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DIAGONALIZABLE EMBEDDING OF COMPOSITION GRAPHS

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ABSTRACT. The concept of diagonalizable quadrilateral embedding was first introduced by Železnik for bipartite graphs and was generalized by Nedela and Škoviera for an arbitrary quadrilateral embedding. In this paper, we show that for an arbitrary connected graph G with at least two vertices, the composition $G(2n)$ has a diagonalizable quadrilateral embedding for all $n \geq 1$.

1. Introduction

Let G be a graph quadrilaterally embedded in an orientable surface S. Following [4], the *diagonal graph* $D(G)$ of G is a graph with the vertex set $V(G)$ in which two vertices are adjacent if and only if they are opposite vertices in some quadrilateral of the embedding of G. We note that each quadrilateral region R gives rise to two *dual* edges in $D(G)$, called the *diagonals* of R. The quadrilateral embedding of G is said to be *diagonalizable* if $D(G)$ has a 1-factor M which contains at most one edge of each pair of dual edges. The set M is called a *diagonal set*. The composition $G(n)$ of a graph G is a graph obtained from G by replacing each vertex v with n independent vertices and each edge uv with a regular complete bipartite graph on $2n$ vertices.

The concept of diagonalizable quadrilateral embedding was first introduced by Železnik [9] for bipartite graphs and was generalized by Nedela and Škoviera [4] for an arbitrary quadrilateral embedding. In these two papers the authors discussed the existence of a diagonalizable quadrilateral embedding for a variety of families of graphs, including the tensor product of two graphs (see also [1], [2]). One of the main results of [4; Proposition 5] is that if G has a diagonalizable quadrilateral embedding in an orientable surface, so has the composition $G(2^n)$ for $n \geq 1$. In this paper, we use the surgical construction

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of White [6] to show that for any connected graph G with at least two vertices, the composition $G(2n)$ has a diagonalizable quadrilateral embedding in an orientable surface for all $n \geq 1$.

All graphs considered are connected, and all surfaces are orientable. For terms not defined here, the reader is referred to [3], [8].

2. The main result

Let G be a connected graph with at least two vertices.

THEOREM 1. *The composition graph $G(2n)$ has a diagonalizable quadrilateral embedding for all $n \geq 1$.*

Proof. We first note that a strong result of White [6] implies that $G(2n)$ has a quadrilateral embedding. We follow his surgical construction to show that this embedding is diagonalizable.

Let $e_1, e_2, \dots, e_q$ be the edges of G. Take q copies of the complete bipartite graph $K_{2n,2n}$ embedded in q disjoint closed orientable surfaces $S_1, S_2, \dots, S_q$ as described in Ringel [5]. For each i, $i = 1, 2, \dots, q$, each embedding of $K_{2n,2n}$ in S_i is quadrilateral with $2n^2$ quadrilateral regions. The $2n^2$ quadrilateral regions can be partitioned into $2n$ mutually disjoint sets $A_1^i, A_2^i, \dots, A_{2n}^i$ of n quadrilateral each (see [7]); each set containing all $4n$ vertices of $K_{2n,2n}$. Furthermore, in any region, the diagonally opposite vertices are in the same part of the vertex set partition of $K_{2n,2n}$. Each of these embeddings can be diagonalized as follows. Add a diagonal in each quadrilateral of the set A_1^i connecting vertices in the first partition set of $K_{2n,2n}$, and add a diagonal in each quadrilateral of the set A_2^i connecting vertices in the second partition set of $K_{2n,2n}$. Thus each quadrilateral embedding of $K_{2n,2n}$ is diagonalizable.

Now if the edges e_i and e_j are adjacent in G, we make $2n$ vertex identifications between S_i and S_j, so that the newly obtained embedding S_{ij} is diagonalizable as follows. Take the $2n$ vertices which are adjacent to the newly added diagonals in the regions of A_1^i. Make similar selection in A_1^j. Now attach n tubes between S_i and S_j, one tube for each region joining the corresponding regions in A_1^i and A_1^j. The first such tube may be attached as follows. If the region R_1^i in A_1^i corresponds to the region R_1^j in A_1^j, we delete the interiors of R_1^i and R_1^j and join the boundary circuits with a topological cylinder T. It is clear how to add the remaining tubes.

We now make two vertex identifications per tube as indicated by the sequence of operations in the figure.

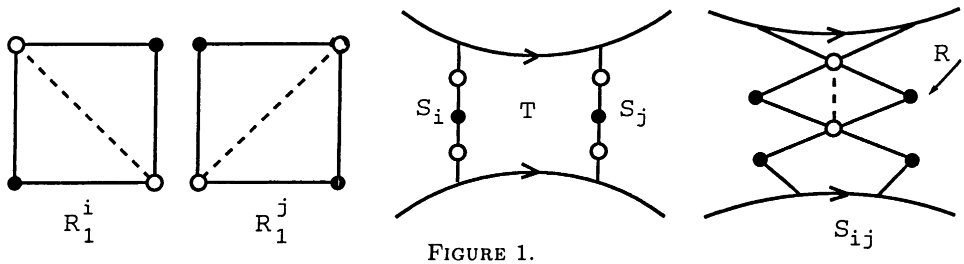

FIGURE 1.

This process destroys the two quadrilaterals R_1^i and R_1^j and creates two new quadrilaterals for each tube. Furthermore, the identification on each tube results in two vertices that are diagonally opposite in the two newly created quadrilaterals. Select one of these two quadrilaterals R, and add a diagonal joining the two identified vertices. Let A_1^{ij} denote the set of selected quadrilaterals R, one from each tube.

If the edge e_k is adjacent to e_i (and to e_j) in G, there are now n quadrilaterals in A_1^{ij} of the new surface S_{ij} with which we make the appropriate $2n$ identifications with the A_1^i of the surface S_k. It is clear that this process may continue until a diagonalizable quadrilateral embedding of the composition $G(2n)$ is reached. □

The above theorem can be applied to many families of graphs. For example, if we take G to be the complete bipartite graph $K_{r,s}$ in the above theorem, and since $K_{2r,2s} = K_{r,s}(2)$ we get [9; Theorem 4].

COROLLARY 2. *The complete bipartite graph $K_{2r,2s}$ has a diagonalizable quadrilateral embedding for all $r \geq 1$ and $s \geq 1$.*

Finally, we note that not all graphs with diagonalizable embedding have the form $G(2n)$ for some positive integer n, as can be clearly seen in the embedding of the 3-cube Q_3. The smallest graph G with diagonalizable quadrilateral embedding which is not of the form $G(2n)$ is of size 6, and consists of the complete graph K_6 minus a path of length 3 (Figure 2-left).

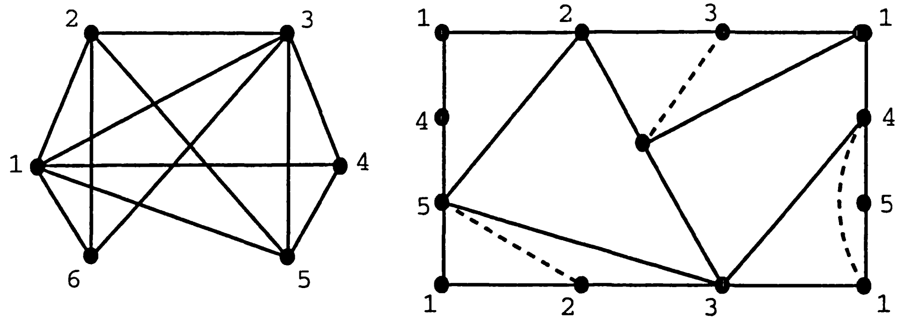

FIGURE 2.

This graph admits a diagonalizable quadrilateral embedding in the torus (Figure 2-right). A 1-factor of the diagonals consists of the 3 diagonals $(1,4)$, $(2,5)$ and $(3,6)$ shown as dotted lines in the figure. Clearly this graph is not of the form $G(2n)$, because it has a vertices of odd degree.

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