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ON THE b-EQUIVALENCE OF MULTILATTICES

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The notion of the b-isomorphism for lattices was investigated by Kolibiar [5]; he proved the following theorem:

(A) *Let M and M' be distributive lattices. Then the following conditions are equivalent:*

(i) *M and M' are b-equivalent;*

(ii) *there are lattices M_1 and M_2 such that M is isomorphic with $M_1 \times M_2$ and M' is isomorphic with $M_1 \times \tilde{M}_2$.*

Klaučová [4] generalized theorem (A) for directed distributive multilattices. Jakubík [2] studied pairs of modular lattices of locally finite lengths with isomorphic unoriented graphs; he proved that two modular lattices M and M' of locally finite lengths have isomorphic unoriented graphs if and only if (ii) is valid. Jakubík [3] also proved that if M and M' are lattices of locally finite lengths such that the unoriented graphs of M and M' are isomorphic and if M is modular, then M' is modular as well.

In this note it will be shown that if M and M' are b-equivalent directed multilattices and if M is distributive, then M' must also be distributive. Hence in the above mentioned theorem of [4] it suffices to assume that M, M' are directed multilattices and that M is distributive.

Let us recall some basic concepts that will be used later.

A multilattice [1] is a poset M in which condition (i) and its dual (ii) are satisfied:

(i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$ and (b) $z \in M$, $z \geqq a$, $z \geqq b$, $z \leqq v$ implies $z = v$. $(a \vee b)_h$ designates the set of all elements $v \in M$ satisfying (i); the symbol $(a \wedge b)_d$ has a dual meaning.

We denote $a \vee b = \cup (a \vee b)_h$, $a \wedge b = \cup (a \wedge b)_d$.

For any multilattice M we denote by $\tilde{M}$ the multilattice dual to M.

A poset A is called upper (lower) directed if for every pair of the elements a, $b \in A$ there exists an element $h \in A$ $(d \in A)$ such that $a \leqq h$, $b \leqq h$ $(d \leqq a, d \leqq b)$. The upper and lower directed poset A is called directed [5].

A multilattice M is said to be distributive iff for every a, b, b', d, $h \in M$ satisfying $d \leqq a, b, b' \leqq h$, $(a \vee b)_h = (a \vee b')_h = h (a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$ [1].

The following definitions have been introduced in [4].

Let M be a directed multilattice $a, b, x \in M$. We say that x is between a and b and write axb if the following condition is satisfied.

(b) $[(a \wedge x) \vee (b \wedge x)]_x = x$, $(a \wedge x) \wedge (b \wedge x) \subset a \wedge b$.

Directed multilattices M, M' are said to be b-equivalent if there exists a bijection f of M onto M' such that, for each $a, b, x \in M$, we have axb iff $f(a)\, f(x)\, f(b)$.

Further we assume that M and M' are directed b-equivalent and that the multilattice M is distributive. If f is the corresponding bijection and $x \in M$, we put $f(x) = x'$. The partial ordering and multioperations in M and M' will be denoted by $\leqslant$, $\vee$, $\wedge$ and $\subseteq$, $\cup$, $\cap$, respectively. Let $u, v \in M$, $u \leqslant v$. The interval $[u, v]$ is the set $\{x \in M: u \leqslant x \leqslant b\}$. We say that the interval $[u, v]$ is preserved (reserved) if $u' \subseteq v'$ $(v' \subseteq u')$ in M'; the interval $[u, u]$ is simultaneously preserved and reversed.

We need the following results (cf. [4]):

Lemma $\mathbf{I_1}$. *Let $a, b \in M$, $a \leqslant b$. Then axb iff $a \leqslant x \leqslant b$.*

Lemma $\mathbf{I_2}$. *Let $a, b, u, v \in M$, $u \leqslant a \leqslant b \leqslant v$ and let the interval $[u, v]$ be preserved (reversed). Then the interval $[a, b]$ is preserved (reversed).*

Lemma $\mathbf{I_3}$. *Let $a, b, x \in M$, $x \leqslant a$, $x \leqslant b$ $(a \leqslant x$, $b \leqslant x)$. Then axb iff $x \in a \wedge b$ $(x \in a \vee b)$.*

Lemma $\mathbf{I_4}$. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$. If the interval $[a, v]$ ($[u, b]$) is preserved and the interval $[b, v]$ ($[u, a]$) is reversed, then the interval $[u, b]$ ($[a, v]$) is preserved and the interval $[u, a]$ ($[b, v]$) is reversed.*

The assertions of Lemma I_5, I_6 were stated in [4] under the assumption that both M and M' are directed distributive multilattices. But it follows from the method of their proofs that they remain valid also without the assumption of distributivity of M'.

Lemma $\mathbf{I_5}$. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$. If the intervals $[a, v]$, $[b, v]$ or the intervals $[u, a]$, $[u, b]$ are preserved (reversed), then the interval $[u, v]$ is preserved (reversed).*

Lemma $\mathbf{I_6}$. *Let $a, b \in M$. Put aR_1b (aR_2b) iff there exists an element $v \in M$, $v \in a \vee b$, such that the intervals $[a, v]$, $[b, v]$ are reversed (preserved). The relations R_1, R_2 are equivalences on M.*

For $a', b' \in M'$ set $a'R_1'b'$ ($a'R_2'b'$) iff there exists an element $v' \in M'$, $v' \in a' \cup b'$ such that the intervals $[a', v']$, $[b', v']$ are reversed (preserved), i.e. $a \geqslant v$, $b \geqslant v$ $(a \leqslant v$, $b \leqslant v)$.

Lemma 1. *Let $a, b \in M$. The relation aR_1b (aR_2b) is satisfied iff $a'R_1'b'$ ($a'R_2'b'$) is valid.*

Proof. Let aR_1b be valid. Then there exists an element $v \in a \vee b$ such that the intervals $[a, v]$, $[b, v]$ are reversed. Choose $u \in a \wedge b$. By the Lemmas I_5 and I_2 the intervals $[u, a]$, $[u, b]$ are reversed. Consequently $u' \supseteq a'$, $u' \supseteq b'$. Moreover by Lemma I_3 we have aub, hence $a'u'b'$ holds. It follows that $u' \in a' \cup b'$ according to Lemma I_3. Thus the relation $a'R_1'b'$ is valid.

Conversely, the assumption $a'R_1'b'$ implies that there exists $v' \in a' \cup b'$ such that the intervals $[a', v']$, $[b', v']$ are reversed. By Lemma I_3 we have $v \in a \wedge b$. Choose $u \in a \vee b$; then from Lemmas I_5, I_2 it follows that the intervals $[a, u]$, $[b, u]$ are reversed and hence aR_1b is valid.

Analogously we can prove the assertion concerning R_2'.

Lemma 2. *Let a', $b' \in M'$, $u' \in a' \cap b'$, $v' \in a' \cup b'$. If the intervals $[a', v']$, $[b', v']$ are preserved (reversed), then the interval $[u', v']$ is preserved (reversed).*

Proof. Let the intervals $[a', v']$, $[b', v']$ be preserved. Choose $r \in a \wedge u$, $s \in b \wedge u$. From Lemma I_3 it follows that aru, bsu. Consequently $a'r'u'$, $b's'u'$. Using Lemma I_1 we obtain that the intervals $[r, a]$, $[s, b]$ are preserved and the intervals $[r, u]$, $[s, u]$ are reversed. Choose $t \in r \wedge s$. By Lemma I_3 we have $a'u'b'$. Hence aub. It follows that $t \in a \wedge b$, $u \in r \vee s$ according to the condition (b). Using Lemma I_5 we infer that the interval $[t, v]$ is preserved. Consequently the intervals $[t, s]$, $[t, r]$ are preserved by Lemma I_2. According to Lemma I_5 the interval $[t, u]$ is simultaneously preserved and reversed. Hence $t = r = s = u$. Thus $u \leq a \leq v$.

If the intervals $[a', v']$, $[b', v']$ are reversed, then choose $w \in a \vee b$. Consider r, s, t as above. By Lemma I_5 the interval $[v, w]$ is reversed, hence the intervals $[a, w]$, $[b, w]$ are reversed according to Lemma I_2. Again from Lemma I_5 it follows that the interval $[t, w]$ is reversed. Consequently the intervals $[r, a]$, $[s, b]$ are reversed. Hence $r = a$, $s = b$, thus $u \geq b \geq v$.

Lemma 2'. *Let a', $b' \in M'$, $u' \in a' \cap b'$, $v' \in a' \cup b'$. If the intervals $[u', a']$, $[u', b']$ are preserved (reversed), then the interval $[u', v']$ is preserved (reversed).*

Proof. Let the intervals $[u', a']$, $[u', b']$ be preserved. Choose $r \in a \wedge v$, $s \in b \wedge v$. Similarly as in the proof of Lemma 2 (by using Lemma I_3 and Lemma I_1) we obtain that the intervals $[r, a]$, $[s, b]$ are reversed and the intervals $[s, v]$, $[r, v]$ are preserved. Choose $w \in a \vee b$, $t \in r \wedge s$. Since avb, we have $t \in a \wedge b$ according to the condition (b). By Lemma I_5 the interval $[u, w]$ is preserved. Therefore the intervals $[a, w]$, $[b, w]$ are preserved by Lemma I_2. Again by Lemma I_5 the interval $[t, w]$ is preserved. Hence the intervals $[r, a]$, $[s, b]$ are preserved. Consequently $r = a$, $s = b$. Thus $v \geq a \geq u$.

Let the intervals $[u', a']$, $[u', b']$ be reversed and let r, s, t be as above. The interval $[t, u]$ is reversed by Lemma I_5. Then the intervals $[t, s]$, $[t, r]$ are reversed according to Lemma I_2. Hence $v = r = s = t$. Thus $v \leq a \leq u$.

Lemma 3. *Let a', $b' \in M'$, $a'R_1'b'$. If $w' \in a' \cup b'$, then the intervals $[a', w']$, $[b', w']$ are reversed.*

Proof. Let $a'R_1'b'$. Then there exists $v' \in a' \cup b'$ such that the intervals $[a', v']$,

$[b', v']$ are reversed. Choose $u' \in a' \cap b'$. The interval $[u', v']$ is reversed by Lemma 2. Hence the intervals $[u', a']$, $[u', b']$ are reversed according to Lemma I_2. If $w' \in a' \cup b'$, then again by Lemma 2 the interval $[u', w']$ is reversed. Therefore the intervals $[a', w']$, $[b', w']$ are reversed by Lemma I_2.

Analogously we can prove:

Lemma 3'. *Let a', $b' \in M'$, $a' R_2' b'$. If $w' \in a' \cup b'$, then the intervals $[a', w']$, $[b', w']$ are preserved.*

Lemma 4. *Let a', $b' \in M'$, $a' R_1' b'$ ($a' R_2' b'$). If $u' \in a' \cap b'$, then the intervals $[u', a']$, $[u', b']$ are reversed (preserved).*

Proof. Let $a' R_1' b'$, $u' \in a' \cap b'$, $v' \in a' \cup b'$. By Lemma 3 the intervals $[a', v']$, $[b', v']$ are reversed. Hence the interval $[u', v']$ is reversed by Lemma 2. Therefore the intervals $[u', a']$, $[u', b']$ are reversed according to Lemma I_2. Similarly we can prove the analogous assertion concerning R_2'.

Lemma 5. *The relations R_1', R_2' are equivalence relations on M' and they satisfy the following conditions*

(i) $R_1' \cdot R_2' = R_2' \cdot R_1'$

(ii) $R_1' \cup R_2' = I'$, $R_1' \cap R_2' = 0'$ (where $0'(I')$ is *the least (greatest) element of the lattice of all equivalence relations on the set M'*).

(iii) *If a', b', $c' \in M'$, $a' \subseteq c'$, $a' R_1' b'$, $b' R_2' c'$, then $a' \subseteq b' \subseteq c'$.*

(iv) *Let a', b', c', $d' \in M'$, $a' R_1' b'$, $c' R_1' d'$, $a' R_2' c'$, $b' R_2' d'$. Then from $a' \subseteq b'$ it follows that $c' \subseteq d'$ and from $a' \subseteq c'$ it follows that $b' \subseteq d'$.*

The Lemma can be proved in the same way as [4, Lemma 9].

The following assertions K_1, K_2 were proved by Kolibiar.

(K_1) [5]. *Let M be a Cartesian product of two posets M_1, M_2. M is a multilattice iff M_1 and M_2 are multilattices. For $x \in M$ we denote by x_1, x_2 the components of $x(x_i \in M_i)$. Let a, b, h, $v \in M$. Then $v \in (a \vee b)_h$, $(v \in (a \wedge b)_h)$ iff $v_i \in (a_i \vee b_i)_{h_i}$ $(v_i \in (a_i \wedge b_i)_{h_i})$ for a_i, b_i, h_i, $v_i \in M_i$ $(i = 1, 2)$.*

(K_2) [6]. *Let A be a quasiordered set. There exists a one-one correspondence between the non trivial direct decompositions of the quasiordered set A into two factors and pairs (R_1, R_2) of non trivial congruence relations R_1, R_2 on A satisfying the properties* (i), (ii), (iii), (iv) *from Lemma 5. To each couple (R_1, R_2) with the mentioned properties there corresponds the decomposition $A \sim A/R_1 \times A/R_2$ and to each element $a \in A$ there corresponds the element (a_1, a_2), where a_i is the equivalence class under R_i $(i = 1, 2)$ containing a.*

Denote $M/R_1 = M_1$, $M/R_2 = M_2$, $M'/R_1' = M_1'$, $M'/R_2' = M_2'$. From the assertion K_2 and from Lemma I_6 it follows that there exists an isomorphism $\psi: M \sim M_1 \times M_2$. According to K_2 and Lemma 5 there exists an isomorphism $\psi': M' \sim M_1' \times M_2'$. Since M, M' are multilattices, we infer that $M_1 \times M_2$, $M_1' \times M_2'$ are multilattices and by K_1, M_1, M_2, M_1', M_2' are multilattices as well. Let φ be a b-equivalence of M

onto M'; then it is obvious that $x=\psi'\varphi\psi^{-1}$ is a b-equivalence of $M_1\times M_2$ onto $M_1'\times M_2'$. In the same way as in [4] we can now prove that M_1 and M_1' are isomorphic, M_2 and M_2' are anti-isomorphic. Thus the following assertion holds.

Theorem 1. *Let M, M' be directed b-equivalent multilattices, φ be an b-equivalence of M onto M' and let M be distributive. Then there exist multilattices M_1, M_2 such that $M\sim M_1\times M_2$, $M'\sim M_1\times \tilde{M}_2$, whereby the elements $x\in M$, $x'\in M'$, $x'=\varphi(x)$ are mapped on the same pair (x_1, x_2), $x_1\in M_1$, $x_2\in M_2$.*

Theorem 2. *Let M and M' be directed b-equivalent multilattices. If M is distributive, then M' is distributive as well.*

Proof. Let M, M' be directed b-equivalent multilattices and let M be distributive. Then by Theorem 1 there exist multilattices M_1, M_2 such that $M\sim M_1\times M_2$, $M'\sim M_1\times \tilde{M}_2$. Since M is distributive, then by the assertion K_1, M_1 and M_2 are distributive also. Consequently $\tilde{M}_2$ is distributive. Thus by the assertion K_1, M' is distributive.

The following assertion has been proved in [4].

(C) *Let M, M' be directed distributive multilattices. M, M' are b-equivalent if and only if there exist multilattices M_1, M_2 such that $M\sim M_1\times M_2$ and $M'\sim M_1\times \tilde{M}_2$.*

The following result is a direct corollary of Theorem 1, Theorem 2 and the assertion (C).

Theorem 3. *Let M, M' be direct multilattices. If M is distributive, then the following conditions are equivalent.*

(a) M and M' are b-equivalent multilattices.

(b) There exist multilattices M_1, M_2 such that $M\sim M_1\times M_2$ and $M'\sim M_1\times \tilde{M}_2$.

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О b-ЭКВИВАЛЕНТНЫХ МУЛТИСТРУКТУРАХ

Мария Томкова

Резюме

В данной статье обобщена одна теорема О. Клаучовой касающаяся пар дистрибутивных мультиструктур. Затем доказано, что если M и M' – b-эквивалентные направленные мультиструктуры и если M – дистрибутивна, тогда M' – также должна быть дистрибутивна.