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A NOTE ON THE MAXIMAL SEMILATTICE OF AN R^*NC-SEMIGROUP DECOMPOSITION

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Let S be a semigroup with an ideal J. By an ideal we mean a two-sided ideal. The principal ideal generated by an element $a \in S$ we denote by $J(a)$.

An element $x \in S$ is called nilpotent with respect to J if $x^n \in J$ for some positive integer n. An ideal I of S is called a nilideal with respect to J if each element of I is nilpotent with respect to J.

An ideal $P \subseteq S$ is called completely prime if for any a, b of S, $ab \in P$ implies that either $a \in P$ or $b \in P$. A subsemigroup U of S is a filter of S if $xy \in U$ implies $x \in U$ and $y \in U$. We consider the empty set a filter and a completely prime ideal of S. By $N(J)$ we denote the set of all nilpotent elements of S with respect to J. The Luh radical $C(J)$ is the intersection of all completely prime ideals of S which contain J. The Clifford radical $R^*(J)$ is the union of all nilideals of S with respect to J. A commutative semigroup, each element of which is idempotent, is called a semilattice. A Congruence ϱ on S is a semilattice congruence if the factor semigroup S/ϱ is a semilattice. By a maximal semilattice decomposition of a semigroup S we mean a partition of S belonging to a minimal semilattice congruence on S. A semigroup S is semilattice indecomposable if the only semilattice congruence on S is the universal congruence.

A semigroup S is called archimedean [6] if for any a, b of S there exists a positive integer n for which $a^n \in SbS$.

We define a relation η on a semigroup S as follows: $a\eta b$ if and only if $a \in N(J(b))$ and $b \in N(J(a))$.

A semigroup S is called an R^*NC-semigroup if for each ideal J of S, $R^*(J) = N(J) = C(J)$ holds.

It is known [2] that S is an R^*NC-semigroup if and only if for an arbitrary ideal J of S the set $N(J)$ is an ideal of S.

In this note we prove that in an R^*NC-semigroup S the relation η is equal to the minimal semilattice congruence and S is a semilattice of archimedean semigroups.

In an arbitrary semigroup S we denote by $U(x)$ the smallest filter of S containing an element x, by $U_x = \{y \in S \mid U(y) = U(x)\}$ a U-class of S and by Y the set of all distinct U-classes of S with the multiplication $U_xU_y = U_{xy}$.

Let T be the family of all completely prime ideals of S. Define an equivalence relation $\mathcal{T}$ on S as follows: $x\mathcal{T}y$ for $x, y \in S$ if and only if $x, y \in I$, or $x, y \notin I$ for all $I \in T$. The equivalence relation $\mathcal{T}$ is a congruence on S ([7], [10]).

Let M be the set of all filters of S without the empty set. Define an equivalence relation $\mathcal{M}$ as follows: $x\mathcal{M}y$ for $x, y \in S$ if and only if $U(x) = U(y)$.

The following is known.

Lemma 1 (M. Petrich [7, Theorem 3, 2]). *Y is the maximal semilattice decomposition of S.*

Lemma 2 (R. Šulka [10, Theorem 1]). *The fulfilment of the following conditions for elements x, y of a semigroup S is equivalent:*

a) $x\mathcal{T}y$,

b) $x\mathcal{M}y$,

c) $U(x) = U(y)$,

d) $C(x) = C(y)$,

e) $C(J(x)) = C(J(y))$.

Lemma 3. *In an R^*NC-semigroup S for elements a, b we have $a\eta b$ if and only if $N(J(a)) = N(J(b))$.*

Proof. Suppose $a\eta b$, i.e. $a \in N(J(B))$ and $b \in N(J(a))$. Then $a \in N(J(b))$ implies $J(a) \subseteq N(J(b))$ and from this by R. Šulka [9, Lemma 2] we obtain $N(J(a)) \subseteq N(N(J(b))) = N(J(b))$. Similarly, from $b \in N(J(a))$ we obtain $N(J(b)) \subseteq N(J(a))$. From both inclusions $N(J(a)) \subseteq N(J(b))$ and $N(J(b)) \subseteq N(J(a))$ we have $N(J(a)) = N(J(b))$.

Conversely, if $N(J(a)) = N(J(b))$, then evidently $a \in N(J(b))$ and $b \in N(J(a))$, therefore $a\eta b$ holds.

Corollary 4. *In an R^*NC-semigroup S for elements x, y we have $x\eta y$ if and only if $x\mathcal{T}y$.*

Proof. If $x\eta y$, then $N(J(x)) = N(J(y))$. However, S is an R^*NC-semigroup and so $N(J(x)) = C(J(x)) = N(J(y)) = C(J(y))$ which by Lemma 2 gives $x\mathcal{T}y$. Conversely, if $x\mathcal{T}y$, then by Lemma 2 and by the definition of an R^*NC-semigroup we obtain $C(J(x)) = N(J(x)) = C(J(y)) = N(J(y))$, which means by Lemma 3 that $x\eta y$.

Remark 1. In general in a semigroup S we have only $\eta \subseteq \mathcal{T}$. For example, let $S_1 = \{0, e_{11}, e_{12}, e_{21}, e_{22}\}$ be a semigroup with the multiplication $e_{ij} \cdot e_{jk} = e_{ik}$, $e_{ij} \cdot e_{mk} = 0 \cdot e_{mk} = e_{ij} \cdot 0 = 0$, $j \neq m$, $i, j, k, m \in \{1, 2\}$. Then we have $0\eta e_{12}$, $e_{12}\eta e_{11}$, however $0\eta e_{11}$ does not hold. Therefore η is not an equivalence relation on S_1 and $\eta \subset \mathcal{T} = S_1 \times S_1$, $\eta \neq \mathcal{T}$.

Theorem 5. *Let S be an R^*NC-semigroup. Then to the congruence η there belongs the maximal semilattice decomposition of S. Moreover, each η-class is an archimedean semigroup.*

Proof. The first statement follows from Corollary 4 and Lemmas 1 and 2. Let now A be an η-class and any $a, b \in A$. Then $a \in N(J(b))$ implies that $a^n = xby$ for some positive integer n and $x, y \in S^1$. Then $a^{n+2} = (ax)b(ya)$. Evidently $a^{n+2} \in J(ax)$, $a^{n+2} \in J(ya)$, thus $a \in N(J(ax))$ and $a \in N(J(ya))$. The set $N(J(a))$ is an ideal of S and so $ax \in N(J(a))$ and $ya \in N(J(a))$. Therefore $ax, ya \in A$. From the preceding we obtain $a^{n+2} = (ax)b(ya) \in AbA$, which means that A is an archimedean semigroup.

Remark 2. A non-commutative archimedean semigroup can contain more than one idempotent. This is shown by the next example.

Let $S_2 = \{a, b\}$ be a semigroup of left-hand zeros, i.e. the semigroup with the multiplication $ab = a^2 = a$, $ba = b^2 = b$. Evidently, S_2 is an archimedean semigroup with two idempotents.

A semigroup S is called a C_2-semigroup if for all x, y, z of S, $xyzyx = yxzxy$ holds. A C_2-semigroup is an R^*NC-semigroup [3].

Theorem 6. *Let S be a C_2-semigroup. Then S is a semilattice of archimedean semigroups each of which contains at most one idempotent.*

Proof. Suppose, that idempotents e, f belongs to some η-class A. Then $N(J(e)) = N(J(f))$, i.e. $e = xfy$ and $f = set$ for some $x, y, s, t \in S^1$. Since S is a C_2-semigroup we have $e = e^3 = xfyxfyxfy = fyx^3(fy)^2 = fu$ and $f = f^3 = setsetset = ets^3(et)^2 = ets^3ete^3t = ets^3(et)^2e = ve$, where $u, v \in S$. Using the preceding we obtain $e = fu = f^2u = fe = ve^2 = ve = f$.

We note that the next theorem is valid in commutative ([1], [8]) and in quasicommutative semigroups ([4], [5]).

Theorem 7. *Let S be an R^*NC-semigroup and suppose that in S the idempotents commute with all elements. Then S is a semilattice of archimedean semigroups each of which has at most one idempotent.*

Proof. Suppose, that idempotents e, f belong to some η-class A. Then $N(J(e)) = N(J(f))$, i.e. $e^m = xfy$ and $f^n = set$ for some positive integers m, n and $x, y, s, t \in S^1$. From this it follows that $e = xfy = xyf^2 = ef = eset = se^2t = f$.

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ЗАМЕТКА К МАКСИМАЛЬНОМУ ПОЛУСТРУКТУРНОМУ РАЗВИЕНИЮ R^*NC-ПОЛУГРУППЫ

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Резюме

Полугруппа S, в которой радикалы Клиффорда и Луга относительно произвольного идеала равны, названа R^*NC-полугруппой. В статье доказано, что R^*NC-полугруппа является полуструктурой архимедовых полугрупп.