and arbitrary dimension $d \ge 1$.

PROPOSITION 3.1. *Let (q_m, q_n) denote the greatest common divisor of q_m and q_n. Then*

$$\int\limits_{U^d} R_N^2(\vec{x},\vec{y},\vec{\alpha})\,\mathrm{d}\vec{\alpha}$$
$$= \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d}\left[(y_i - x_i)^2 + \frac{(q_m,q_n)^2}{q_m\cdot q_n} T\left(x_i, y_i, \frac{q_m}{(q_m,q_n)}, \frac{q_n}{(q_m,q_n)}\right)\right] \tag{3.2}$$
$$- \prod_{i=1}^{d}(y_i - x_i)^2,$$

where

$$\begin{aligned} T(x_i, y_i, a, b) = \big(\{y_i a\} - \{x_i a\}\big)\big(\{x_i b\} - \{y_i b\}\big) - \{y_i a\} + \{x_i a\} \\ + \max\big(\{y_i a\}, \{x_i b\}\big) - \max\big(\{x_i a\}, \{x_i b\}\big) \\ + \min\big(\{y_i a\}, \{y_i b\}\big) - \min\big(\{x_i a\}, \{y_i b\}\big)\,. \end{aligned}$$

Proof. Since for every pair of real numbers x and y with $0 \le x \le y \le 1$ and $a \in \mathbb{N}$ we have

$$\int_0^1 c_{[x,y)}\big(\{a\alpha\}\big)\,\mathrm{d}\alpha = y - x\,,$$

it follows from definition (3.1) that

$$\begin{aligned} &\int_{U^d} R_N^2(\vec{x}, \vec{y}, \vec{\alpha})\,\mathrm{d}\vec{\alpha} \\ &= \int_{U^d} \Big(\sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}\big(\{q_n \vec{\alpha}\}\big)\Big)^2\,\mathrm{d}\vec{\alpha} \\ &\qquad - 2\prod_{i=1}^{d}(y_i - x_i)\sum_{n=1}^{N}\int_{U^d} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}\big(\{q_n \alpha_i\}\big)\,\mathrm{d}\vec{\alpha} + \prod_{i=1}^{d}(y_i - x_i)^2 \\ &= \int_{U^d} \Big(\sum_{n=1}^{N} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}\big(\{q_n \alpha_i\}\big)\Big)^2\,\mathrm{d}\vec{\alpha} \\ &\qquad - 2\prod_{i=1}^{d}(y_i - x_i)\sum_{n=1}^{N} k_n \prod_{i=1}^{d}(y_i - x_i) + \prod_{i=1}^{d}(y_i - x_i)^2 \\ &= \int_{\alpha_1=0}^{1}\cdots\int_{\alpha_d=0}^{1} \Big(\sum_{n=1}^{N} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}\big(\{q_n \alpha_i\}\big)\Big)^2\,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_d - \prod_{i=1}^{d}(y_i - x_i)^2\,. \end{aligned} \tag{3.3}$$

It has been shown in [10] that for every pair of real numbers x and y with $0 \le x \le y \le 1$ and positive integers q_m, q_n

$$\begin{aligned} &\int_{\alpha=0}^{1} c_{[x,y)}\big(\{q_m \alpha\}\big) c_{[x,y)}\big(\{q_n \alpha\}\big)\,\mathrm{d}\alpha \\ &\qquad = (y-x)^2 + \frac{(q_m, q_n)^2}{q_m \cdot q_n} T\Big(x, y, \frac{q_m}{(q_m, q_n)}, \frac{q_n}{(q_m, q_n)}\Big)\,, \end{aligned}$$

which can now be used to calculate the integrals in (3.3) by using

$$\int_{\alpha_1=0}^{1} \cdots \int_{\alpha_d=0}^{1} \prod_{i=1}^{d} c_{[x_i,y_i)}(\{q_m\alpha_i\}) \prod_{i=1}^{d} c_{[x_i,y_i)}(\{q_n\alpha_i\})\, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_d$$

$$= \prod_{i=1}^{d} \int_{\alpha_i=0}^{1} c_{[x_i,y_i)}(\{q_m\alpha_i\}) c_{[x_i,y_i)}(\{q_n\alpha_i\})\, \mathrm{d}\alpha_i\,,$$

which completes the proof of (3.2). □

Proposition 3.1 can now be used to calculate the L^2-norm of the L^2-discrepancy $D_N^{(2)}(\{q_n\vec{\alpha}\})$ with arbitrary weights k_i:

THEOREM 3.1. *We have*

$$\left\| D_N^{(2)}(\{q_n\vec{\alpha}\}) \right\|_2 = \sqrt{\left(\frac{1}{6^d} - \frac{1}{12^d}\right) \sum_{\substack{m,n=1 \\ q_n=q_m}}^{N} k_n k_m}\,. \tag{3.4}$$

P r o o f . By changing the order of integration we get

$$\int_{U^d} \left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 \mathrm{d}\vec{\alpha} = \iint_{J^d} \int_{U^d} R_N^2(\vec{x},\vec{y},\vec{\alpha})\, \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{y}\,,$$

which by (3.2) yields

$$\int_{U^d} \left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 \mathrm{d}\vec{\alpha}$$

$$= \sum_{m,n=1}^{N} k_n k_m \iint_{J^d} \prod_{i=1}^{d} \left[(y_i - x_i)^2 + \frac{(q_m,q_n)^2}{q_m \cdot q_n} T\left(x_i, y_i, \frac{q_m}{(q_m,q_n)}, \frac{q_n}{(q_m,q_n)}\right)\right] \mathrm{d}\vec{x}\, \mathrm{d}\vec{y}$$

$$- \iint_{J^d} \prod_{i=1}^{d} (y_i - x_i)^2\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{y}$$

$$= \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d} \iint_{0\le x_i\le y_i\le 1} \Bigg[(y_i - x_i)^2$$

$$+ \frac{(q_m,q_n)^2}{q_m \cdot q_n} T\left(x_i, y_i, \frac{q_m}{(q_m,q_n)}, \frac{q_n}{(q_m,q_n)}\right)\Bigg]\, \mathrm{d}x_i\, \mathrm{d}y_i$$

$$- \prod_{i=1}^{d} \iint_{0\le x_i\le y_i\le 1} (y_i - x_i)^2\, \mathrm{d}x_i\, \mathrm{d}y_i\,.$$

But from [10; Corollary 2] it follows that for all $a, b \in \mathbb{N}$

$$\iint\limits_{0 \le x_i \le y_i \le 1} T(x_i, y_i, a, b)\, \mathrm{d}x_i\, \mathrm{d}y_i = \begin{cases} 0 & \text{for } a \ne b\,, \\ \frac{1}{12} & \text{for } a = b\,, \end{cases}$$

so that

$$\begin{aligned}
&\int\limits_{U^d} \Big(D_N^{(2)}(\{q_n\vec{\alpha}\})\Big)^2\, \mathrm{d}\vec{\alpha} \\
&= \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} k_n k_m \prod_{i=1}^{d} \left(\frac{1}{12} + \frac{1}{12}\frac{(q_m, q_n)^2}{q_m q_n}\right) + \sum_{\substack{m,n=1 \\ q_n \ne q_m}}^{N} k_n k_m \prod_{i=1}^{d} \frac{1}{12} - \frac{1}{12^d} \\
&= \sum_{m,n=1}^{N} k_n k_m \frac{1}{12^d} - \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} k_n k_m \frac{1}{12^d} + \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} k_n k_m \frac{1}{6^d} - \frac{1}{12^d} \\
&= \left(\frac{1}{6^d} - \frac{1}{12^d}\right) \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} k_n k_m\,.
\end{aligned}$$

□

EXAMPLE 1. For equal weights $k_n = \frac{1}{N}$ $(n = 1, \dots, N)$ and $d = 1$ in (3.4), we obtain

$$\int\limits_0^1 \Big(D_N^{(2)}(\{q_n \alpha\})\Big)^2\, \mathrm{d}\alpha = \frac{1}{12N^2} \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} 1\,,$$

which is given in [10; Theorem 2].

EXAMPLE 2. If $(q_n)_{n \ge 1}$ is a sequence of distinct positive integers, equation (3.4) gives

$$\int\limits_{U^d} \Big(D_N^{(2)}(\{q_n\vec{\alpha}\})\Big)^2\, \mathrm{d}\vec{\alpha} = \left(\frac{1}{6^d} - \frac{1}{12^d}\right) \sum_{n=1}^{N} k_n^2\,. \tag{3.5}$$

Remark. In [9], Schoissengeier pointed out that for $q_n = n$ $(n = 1, \dots, N)$, equal weights $k_n = \frac{1}{N}$ and $d = 1$, the asymptotic order of the L^2-norm of D_N is $1/\sqrt{N}$. Equation (3.5) shows that in this case $1/\sqrt{N}$ is also the right order of magnitude of the L^2-norm of $D_N^{(2)}$ and, more generally, that this asymptotic result also holds for arbitrary, but fixed dimension d and arbitrary sequences of distinct positive integers $(q_n)_{n \ge 1}$.

In the theory of uniform distribution it is of particular interest to consider the discrepancy of sequences with the underlying set system J_0^d consisting of intervals of the form $[0,\vec{y}) = [0,y_1) \times [0,y_2) \times \cdots \times [0,y_d)$ with $0 \le y_i \le 1$, $i = 1,\dots,d$, which is called the *star discrepancy* D_N^* of $\omega = \big(\{q_n\vec{\alpha}\}\big)_{n\ge 1}$, so

$$D_N^*(\omega) = \sup_{[0,\vec{y})\in J_0^d} \left| \sum_{n=1}^N k_n c_{[0,\vec{y})}\big(\{q_n\vec{\alpha}\}\big) - \lambda_d\big([0,\vec{y})\big) \right|,$$

and correspondingly

$$D_N^{*(p)}(\omega) := \left(\int_{J_0^d} \left| \sum_{n=1}^N k_n c_{[0,\vec{y})}\big(\{q_n\vec{\alpha}\}\big) - \lambda_d\big([0,\vec{y})\big) \right|^p \mathrm{d}\vec{y} \right)^{\frac{1}{p}}.$$

By Roth's theorem (see e.g. [1]), for any dimension d there exists an absolute constant $c_d > 0$ such that $\big(D_N^{*(2)}\big)^2 \ge c_d \frac{(\log N)^{d-1}}{N^2}$ for any N points $\vec{x}_1,\dots,\vec{x}_N \in [0,1]^d$.

Proposition 3.1 allows us to investigate the average value of $D_N^{*(2)}(\omega)$ (with respect to the L^2-norm):

THEOREM 3.2. *We have*

$$\int_{U^d} \Big(D_N^{*(2)}\big(\{q_n\vec{\alpha}\}\big)\Big)^2 \mathrm{d}\vec{\alpha}$$
$$= \sum_{m,n=1}^N k_m k_n \left(\frac{1}{3} + \frac{1}{12}\frac{(q_m,q_n)^2}{q_m\cdot q_n} \right)^d + \sum_{\substack{m,n=1\\ q_n=q_m}}^N k_m k_n \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d \right) - \frac{1}{3^d}. \tag{3.6}$$

Proof. We proceed similarly to the proof of Theorem 3.1. As we now have $x_i = 0$ $(i = 1,\dots,d)$, we see from (3.2) that

$$\int_{U^d} \Big(D_N^{*(2)}\big(\{q_n\vec{\alpha}\}\big)\Big)^2 \mathrm{d}\vec{\alpha}$$
$$= \int_{y_1=0}^1 \cdots \int_{y_d=0}^1 \int_{U^d} R_N^2(0,\vec{y},\vec{\alpha})\, \mathrm{d}\vec{\alpha}\, \mathrm{d}y_1\cdots \mathrm{d}y_d$$
$$= \int_{y_1=0}^1 \cdots \int_{y_d=0}^1 \sum_{m,n=1}^N k_n k_m \prod_{i=1}^d \left[y_i^2 + \frac{(q_m,q_n)^2}{q_m q_n} \left(\min\left(\left\{\frac{q_m y_i}{(q_m,q_n)}\right\}, \left\{\frac{q_n y_i}{(q_m,q_n)}\right\} \right) \right.\right.$$
$$\left.\left. - \left\{\frac{q_m y_i}{(q_m,q_n)}\right\}\left\{\frac{q_n y_i}{(q_m,q_n)}\right\} \right) \right] \mathrm{d}y_1\cdots \mathrm{d}y_d - \int_{y_1=0}^1 \cdots \int_{y_d=0}^1 \prod_{i=1}^d y_i^2\, \mathrm{d}y_1\cdots \mathrm{d}y_d$$

$$= \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d} \int_{y_i=0}^{1} \left[y_i^2 + \frac{(q_m, q_n)^2}{q_m q_n} \left(\min\left(\left\{ \frac{q_m y_i}{(q_m, q_n)} \right\}, \left\{ \frac{q_n y_i}{(q_m, q_n)} \right\} \right) \right.\right.$$
$$\left.\left. - \left\{ \frac{q_m y_i}{(q_m, q_n)} \right\} \left\{ \frac{q_n y_i}{(q_m, q_n)} \right\} \right) \right] \mathrm{d}y_i - \frac{1}{3^d}.$$

Since

$$\int_0^1 \Big(\min(\{ay_i\}, \{by_i\}) - \{ay_i\}\{by_i\} \Big) \, \mathrm{d}y_i = \begin{cases} \frac{1}{12} & \text{for } a \neq b, \\ \frac{1}{6} & \text{for } a = b \end{cases}$$

for arbitrary $a, b \in \mathbb{N}$ (see [10]), we conclude that

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n \vec{\alpha}\}) \right)^2 \mathrm{d}\vec{\alpha}$$
$$= \sum_{\substack{m,n=1 \\ q_m \neq q_n}}^{N} k_n k_m \left(\frac{1}{3} + \frac{1}{12} \frac{(q_m, q_n)^2}{q_m q_n} \right)^d + \sum_{\substack{m,n=1 \\ q_m = q_n}}^{N} k_n k_m \frac{1}{2^d} - \frac{1}{3^d}$$
$$= \sum_{m,n=1}^{N} k_m k_n \left(\frac{1}{3} + \frac{1}{12} \frac{(q_m, q_n)^2}{q_m \cdot q_n} \right)^d + \sum_{\substack{m,n=1 \\ q_n = q_m}}^{N} k_m k_n \left(\frac{1}{2^d} - \left(\frac{5}{12} \right)^d \right) - \frac{1}{3^d}.$$

□

EXAMPLE 3. For $d = 1$ equation (3.6) gives

$$\int_{y=0}^{1} \int_{\alpha=0}^{1} R_N^2(0, y, \alpha) \, \mathrm{d}y \, \mathrm{d}\alpha = \frac{1}{12} \sum_{m,n=1}^{N} k_m k_n \frac{(q_m, q_n)^2}{q_m q_n} + \frac{1}{12} \sum_{\substack{m,n=1 \\ q_m = q_n}}^{N} k_m k_n,$$

which for $k_n = \frac{1}{N}$ $(n = 1, \dots, N)$ was already derived in [10].

EXAMPLE 4. If $(q_n)_{n \geq 1}$ is a sequence of distinct positive integers, then it follows from (3.6) that

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n \vec{\alpha}\}) \right)^2 d\vec{\alpha}$$
$$= \sum_{m,n=1}^{N} k_m k_n \left(\frac{1}{3} + \frac{1}{12} \frac{(q_m, q_n)^2}{q_m q_n} \right)^d + \left(\frac{1}{2^d} - \left(\frac{5}{12} \right)^d \right) \sum_{n=1}^{N} k_n^2 - \frac{1}{3^d}. \tag{3.7}$$

Remark. Gál [3] showed that

$$\sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} \ll N(\log\log N)^2$$

for every finite sequence $(q_n)_{n\geq 1}$ of distinct positive integers.[1] This bound is also tight. Thus for equal weights $k_n = \frac{1}{N}$ $(n = 1, \ldots, N)$ and arbitrary, but fixed $d \geq 1$ we can determine the asymptotic behavior of (3.7):

$$\begin{aligned}
&\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 \mathrm{d}\vec{\alpha} \\
&= \frac{1}{3^d} + \frac{d}{12N^2 \cdot 3^{d-1}} \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} + \frac{1}{N^2} \sum_{j=0}^{d-2} \binom{d}{j} \frac{1}{3^j} \sum_{m,n=1}^{N} \left(\frac{(q_m, q_n)^2}{q_m q_n}\right)^{d-j} \\
&\qquad + \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d\right) \frac{1}{N} - \frac{1}{3^d} \\
&\ll \frac{(\log\log N)^2}{N},
\end{aligned}$$

so that the L^2-norm of $D_N^{*(2)}(\{q_n\vec{\alpha}\})$ is of asymptotic order $\frac{\log\log N}{\sqrt{N}}$.

Appendix

Let $g(i,j)$ be a superadditive function, i.e. a function satisfying

$$\begin{aligned}
g(i,j) &\geq 0 && \text{for all} \quad 1 \leq i \leq j \leq n, \\
g(i,j) &\leq g(i,j+1) && \text{for all} \quad 1 \leq i \leq j \leq n, \\
g(i,j) + g(j+1,k) &\leq g(i,k) && \text{for all} \quad 1 \leq i \leq j \leq n.
\end{aligned}$$

The following lemma is a special case of [6; Corollary 3.1]:

Lemma A.1. *Let $X_1, \ldots, X_n$ be arbitrary random variables and put $S(i,j) = X_i + \cdots + X_j$ and $M(i,j) = \max\{|S(i,i)|, |S(i,i+1)|, \ldots, |S(i,j)|\}$ for $1 \leq i \leq j \leq n$. Suppose that there exists a superadditive function $g(i,j)$ such that*

$$\mathbb{E}|S(i,j)|^\gamma \leq g(i,j) \qquad \textit{for all} \quad 1 \leq i \leq j \leq n$$

for a given real $\gamma \geq 1$. Then

$$\mathbb{E}M^\gamma(1,n) \leq g(1,n)\left(\lfloor \log n \rfloor + 1\right)^\gamma.$$

[1] This result was extended for weighted sums in Dyer and Harman [2].

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Received February 14, 2001

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