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ODD PERFECT NUMBERS

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ABSTRACT. Much work on odd perfect numbers has been done by Euler. He found that each odd perfect number can be written as a product of two terms of a particular form. In this article, one of these terms has been expanded further to give a new form to odd perfect numbers.

According to Euler, a prime decomposition of an odd perfect number n, i.e. one such that $\sigma(n) = 2n$ and $2 \nmid n$, must be of the form $n = p^{4\alpha+1} \cdot \prod_{i=1}^{k} p_i^{2\alpha_i}$ with $p \equiv 1 \pmod 4$ and $(p, p_i) = 1$. Here $p^{4\alpha+1}$ is called the Euler factor of n. For the proof see [1; p. 424]. Clearly, from $\sigma(n) = 2n$ it follows that $\frac{\sigma(p^{4\alpha+1})}{2} \cdot \prod_{i=1}^{k} \sigma(p_i^{2\alpha_i}) = p^{4\alpha+1} \cdot \prod_{i=1}^{k} p_i^{2\alpha_i}$ and since $p \nmid \sigma(p^{4\alpha+1})$ we have $p^{4\alpha+1} \mid \prod_{i=1}^{k} \sigma(p_i^{2\alpha_i})$. The purpose of this note is to establish that $p^{4\alpha+1} < \prod_{i=1}^{k} \sigma(p_i^{2\alpha_i})$.

THEOREM. *An odd perfect number, must be of the form $n = p^{4\alpha+1} \cdot \frac{\sigma(p^{4\alpha+1})}{2} \cdot d$, where $d > 1$.*

Proof. Suppose, on the contrary, that $p^{4\alpha+1} = \prod_{i=1}^{k} \sigma(p_i^{2\alpha_i})$ or equivalently $\frac{\sigma(p^{4\alpha+1})}{2} = \prod_{i=1}^{k} p_i^{2\alpha_i}$. We investigate two different cases:

i) If $\alpha = 0$, then $k = 1$. Thus $p = \sigma(p_1^{2\alpha_1})$ and $\frac{1+p}{2} = p_1^{2\alpha_1}$ which is a contradiction.

ii) Let $\alpha > 0$. Since $\frac{\sigma(p^{4\alpha+1})}{2} = \sigma(p^{2\alpha}) \cdot \frac{1+p^{2\alpha+1}}{2}$ and $\left(\sigma(p^{2\alpha}), \frac{1+p^{2\alpha+1}}{2}\right) = 1$ we have $\sigma(p^{2\alpha}) = a^2$ and $\frac{1+p^{2\alpha+1}}{2} = b^2$. Moreover $\sigma(a^2) \cdot \sigma(b^2) = p^{4\alpha+1}$ and thus $\sigma(a^2) = p^{\delta_a}$ and $\sigma(b^2) = p^{\delta_b}$, for some positive integers δ_a and δ_b, $\delta_a + \delta_b = 4\alpha + 1$.

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Using $\sigma(a^2) = p^{\delta_a}$ and $\sigma(p^{2\alpha}) = a^2$ we find that $\delta_a \geq 2\alpha + 1$, and using $\sigma(b^2) = p^{\delta_b}$ and $\frac{1+p^{2\alpha+1}}{2} = b^2$ we find that $\delta_b \geq 2\alpha + 1$.

The inequality $\delta_a + \delta_b \geq 4\alpha + 2$ gives a contradiction. □

REFERENCES

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