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ON LEBESGUE PSEUDONORMS ON $C_0(T)$

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Let T be a locally compact Hausdorff topological space and let $\sigma(\mathscr{B}_0)$ denote the σ-ring of all Baire measurable subsets of T. Denote by $C_0(T)$ the Banach space of all continuous functions on T tending to zero at infinity with the usual supremum norm $\|\cdot\|_T$. Let further Y be a Banach space and Y^* its dual. (All considered Banach spaces are either real or complex.)

Definition. *We say that a mapping p: $C_0(T)\to[0, +\infty)$ is a Lebesgue pseudonorm on $C_0(T)$ if it has the following properties:*

1) $p(f)=p(|f|)$,
2) $|f|\leqq|g|\Rightarrow p(f)\leqq p(g)$,
3) $p(af)=|a|\cdot p(f)$ *for each scalar* a,
4) $p(f+g)\leqq p(f)+p(g)$, *and*
5) *if* $g, f_n\in C_0(T)$, $n=1, 2, \ldots$, *and* $\sum_{n=1}^{\infty}|f_n|\leqq|g|$ *then* $p(f_n)\to 0$.

There is a remarkable result, see [7, 24H], which is valid in the more general context of arbitrary Riesz spaces with a linear space topology such that every order-bounded set is bounded, that condition 5) may be replaced by the following "disjointness" condition:

5d) *if* $g, f_n\in C_0(T)$, $n=1, 2, \ldots$, $f_n\cdot f_m=0$ *for* $n\neq m$, *and* $|f_n|\leqq|g|$ *for each* n, *then* $p(f_n)\to 0$.

(For more information about Lebesgue topologies on a general Riesz space see [7, section 24] and also [8].)

According to the Lebesgue Dominated Convergence Theorem each countably additive Baire measure μ: $\sigma(\mathscr{B}_0)\to$ the scalars of $C_0(T)$, induces by the equality

$$\hat{\mu}(f)=\int_T|f|\,\mathrm{d}v(\mu, \cdot)=\sup\left\{\left|\int_T g\,\mathrm{d}\mu\right|;\ g\in C_0(T),\ |g|\leqq|f|\right\},\quad f\in C_0(T),$$

a Lebesgue pseudonorm on $C_0(T)$. Hence by the Riesz Representation Theorem each bounded linear functional $F\in C_0(T)^*=ca(\sigma(\mathscr{B}_0))$ — the Banach space of all countably additive scalar valued Baire measures on $\sigma(\mathscr{B}_0)$ with the total variation norm, induces by the equality

$$\hat{F}(f)=\sup\{|F(g)|;\ g\in C_0(T),\ |g|\leqq|f|\}=\int_T |f|\,\mathrm{d}v(\mu_F,\cdot),\quad f\in C_0(T),$$

where μ_F is the representing Baire measure of F, a Lebesgue pseudonorm on $C_0(T)$.

Let U: $C_0(T)\to Y$ be a bounded linear operator and for $y^*\in Y^*$ let μ_{y^*} denote the representing Baire measure of the linear functional y^*U.

For $f\in C_0(T)$ put $\hat{U}f=\sup\{|Ug|;\ g\in C_0(T),\ |g|\leqq|f|\}$, and for $E\in\sigma(\mathscr{B}_0)$ put $\hat{\mu}(E)=\sup_{|y^*|\leqq 1} v(\mu_{y^*}, E)$.

Then clearly $\hat{U}=|U|<+\infty$, and $\hat{U}$ has the properties 1), 2) and 3) of the Definition above. Further, by the Hahn—Banach Theorem and the Riesz Representation Theorem we have the equalities

$$\hat{U}f=\sup\{\sup_{|y^*|\leqq 1}\{|y^*Ug|;\ g\in C_0(T),\ |g|\leqq|f|\}\}=$$
$$=\sup_{|y^*|\leqq 1}\{\sup\{|y^*Ug|;\ g\in C_0(T),\ |g|\leqq|f|\}\}=$$
$$=\sup_{|y^*|\leqq 1}\int_T |f|\,\mathrm{d}v(\mu_{y^*},\cdot)$$

for each $f\in C_0(T)$, hence $\hat{U}$ is also subadditive.

Obviously $\hat{\mu}(\emptyset)=0$, $\hat{\mu}$ is monotone and countably subadditive. Since each measure $v(\mu_{y^*},\cdot)$, $y^*\in Y^*$ has the Fatou property, i.e., $E_n\in\sigma(\mathscr{B}_0)$, $n=1,2,\dots$ and $E_n\nearrow E\Rightarrow v(\mu_{y^*}, E_n)\nearrow v(\mu_{y^*}, E)$, $\hat{\mu}$ also has the Fatou property.

Let U^*: $Y^*\to C_0(T)^*=ca(\sigma(\mathscr{B}_0))$ be the conjugate of U. Then

$$\hat{\mu}(T)=\sup_{|y^*|\leqq 1} v(\mu_{y^*}, T)=\sup_{|y^*|\leqq 1}|y^*U|=\sup_{|y^*|\leqq 1}|U^*y^*|=|U^*|=|U|<+\infty.$$

According to Theorems VI. 4.8, IV. 9.1 and IV. 9.2 in [6] (for a short proof of IV. 9.2 see [9]) U is weakly compact $\Leftrightarrow U^*$ is weakly compact $\Leftrightarrow\hat{\mu}$: $\sigma(\mathscr{B}_0)\to[0,|U|]$ is continuous, i.e., $E_n\in\sigma(\mathscr{B}_0)$, $n=1,2,\dots$ and $E_n\searrow\emptyset\ \Rightarrow\ \hat{\mu}(E_n)\to 0\Leftrightarrow$ there is a countably additive measure λ: $\sigma(\mathscr{B}_0)\to[0,1]$ such that $\hat{\mu}$ is absolutely λ-continuous $\Leftrightarrow\hat{\mu}$ is exhaustive, i.e., if $E_n\in\sigma(\mathscr{B}_0)$, $n=1,2,\dots$ are pairwise disjoint, then $\hat{\mu}(E_n)\to 0$.

Let U be weakly compact. Then from the exhaustivity of $\hat{\mu}$ on $\sigma(\mathscr{B}_0)$ it is easy to see that $\hat{U}$ has the property 5d) stated above, hence $\hat{U}$ is a Lebesgue pseudonorm on $C_0(T)$. The converse is also true, see Theorem 3.3 in [11], where a lot of other characterizations of weak compactness of U is proved.

Let $\mathscr{A}\subset 2^T$. We say that a set function v: $\mathscr{A}\to Y$ has the property (p), or better that v is uniformly exhaustive, if for each $\varepsilon>0$ there is a positive integer N_ε such that for any collection of pairwise disjoint sets $A_1, \ldots, A_{N_\varepsilon} \in \mathscr{A}$ there is at least one $n \in \{1, \ldots, N_\varepsilon\}$ for which $|v(A_n)| \leqq \varepsilon$, see [5, Def. 4]. We say that $f, g \in C_0(T)$ are orthogonal if $f \cdot g = 0$.

In [3] we announced the following characterization of weak compactness of U:

Theorem 1. *For a bounded linear operator U: $C_0(T) \to Y$ the following conditions are equivalent:*

1) *U is weakly compact,*

2) *$\hat{\mu}$: $\sigma(\mathscr{B}_0) \to [0, |U|]$ is uniformly exhaustive, and*

3) *U has the following property (p): for every $\varepsilon > 0$ there is a positive integer N_ε such that for any collection $f_1, \ldots, f_{N_\varepsilon} \in C_0(T)$ with $\|f_n\|_T \leqq 1$ and $f_n \cdot f_m = 0$ for $n \neq m$, $n, m = 1, \ldots, N_\varepsilon$ there is at least one $n \in \{1, \ldots, N_\varepsilon\}$ for which $|Uf_n| \leqq \varepsilon$.*

We now prove this result, and in Theorem 2 below we give an extension of it. (Theorem 2 from [3] will be proved elsewhere.)

Proof. 1)$\Rightarrow$2). Let $\varepsilon > 0$ and let λ: $\sigma(\mathscr{B}_0) \to [0, 1]$ be a countably additive measure such that $\hat{\mu}$ is absolutely λ-continuous. Then there is a $\delta > 0$ such that $E \in \sigma(\mathscr{B}_0)$ and $\lambda(E) < \delta \Rightarrow \hat{\mu}(E) \leqq \varepsilon$. Take a positive integer $N_\varepsilon \geqq \left[\frac{1}{\delta}\right] + 1$. Then for any collection of pairwise disjoint sets $E_1, \ldots, E_{N_\varepsilon} \in \sigma(\mathscr{B}_0)$ there must be at least one $n \in \{1, \ldots, N_\varepsilon\}$ for which $\lambda(E_n) < \delta$, since otherwise we have the contradiction $1 \geqq \lambda(T) \geqq \sum_{i=1}^{N_\varepsilon} \lambda(E_i) > 1$. Thus $\hat{\mu}(E_n) \leqq \varepsilon$ for at least one $n \in \{1, \ldots, N_\varepsilon\}$, hence $\hat{\mu}$ is uniformly exhaustive on $\sigma(\mathscr{B}_0)$.

2)$\Rightarrow$3). Let $\varepsilon > 0$ and take a positive integer N_ε so that for any collection of pairwise disjoint sets $E_1, \ldots, E_{N_\varepsilon} \in \sigma(\mathscr{B}_0)$ there is at least one $n \in \{1, \ldots, N_\varepsilon\}$ for which $\hat{\mu}(E_n) \leqq \varepsilon$. Take arbitrary $f_1, \ldots, f_{N_\varepsilon} \in C_0(T)$ with $\|f_i\|_T \leqq 1$ and $f_i \cdot f_j = 0$ for $i \neq j$, $i, j = 1, \ldots, N_\varepsilon$. Since by the Hahn—Banach Theorem and the Riesz Representation Theorem

$$|Uf_i| = \sup_{|y^*| \leqq 1} |y^* Uf_i| = \sup_{|y^*| \leqq 1} \left| \int_T f_i \, d\mu_{y^*} \right| \leqq \sup_{|y^*| \leqq 1} \int_T |f_i| \, dv(\mu_{y^*}, \cdot) \leqq$$

$$\leqq \sup_{|y^*| \leqq 1} v(\mu_{y^*}, \{t;\, t \in T, f_i(t) \neq 0\}) = \hat{\mu}(\{t;\, t \in T, f_i(t) \neq 0\})$$

for each $i = 1, \ldots, N_\varepsilon$, and since the sets $E_i = \{t;\, t \in T, f_i(t) \neq 0\}$, $i = 1, \ldots, N_\varepsilon$ are pairwise disjoint, there must be at least one $n \in \{1, \ldots, N_\varepsilon\}$ for which $|Uf_n| \leqq \hat{\mu}(E_n) \leqq \varepsilon$.

3)$\Rightarrow$1). Clearly $\hat{U}$ has also the property (p). Denote by $\mathscr{U}_0$ the lattice of all open Baire subsets of T and by $\mathscr{C}_0$ the lattice of all compact G_δ subsets of T. Let $V \in \mathscr{U}_0$ and let $y^* \in Y^*$. Then

$$v(\mu_{y^*}, V) = \sup \left\{ \int_T |f| \, dv(\mu_{y^*}, \cdot);\, f \in C_0(T), |f| \leqq \chi_V \right\}$$

by the regularity of the Baire measure $v(\mu_{y^*}, \cdot)$ and Theorem B in § 50 in [10], hence $\hat{\mu}(V) = \sup \{\hat{U}f;\ f \in C_0(T),\ |f| \leq \chi_V\}$. The last equality implies that $\hat{\mu}$: $\mathscr{U}_0 \to [0, |U|]$ is uniformly exhaustive. Since any finite collection of pairwise disjoint compact G_δ sets can be mutually separated by the same number of pairwise disjoint sets from $\mathscr{U}_0$, see Theorem D in § 50 in [10], $\hat{\mu}$: $\mathscr{C}_0 \to [0, |U|]$ is also uniformly exhaustive. Since $v(\mu_{y^*}, E) = \sup \{v(\mu_{y^*}, C);\ C \in \mathscr{C}_0,\ C \subset E\}$ for each $y^* \in Y^*$ and each $E \in \sigma(\mathscr{B}_0)$ by the regularity of the Baire measure $v(\mu_{y^*}, \cdot)$, $\hat{\mu}(E) = \sup \{\hat{\mu}(C);\ C \in \mathscr{C}_0,\ C \subset E\}$ for each $E \in \sigma(\mathscr{B}_0)$. Thus $\hat{\mu} : \sigma(\mathscr{B}_0) \to [0, |U|]$ is uniformly exhaustive, hence U is weakly compact. The theorem is proved.

Remark 1. Let X be a Banach space and consider the Banach space $C_0(T, X)$ of all X-valued continuous functions on T tending to zero at infinity with the supremum norm. It is well known that $C_0(T, X)^* = cabv(\sigma(\mathscr{B}_0), X^*)$ — the Banach space of all countably additive X^*-valued Baire measures with bounded variations. Since reflexive Banach spaces have the Radon—Nikodým property, a subset $M \subset cabv(\sigma(\mathscr{B}_0), X^*)$ is relatively weakly compact if and only if the subset $\{v(\mu, \cdot);\ \mu \in M\} \subset ca(\sigma(\mathscr{B}_0))$ is relatively weakly compact, see [1], [2] and [4]. Hence for reflexive Banach spaces X Theorem 1 remains valid if $C_0(T)$ is replaced by $C_0(T, X)$. We note that the implications 1)$\Rightarrow$2)$\Leftrightarrow$3) of Theorem 1 hold for $C_0(T, X)$ for any Banach space X, see [1], [2] and Theorem 3 in [4] in this connection. In fact, above we proved that for any bounded linear operator U: $C_0(T, X) \to Y$, X being an arbitrary Banach space, the following conditions are equivalent:

1) $\hat{\mu}$ (=the semivariation of the representing measure of U) is continuous on $\sigma(\mathscr{B}_0)$,
2) $\hat{\mu}$ is uniformly exhaustive on $\sigma(\mathscr{B}_0)$, and
3) U has the property (p) in Theorem 1.

Theorem 2. *Let p: $C_0(T) \to [0, +\infty)$ have the properties 1)—4) of the Definition above, let $p(1) = \sup \{p(f);\ f \in C_0(T),\ |f| \leqq 1\} < +\infty$, and let p have the property (p) from Theorem 1. Then for every $\varepsilon > 0$ there is a positive integer M_ε such that for any collection $f_1, \ldots, f_{M_\varepsilon} \in C_0(T)$ with $\sum_{n=1}^{M_\varepsilon} |f_n| \leqq 1$ there is at least one $n \in \{1, \ldots, M_\varepsilon\}$ for which $p(f_n) \leqq \varepsilon$*

Proof Suppose the contrary. Then there is an $\varepsilon > 0$ such that for each positive integer M there are M functions $f_1, \ldots, f_M \in C_0(T)^+ = \{f;\ f \in C_0(T),\ f \geqq 0\}$ such that $\sum_{n=1}^{M} f_n \leqq 1$ and $p(f_n) > \varepsilon$ for each $n = 1, \ldots, M$

Let k be the smallest positive integer for which $p(1) < \frac{\varepsilon}{2} \cdot \frac{k}{2}$. Since $p(1) > \varepsilon$, $k \geqq 5$.

If now $f \in C_0(T)^+$, $f \leqq 1$ and $p(f) > \varepsilon$, then $\frac{2}{k} < \max_{t \in T} f(t) = \|f\|_T$ (otherwise we have the contradiction

$$\frac{\varepsilon}{2} > \frac{2}{k} \cdot p(1) \geqq p(\|f\|_T) \geqq p(f) > \varepsilon).$$

In this proof let N_δ for $\delta > 0$ denote the smallest positive integer corresponding to δ according to the property (p) of p. Put $M = N_{\varepsilon/8} + \ldots + N_{\varepsilon\, 2^{k+1}}$. Then by assumption there are functions $f_1, \ldots, f_M \in C_0(T)^+$ such that $\sum_{n=1}^{M} f_n \leqq 1$ and $p(f_n) > \varepsilon$ for each $n = 1, \ldots, M$. To each f_n we construct two functions φ_n and ψ_n in the following way: We put

$$E_{n,0} = \left\{ t\colon t \in T,\ f_n(t) \leqq \frac{5}{4k} \right\}, \quad E_{n,1} = \left\{ t\colon t \in T,\ f_n(t) \geqq \frac{6}{4k} \right\},$$

$$F_{n,0} = \left\{ t\colon t \in T,\ f_n(t) \leqq \frac{6}{4k} \right\}, \quad \text{and} \quad F_{n,1} = \left\{ t\colon t \in T,\ f_n(t) \geqq \frac{7}{4k} \right\}.$$

Then $E_{n,0} \cap E_{n,1} = \emptyset$, $E_{n,1} \neq \emptyset$ $\left(\frac{2}{k} < \|f_n\|_T \right)$, $E_{n,0}$ is a closed and $E_{n,1}$ a compact subset of T. We put $\varphi_n = 1$ if $E_{n,0} = \emptyset$. If $E_{n,0} \neq \emptyset$, then according to Theorem B in § 50 in [10] we take a function $\varphi_n \in C_0(T)^+$ such that $\varphi_n \leqq 1$, $\varphi_n(t) = 0$ for $t \in E_{n,0}$, and $\varphi_n(t) = 1$ for $t \in E_{n,1}$. Similarly we put $\psi_n = 1$ if $F_{n,0} = \emptyset$, and if $F_{n,0} \neq \emptyset$, then we take a function $\psi_n \in C_0(T)^+$ such that $\psi_n \leqq 1$, $\psi_n(t) = 0$ for $t \in F_{n,0}$, and $\psi_n(t) = 1$ for $t \in F_{n,1}$.

Clearly

$$t \in T,\ \varphi_n(t) > 0 \Rightarrow f_n(t) > \frac{1}{k}, \quad (1 - \varphi_n)\psi_n = 0,$$

and

$$f_n = \psi_n f_n + (1 - \psi_n) f_n < \psi_n f_n + \frac{2}{k}.$$

The last inequality implies that

$$\varepsilon < p(f_n) \leqq p(\psi_n f_n) + \frac{2}{k} p(1) \leqq p(\psi_n f_n) + \frac{\varepsilon}{2},$$

hence $p(\psi_n f_n) > \frac{\varepsilon}{2}$ for each $n = 1, \ldots, M$.

Put $n_{1,1} = 1$. Let $n_{1,2}$ be the first $n \in \{1, \ldots, M\}$ for which

$$p((1 - \varphi_{n_{1,1}})\psi_n f_n) > \frac{\varepsilon}{4 \cdot 2},$$

if it exists. Let $n_{1,3}$ be the first $n \in \{1, \ldots, M\}$ for which

$$p((1 - \varphi_{n_{1,1}})(1 - \varphi_{n_{1,2}})\psi_n f_n) > \frac{\varepsilon}{4 \cdot 2},$$

if it exists. In general, let $n_{1,r}$ be the first $n \in \{1, \ldots, M\}$ for which

$$p((1-\varphi_{n_{1,1}}) \ldots (1-\varphi_{n_{1,r-1}})\psi_n f_n) > \frac{\varepsilon}{4 \cdot 2},$$

if it exists. Since the functions $\psi_{n_{1,1}}, (1-\varphi_{n_{1,1}})\psi_{n_{1,2}}, \ldots, (1-\varphi_{n_{1,1}}) \ldots (1-\varphi_{n_{1,r-1}})\psi_{n_{1,r}}$ are pairwise orthogonal elements of $C_0(T)^+$ with values in $[0, 1]$, continuing in this manner, owing to the property (p) of p we may arrive only to some $r_1 < N_{\frac{\varepsilon}{4 \cdot 2}}$.

Put

$$J_1 = \{n_{1,1}, \ldots, n_{1,r_1}\}, \quad \text{and}$$
$$\alpha_1 = \varphi_{n_{1,1}} + (1-\varphi_{n_{1,1}})\varphi_{n_{1,2}} + \ldots + (1-\varphi_{n_{1,1}}) \ldots (1-\varphi_{n_{1,r_1-1}})\varphi_{n_{1,r_1}}.$$

Since

$$1-\alpha_1 = (1-\varphi_{b_{1,1}}) \cdot \ldots \cdot (1-\varphi_{n_{1,r_1}}), \quad p((1-\alpha_1)\psi_n f_n) \leq \frac{\varepsilon}{4 \cdot 2}$$

for each $n \in \{1, \ldots, M\} - J_1$. Thus

$$\frac{\varepsilon}{2} < p(\psi_n f_n) \leqq p(\alpha_1 \psi_n f_n) + p((1-\alpha_1)\psi_n f_n) \leqq p(\alpha_1 \psi_n f_n) + \frac{\varepsilon}{4\ 2},$$

hence $p(\alpha_1 \psi_n f_n) > \frac{\varepsilon}{2} - \frac{\varepsilon}{4\ 2}$ for each $n \in \{1, \ldots, M\} - J_1$.

Let $n_{2,1}$ be the smallest number from $\{1, \ldots, M\} - J_1$. Let $n_{2,2}$ be the first $n \in \{1, \ldots, M\} - J_1$ for which

$$p((1-\varphi_{n_{2,1}})\alpha_1 \psi_n f_n) > \frac{\varepsilon}{4 \cdot 2^2},$$

if it exists. Let $n_{2,3}$ be the first $n \in \{1, \ldots, M\} - J_1$ for which

$$p((1-\varphi_{n_{2,1}})(1-\varphi_{n_{2,2}})\, \alpha_1 \psi_n f_n) > \frac{\varepsilon}{4 \cdot 2^2},$$

if it exists etc. Since the functions $\psi_{n_{2,1}}, (1-\varphi_{n_{2,1}})\psi_{n_{2,2}}, \ldots, (1-\varphi_{n_{2,1}}) \cdot \ldots \cdot (1-\varphi_{n_{2,r-1}})\psi_{n_{2,r}}$ are pairwise orthogonal elements of $C_0(T)^+$ with values in $[0, 1]$, continuing in this manner, owing to property (p) of p, we may arrive only to some $r_2 < N_{\frac{\varepsilon}{4 \cdot 4}}$.

Put

$$J_2 = \{n_{2,1}, \ldots, n_{2,r_2}\}, \text{ and}$$

$$\alpha_2 = \varphi_{n_{2,1}} + (1-\varphi_{n_{2,1}})\varphi_{n_{2,2}} + \ldots + (1-\varphi_{n_{2,1}}) \cdot \ldots \cdot (1-\varphi_{n_{2,r_2-1}})\varphi_{n_{2,r_2}}.$$

Then $J_1 \cap J_2 = \emptyset$, $\{1, \ldots, M\} - (J_1 \cup J_2) \neq \emptyset$, and similarly as above

$$p(\alpha_2 \alpha_1 \psi_n f_n) > \frac{\varepsilon}{2} - \frac{\varepsilon}{4\ 2} - \frac{\varepsilon}{4 \cdot 2^2}$$

for each $n \in \{1, \ldots, M\} - (J_1 \cup J_2)$.

Continuing in this way we obtain pairwise disjoint sets $J_1, \ldots, J_{k-1} \subset \{1, \ldots, M\}$ such that $1 \leqq \operatorname{card} J_i < N_{\frac{\varepsilon}{4\cdot 2}}$ for each $i = 1, \ldots, k-1$, hence $\{1, \ldots, M\} - (J_1 \cup \ldots \cup J_{k-1}) \neq \emptyset$, and functions $\alpha_1, \ldots, \alpha_{k-1}$ of the form

$$\alpha_i = \varphi_{n_{i,1}} + (1 - \varphi_{n_{i,1}})\varphi_{n_{i,2}} + \ldots + (1 - \varphi_{n_{i,1}}) \cdot \ldots \cdot (1 - \varphi_{n_{i,r_i-1}})\varphi_{n_{i,r_i}},$$

$i = 1, \ldots, k-1$, such that

$$p(\alpha_{k-1}\alpha_{k-2} \cdot \ldots \cdot \alpha_1 \psi_n f_n) > \frac{\varepsilon}{2} - \frac{\varepsilon}{4\cdot 2} - \ldots - \frac{\varepsilon}{4\cdot 2^{k-1}} > \frac{\varepsilon}{4}$$

for each $n \in \{1, \ldots, M\} - (J_1 \cup \ldots \cup J_{k-1})$.

Take some $n_0 \in \{1, \ldots, M\} - (J_1 \cup \ldots \cup J_{k-1})$. Then by the last inequality there must be a point $t_0 \in T$ such that

$$\alpha_{k-1}(t_0) \cdot \ldots \cdot \alpha_1(t_0)\psi_{n_0}(t_0) > 0.$$

But then $\psi_{n_0}(t_0) > 0$, hence $f_{n_0}(t_0) > \frac{1}{k}$. Further $\alpha_i(t_0) > 0$ for each $i = 1, \ldots, k-1$, hence by the definition of α_i there exists an $n_{i,j_i} \in J_i$ such that $\varphi_{n_{i,j_i}}(t_0) > 0$. But then $f_{n_{i,j_i}}(t_0) > \frac{1}{k}$. Hence

$$\sum_{n=1}^{M} f_n(t_0) \geqq \sum_{i=1}^{k-1} f_{n_{i,j_i}}(t_0) + f_{n_0}(t_0) > 1,$$

which contradicts the assumption $\sum_{n=1}^{M} f_n \leqq 1$. The theorem is proved.

Corollary. *Let p: $C_0(T) \to [0, +\infty)$ have the properties 1)—4) of the Definition above, and let, for each $g \in C_0(T)^+$ and $\varepsilon > 0$, there exist a positive integer $N_{g,\varepsilon}$ such that for any collection $f_1, \ldots, f_{N_{g,\varepsilon}} \in C_0(T)^+$ of pairwise orthogonal functions with $\sum_{n=1}^{N_{g,\varepsilon}} f_n \leqq g$ there is at least one $n \in \{1, \ldots, N_{g,\varepsilon}\}$ for which $p(f_n) \leqq \varepsilon$. Then the same is true without assuming pairwise orthogonality.*

Proof. For $g \in C_0(T)^+$ it is enough to put $p_g(f) = p(g \wedge |f|)$, $f \in C_0(T)$, and apply the theorem.

Remark 2. Let T be an arbitrary set and let $\mathscr{R} \subset 2^T$ be a ring. Then it is easy to prove that Theorem 2 and its Corollary remains valid if $C_0(T)$ is replaced by $S(\mathscr{R})$ — the space of all $\mathscr{R}$ — simple scalar valued functions on T. There are many other spaces for which the assertion of Theorem 2 and its Corollary are valid. Nevertheless the author was unable to solve the following

Problem. *Does Theorem 2 hold if $C_0(T)$ is replaced by an arbitrary Riesz space?*

Remark 3. In a forthcoming paper the assertion of Theorem 2 will be proved for arbitrary Lebesgue pseudonorm on $C_0(T)$ or on $S(\mathscr{R})$. The proof essentially uses the Hahn—Banach theorem, see section 5.3 in [8].

Remark 4. By a slight modification of the proof of Theorem 2 we can achieve that the assertion of Theorem 2 remains to hold if the property 3) of p: $p(af)=|a|\cdot p(f)$ for each scalar a, is weakened to 3w): $\lim_{n\to\infty} p\left(\frac{f}{n}\right)=0$, and the subadditivity of p is weakened to 4w): for each $\varepsilon>0$ there is a $\delta>0$ such that $p(f+g) \leqq p(f)+\varepsilon$ whenever $p(g)<\delta$. The same is true if $C_0(T)$ is replaced by $S(\mathscr{R})$. We note that for so weakened p the validity of the result in Remark 3 is an open question.

Remark 5. The given method of proof of Theorem 2 may be applied to prove that condition 5) in the Definition above may be replaced by condition 5d), compare with 24H in [7] Namely, suppose 5d) $\not\Rightarrow$ 5). Then there are $g\in C_0(T)^+$, $\varepsilon>0$ and a sequence $f_n\in C_0(T)^+$, $n=1, 2, \ldots$ such that $\sum_{n=1}^{\infty} f_n \leqq 1$ and $p_g(f_n)= p(g\wedge f_n)>\varepsilon$ for each $n=1, 2, \ldots$. Take a positive integer k so that $\frac{2}{k}\cdot p_g(1)<\frac{\varepsilon}{2}$ and construct the functions φ_n and ψ_n, $n=1, 2, \ldots$ as in the proof of Theorem 2. Since the functions $\psi_1, (1-\varphi_1)\psi_2, \ldots, (1-\varphi_1\cdot\ldots\cdot(1-\varphi_{n-1})\psi_n, \ldots$ are pairwise orthogonal elements of $C_0(T)^+$ with values in [0, 1], by 5d) there is a positive integer $r_1\geqq 1$ such that

$$p_g((1-\varphi_1)\cdot\ldots\cdot(1-\varphi_{r_1})\psi_n f_n)\leqq\frac{\varepsilon}{4\cdot 2}$$

for each $n>r_1$. Put $\alpha_1=\varphi_1+(1-\varphi_1)\varphi_2+\ldots+(1-\varphi_1)\cdot\ldots\cdot(1-\varphi_{r_1-1})\varphi_{r_1}$. Then

$$p_g(\alpha_1\psi_n f_n)>\frac{\varepsilon}{2}-\frac{\varepsilon}{4\cdot 2}$$

for each $n>r_1$. Since the functions $\psi_{r_1+1}, (1-\varphi_{r_1+1})\psi_{r_1+2}, \ldots, (1-\varphi_{r_1+1})\cdot\ldots\cdot(1-\varphi_{r_1+n-1})\psi_{r_1+n}, \ldots$ are pairwise orthogonal ... etc. Continuing in this way we obtain a contradiction with the assumption $\sum_{n=1}^{\infty} f_n\leqq 1$.

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ОБ ПОЛУНОРМАХ ЛЕБЕГА НА $C_0(T)$

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Резюме

Пусть T есть локально компактное хаусдорфово пространство. Обозначим $C_0(T)$ банахово пространство всех непрерывных скалярных функций на T стремящихся к нулю в бесконечности с равномерной нормой. Далее, пусть Y есть банахово пространство. В работе доказана и расширена следующая характеризация слабо компактных линейных операторов U: $C_0(T) \to Y$, анонсированная в [3, Теорема 1]:

Теорема 1. *Ограниченный линейный оператор* U: $C_0(T) \to Y$ *является слабо компактным тогда и только тогда, когда он имеет следующее свойство:*

(p) *для каждого* $\varepsilon > 0$ *существует натуральное число* N_ε *такое, что для любого набора* $f_1, f_2, \ldots, f_{N_\varepsilon} \in C_0(T)$ *с* $\|f_n\|_T \leqq 1$ *и* $f_n \cdot f_m = 0$ *для* $n \neq m$, n, $m = 1, 2, \ldots, N_\varepsilon$ *существует хотя бы одно* $n \in \{1, 2, \ldots, N_\varepsilon\}$, *для которого* $|Uf_n| \leqq \varepsilon$.