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INDIVIDUAL ERGODIC THEOREM ON A LOGIC

SYLVIA PULMANNOVÁ

A generalization of the individual ergodic theorem on a logic, formulated and proved by Dvurečenskij and Riečan [1], is given. It is shown that x-measurability is not a necessary condition for the validity of the individual ergodic theorem.

Let $\mathscr{L}$ be a logic, that is, let $\mathscr{L}$ be a σ-lattice with the first and last elements 0 and 1, respectively, with an orthocomplementation $\perp: a \mapsto a^{\perp}$, a, $a^{\perp} \in \mathscr{L}$, which satisfies (*i*) $(a^{\perp})^{\perp} = a$ for all $a \in \mathscr{L}$, (*ii*) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$, (*iii*) $a \vee a^{\perp} = 1$ for all $a \in \mathscr{L}$; and with the orthomodular law: if $a \leq b$, then $b = a \vee (b \wedge a^{\perp})$.

Two elements a, $b \in \mathscr{L}$ are *orthogonal* ($a \perp b$) if $a \leq b^{\perp}$; and they are *compatible* ($a \leftrightarrow b$) if there are mutually orthogonal elements a_1, b_1, $c \in \mathscr{L}$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logics with the last elements 1_1 and 1_2, respectively. A map τ: $\mathscr{L}_1 \to \mathscr{L}_2$ is a *σ-homomorphism* if (*i*) $\tau(1_1) = 1_2$, (*ii*) if $a \perp b$, then $\tau(a) \perp \tau(b)$, a, $b \in \mathscr{L}_1$, (*iii*) $\bigvee_{i=1}^{\infty} \tau(a_i) = \tau\left(\bigvee_{i=1}^{\infty} a_i\right)$ for any sequence $\{a_i\} \subset \mathscr{L}_1$.

An *observable* on $\mathscr{L}$ is a σ-homomorphism from the Borel σ-algebra $\mathscr{B}(R_1)$ into $\mathscr{L}$. If f: $R_1 \to R_1$ is a Borel measurable function, then $f \circ x$: $E \to x(f^{-1}(E))$, $E \in \mathscr{B}(R_1)$ is an observable. Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for any E, $F \in \mathscr{B}(R_1)$.

A subset $S \subset L$ is a *sublogic* of $\mathscr{L}$ if (*i*) $a \in S$ implies $a^{\perp} \in S$, (*ii*) $\{a_i\} \subset S$ implies $\bigvee_{i=1}^{\infty} a_i \in S$. A sublogic of $\mathscr{L}$ which is distributive is a *Boolean sub-σ-algebra* of $\mathscr{L}$. The range $R(x) = \{x(E): E \in \mathscr{B}(R_1)\}$ of an observable x is a sub-σ-algebra of $\mathscr{L}$.

The *state* on $\mathscr{L}$ is a map m: $\mathscr{L} \to [0, 1]$ such that (*i*) $m(1) = 1$, (*ii*) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1} m(a_i)$ if $a_i \perp a_j$, $i \neq j$. If x is an observable, then the expectation $m(x)$ of x in a state m is defined by the equality

$$m(x) = \int t m_x\,(\mathrm{d}t)$$

if the integral exists, where $m_x(E) = m(x(E))$, $E \in \mathscr{B}(R_1)$.

Let m be a state and $\tau\colon \mathcal{L}\to\mathcal{L}$ be a σ-homomorphism. We say that τ is *m-preserving* if $m(\tau(a))=m(a)$ for any $a\in\mathcal{L}$. An m-preserving σ-homomorphism $\tau\colon \mathcal{L}\to\mathcal{L}$ is *ergodic* in m if $\tau(a)=a$ implies $m(a)\in\{0, 1\}$.

Let x be an observable. A σ-homomorphism $\tau\colon \mathcal{L}\to\mathcal{L}$ is *x-measurable* if $\tau(R(x))\subset R(x)$ (see [1]). If we set $\tau\circ x(E)=\tau(x(E))$, $E\in\mathcal{B}(R_1)$, then the map $\tau\circ x\colon \mathcal{B}(R_1)\to\mathcal{L}$ is an observable.

Let m be a state. We say that a sequence of observables $\{x_n\}$ converges to the null observable $o(o\{0\}=1)$ almost everywhere in m (a.e. $[m]$, see [2]) if

$$m\,(\limsup_n x_n(\langle -\varepsilon, \varepsilon\rangle)^c)=0 \quad \text{for any} \quad \varepsilon>0.$$

The following theorem was proved in [1].

Theorem 1. *Let x be an observable, $\tau\colon \mathcal{L}\to\mathcal{L}$ an x-measurable σ-homomorphism of the logic $\mathcal{L}$, ergodic in a state m. Let $m(x)=0$. Then*

$$\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x\to o \quad \text{a.e.} \quad [m].$$

Theorem 1 was generalized for the case in which $m(x)\neq 0$ and τ is m-preserving but not necessarily ergodic [5]. The following theorem generalizes the result of [5] by relaxing the condition of x-measurability. We require only that the range of x be contained in an invariant countably generated sub-σ-algebra of $\mathcal{L}$. This we believe may become useful as soon as we intend to apply the theorem in the realm of quantum theories.

Theorem 2. *Let B be a countably generated sub-σ-algebra of $\mathcal{L}$. Let m be a state on $\mathcal{L}$ and let τ be an m-preserving σ-homomorphism of $\mathcal{L}$ such that $\tau(B)\subset B$. Let x be an observable such that $R(x)\subset B$ and $m(x)<\infty$. Then there is an observable x^* such that $R(x^*)\subset B$, $\tau\circ x^*=x^*$* a.e. $[m]$, *$m(x)=m(x^*)$ and*

$$\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x-x^*\to o \quad \text{a.e.} \quad [m].$$

Proof. By [6] there is an observable y such that $R(y)=B$. As $R(x)\subset R(y)$, there exists a Borel measurable function $f\colon R_1\to R_1$ such that $x=f\circ y$ [6]. Now by the proof of Theorem 1 there is a Borel measurable transformation $T\colon R_1\to R_1$ such that $\tau\circ y=T\circ y$, i.e. $\tau\circ y(E)=y(T^{-1}(E))$, $E\in B(R_1)$. Then we have

$$\begin{aligned}\tau\circ x(E)&=\tau(f\circ y(E))=\tau(y(f^{-1}(E)))=T\circ y(f^{-1}(E))=\\&=y(T^{-1}(f^{-1}(E)))=y((f\circ T)^{-1}(E)).\end{aligned}$$

Let us set

$$s_n = \frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i$$

In view of the definition of the sum of compatible observables [6], the observables $y_n = s_n \circ y$ are the sums

$$\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x.$$

Since T is the measure m_y — preserving transformation from R_1 into itself, from the validity of the individual ergodic theorem (see [3]) on the dynamical system $(R_1, \mathscr{B}(R_1), m_y, T)$ applicated to the function $f(t)$, $t \in R_1$, we get that there is a Borel measurable function f^* which is T — invariant,

$$\int f^*(t) m_y\,(\mathrm{d}t) = \int f(t) m_y\,(\mathrm{d}t) = m(x),$$

and

$$s_n(t) \to f^*(t) \quad \text{a.e.} \quad [m_y].$$

Since it may be shown that $s_n \circ y - f^* \circ y \to o$ a.e. $[m]$ if and only if $s_n(t) \to f^*(t)$ a.e. $[m_y]$ (see [2]), we finish the proof by setting $x^* = f^* \circ y$.

Q.E.D.

Lemma 3. *Let $M \subset \mathscr{L}$ be such that $\tau(M) \subset M$, where τ is a σ-homomorphism of $\mathscr{L}$. Let $\mathscr{L}_0$ be the minimal sublogic of $\mathscr{L}$ containing M. Then $\tau(\mathscr{L}_0) \subset \mathscr{L}_0$.*

Proof. Let $S = \{b \in \mathscr{L}_0 : \tau(b) \in \mathscr{L}_0\}$. It can be easily checked that S is a sublogic of $\mathscr{L}$, and $M \subset S$. From this we get $S = \mathscr{L}_0$.

Q.E.D

Theorem 4. *Let m be a state on $\mathscr{L}$, τ be an m-preserving σ-homomorphism of $\mathscr{L}$, and let x be an observable such that $m(x) < \infty$ and $\tau^i \circ x$ are pairwise compatible. Then there is an observable x^* such that $\tau \circ x^* = x^*$ a.e. $[m]$, $m(x^*) = m(x)$ and*

$$\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x - x^* \to o \quad \text{a.e.} \quad [m].$$

Proof. Let us set $M = \bigcup_{i=0}^{\infty} R(\tau^i \circ x)$. As $\tau(M) \subset M$, we obtain by Lemma 3 that $\tau(\mathscr{L}_0) \subset \mathscr{L}_0$, where $\mathscr{L}_0$ is the sublogic of $\mathscr{L}$ generated by M. For any a, $b \in M$ we have $a \leftrightarrow b$ in $\mathscr{L}$. Since $\mathscr{L}_0$ is a lattice, $a \leftrightarrow b$ also in $\mathscr{L}_0$. By the proof as in [4], $\mathscr{L}_0$ is a Boolean sub-σ-algebra of $\mathscr{L}$. As each $R(\tau^i \circ x)$ is countably generated, $\mathscr{L}_0$ is also countably generated. The statement of the theorem follows from Theorem 2 if we set $B = \mathscr{L}_0$. Q.E.D.

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ИНДИВИДУАЛЬНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА НА ЛОГИКЕ

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Резюме

В статье исследуется индивидуальная эргодическая теорема на логике. Приводится обобщение результата Двуреченского и Риечана, показывающее, что x-измеримость гомоморфизма логики не является необходимым условием для этой теоремы.