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NEW CHARACTERIZATIONS OF REGULAR OPEN SETS, SEMI–REGULAR SETS, AND EXTREMALLY DISCONNECTEDNESS

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ABSTRACT. In this paper, each of regular open sets, semi-regular sets, and extremally disconnectedness are further examined and characterized.

1. Introduction

In the study of mathematics, there are many solution techniques and strategies. One technique that has been utilized in the study of minimal topological spaces, as well as other areas, is the construction and consideration of associated sets and associated topologies for a given topological space. As one example, in 1937 ([22]), *regular open sets* were introduced and used to define the *semiregularization space* of a topological space. Let (X,T) be a space and let $A \subset X$. Then A is *regular open*, denoted by $A \in RO(X,T)$, if and only if $A = \operatorname{Int}(\operatorname{Cl}(A))$. In the 1937 (paper [22]), it was shown that $RO(X,T)$ is a base for a topology T_s on X coarser than T, and (X,T_s) is called the *semiregularization space* of (X,T). The set A is *regular closed*, denoted by $A \in RC(X,T)$, if and only if one of the following equivalent conditions is satisfied ([23]):

(a) $A = \operatorname{Cl}(\operatorname{Int}(A))$ and
(b) $X \setminus A \in RO(X,T)$.

There are many other instances in which associated sets and associated topologies have been used to better understand and further investigate mathematical properties.

The study of semi open sets and related sets and properties began in 1963 ([18]). Let (X,T) be a space and let $A \subset X$. Then A is *semi open*, denoted by $A \in SO(X,T)$, if and only if there exists $O \in T$ such that $O \subset A \subset$

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$\mathrm{Cl}(O)$. The introduction of semi open sets raised many basic general topological questions, which has thus far led to a productive study in which many new mathematical tools have been added to the general topology tool box, many new properties have been defined and examined, many new gems have been discovered for old properties, additional associated sets and associated topologies have been introduced, examined, and utilized, and, very importantly, additional basic general topological questions continue to arise.

Semi closed sets and the semi closure operator were added to the literature in 1970 ([1]). Let (X,T) be a space and let $A, B \subset X$. Then A is *semi closed* if and only if $X \setminus A$ is semi open, and the *semi closure* of B, denoted by $\mathrm{scl}\, B$, is the intersection of all semi closed sets containing B. In 1978 ([3]), the subset A was defined to be *regular semi open*, denoted by $A \in \mathrm{RSO}(X,T)$, if and only if there exists a regular open set U such that $U \subset A \subset \mathrm{Cl}(U)$. Semi open sets were used in 1980 ([6]), to define semi compact spaces. The space (X,T) is *semi compact* if and only if every cover of X by semi open sets has a finite subcover. In 1984 ([4]), the semi interior and semi closure operators were used to define semi regular open sets. The *semi interior* of A, denoted by $\mathrm{sint}\, A$, is the union of all semi open sets contained in A, and B is *semi regular open*, denoted by $B \in \mathrm{SRO}(X,T)$, if and only if $B = \mathrm{sint}(\mathrm{scl}\, B)$. Also, the question of whether or not semi compactness could be reduced to compactness led to a new associated topology in 1984 ([21]). The topology TSO on X with subbase $SO(X,T)$ is called the *semi open set generated topology* of (X,T). In the 1984 (paper [21]), it was shown that (X,T) is semi compact if and only if (X,TSO) is compact. Semi-regular sets and s-closedness were introduced in 1987 ([5]). The subset A is *semi-regular*, denoted by $A \in \mathrm{SR}(X,T)$, if and only if A is both semi open and semi closed, and (X,T) is *s-closed* if and only if every cover of X by semi open sets has a finite subcollection whose semi closures cover X. In the 1987 (investigation [5]), it was shown that $\mathrm{RSO}(X,T) = \mathrm{SRO}(X,T) = \mathrm{SR}(X,T)$. The further investigation of s-closedness in 1991 ([7]), led to the introduction and investigation of another associated topology. The topology TSR on X with subbase $\mathrm{SR}(X,T)$ is called the *semi-regular set generated topology* of (X,T), and (X,T) is *s-closed* if and only if (X,TSR) is compact. In this paper, the associated sets and associated topologies given above are used to further investigate and characterize *regular open sets, semi-regular sets*, and *extremally disconnectedness*. The space (X,T) is *extremally disconnected* if and only if $\mathrm{Cl}(O) \in T$ for each $O \in T$ ([23]).

2. Regular open sets and semi-regular sets

The study of semi open sets and related properties has led to many new characterizations of regular open sets. In 1978, open sets and the semi closure operator were used to define feebly open sets. Let (X,T) be a space and let $A \subset X$. Then A is *feebly open*, denoted by $A \in \mathrm{FO}(X,T)$, if and only if there exists

$O \in T$ such that $O \subset A \subset \operatorname{scl} A$ ([19]). The further study of feebly open sets showed that $\mathrm{FO}(X,T)$ is a topology on X, $T \subset \mathrm{FO}(X,T) = \mathrm{FO}(X, \mathrm{FO}(X,T))$ ([8]), that $\mathrm{FO}(X,T)$ is the finest topology on X having the same semi open sets as (X,T) ([9]), and $RO(X,T) = \{\operatorname{scl} A \mid A \in \mathrm{FO}(X,T)\} \overset{[10]}{=} \{\operatorname{scl} O \mid O \in T\}$ $\overset{[11]}{=} \{\mathrm{Int}(\mathrm{Cl}(O)) \mid O \in T\} \overset{[12]}{=} \{\mathrm{Ext}(O) \mid O \in \mathrm{FO}(X,T)\} = \{\mathrm{Ext}(O) \mid O \in T\}$ ([13]). Of course, for each new characterization of regular open sets, there is a corresponding characterization of $\mathrm{RC}(X,T)$. Below additional characterizations of $RO(X,T)$ and $\mathrm{RC}(X,T)$ are given.

THEOREM 2.1. *Let (X,T) be a space. Then $RO(X,T) = \{\mathrm{Ext}(O) \mid O \in SO(X,T)\} = \{\mathrm{Ext}(O) \mid O \in \mathrm{SR}(X,T)\}$, and $\mathrm{RC}(X,T) = \{\mathrm{Cl}(O) \mid O \in SO(X,T)\} = \{\mathrm{Cl}(O) \mid O \in \mathrm{SR}(X,T)\}$.*

P r o o f. Since $T \subset SO(X,T)$, then $RO(X,T) \subset \{\mathrm{Ext}(O) \mid O \in SO(X,T)\}$. Let $O \in SO(X,T)$. Let $U \in T$ such that $U \subset O \subset \mathrm{Cl}(U)$. Then $\mathrm{Cl}(O) = \mathrm{Cl}(U)$, which implies $\mathrm{Ext}(O) = \mathrm{Ext}(U)$. Thus $\{\mathrm{Ext}(O) \mid O \in SO(X,T)\} \subset \{\mathrm{Ext}(O) \mid O \in T\} = RO(X,T)$, which implies $RO(X,T) = \{\mathrm{Ext}(O) \mid O \in SO(X,T)\}$. Since $\mathrm{SR}(X,T) \subset SO(X,T)$, then $\{\mathrm{Ext}(O) \mid O \in \mathrm{SR}(X,T)\} \subset RO(X,T)$. Let $O \in T$. Since $\mathrm{SR}(X,T) = \{U \subset X \mid \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \subset U \subset \mathrm{Cl}(\mathrm{Int}(U))\}$ ([14]), then $\mathrm{Int}(\mathrm{Cl}(O)) \in \mathrm{SR}(X,T)$ and $\mathrm{Ext}(O) = \mathrm{Ext}(\mathrm{Int}(\mathrm{Cl}(O))) \in \{\mathrm{Ext}(V) \mid V \in \mathrm{SR}(X,T)\}$. Thus $RO(X,T) \subset \{\mathrm{Ext}(O) \mid O \in \mathrm{SR}(X,T)\}$, which implies $RO(X,T) = \{\mathrm{Ext}(O) \mid O \in \mathrm{SR}(X,T)\}$. □

Let (X,T) be a space and let $A \subset X$. Then A is *semi-regular closed*, denoted by $A \in \mathrm{SRC}(X,T)$, if and only if $X \setminus A$ is semi-regular.

THEOREM 2.2. *Let (X,T) be a space. Then $\mathrm{SR}(X,T) = \mathrm{SRC}(X,T) = \{O \subset X \mid \mathrm{Int}(\mathrm{Cl}(U)) \subset O \subset \mathrm{Cl}(U)$ for some $U \in T\} = \{O \subset X \mid \operatorname{scl} A \subset O \subset \mathrm{Cl}(A)$ for some $A \in \mathrm{FO}(X,T)\} = \{O \subset X \mid \operatorname{scl} A \subset O \subset \mathrm{Cl}(A)$ for some $A \in T\} = \{O \subset X \mid \mathrm{Ext}(A) \subset O \subset \mathrm{Cl}(\mathrm{Ext}(A))$ for some $A \in \mathrm{FO}(X,T)\} = \{O \subset X \mid \mathrm{Ext}(A) \subset O \subset \mathrm{Cl}(\mathrm{Ext}(A))$ for some $A \in T\} = \{O \subset X \mid \mathrm{Ext}(A) \subset O \subset \mathrm{Cl}(\mathrm{Ext}(A))$ for some $A \in SO(X,T)\} = \{O \subset X \mid \mathrm{Ext}(A) \subset O \subset \mathrm{Cl}(\mathrm{Ext}(A))$ for some $A \in \mathrm{SR}(X,T)\}$.*

The proof is straightforward using the definitions and results above and is omitted.

Since for a space (X,T), $RO(X,T) = RO(X, \mathrm{FO}(X,T))$ ([10]), then the next result follows immediately from the results above.

COROLLARY 2.1. *Let (X,T) be a space. Then*

$$RO(X,T) = \{\mathrm{Int}_{\mathrm{FO}(X,T)}(\mathrm{Cl}_{\mathrm{FO}(X,T)}(O)) \mid O \in \mathrm{FO}(X,T)\}.$$

Corollary 2.1 raised the question of whether or not for a space (X,T) the characterization of $RO(X,T)$ given in Corollary 2.1 is the same as one of the characterizations given earlier. Also, since for a space (X,T), $SO(X,T) = SO\big(X,\mathrm{FO}(X,T)\big)$, and thus $\mathrm{SR}(X,T) = \mathrm{SR}\big(X,\mathrm{FO}(X,T)\big)$, the question of whether or not new characterizations of $RO(X,T)$ and $\mathrm{SR}(X,T)$ could be obtained using the topology $\mathrm{FO}(X,T)$ was raised. These questions are resolved by the next result.

Theorem 2.3. *Let (X,T) be a space and let $A \in \mathrm{FO}(X,T)$. Then $\mathrm{scl}_T A = \mathrm{Int}_T\big(\mathrm{Cl}_T\big(\mathrm{Int}_T(A)\big)\big) = \mathrm{scl}_{\mathrm{FO}(X,T)} A = \mathrm{Int}_{\mathrm{FO}(X,T)}\big(\mathrm{Cl}_{\mathrm{FO}(X,T)}(A)\big)$, $\mathrm{Cl}_T(A) = \mathrm{Cl}_{\mathrm{FO}(X,T)}(A)$, and $\mathrm{Ext}_T(A) = \mathrm{Ext}_{\mathrm{FO}(X,T)}(A)$.*

P r o o f. Since $A \in \mathrm{FO}(X,T)$, then $\mathrm{scl}_T A = \mathrm{Int}_T\big(\mathrm{Cl}_T\big(\mathrm{Int}_T(A)\big)\big)$ ([15]). Thus, since $A \in \mathrm{FO}(X,T) = \mathrm{FO}\big(X,\mathrm{FO}(X,T)\big)$,

$$\begin{aligned}\mathrm{scl}_{\mathrm{FO}(X,T)} A &= \mathrm{Int}_{\mathrm{FO}(X,T)}\big(\mathrm{Cl}_{\mathrm{FO}(X,T)}\big(\mathrm{Int}_{\mathrm{FO}(X,T)}(A)\big)\big)\\ &= \mathrm{Int}_{\mathrm{FO}(X,T)}\big(\mathrm{Cl}_{\mathrm{FO}(X,T)}(A)\big)\,.\end{aligned}$$

Since $\mathrm{scl}_T U = \mathrm{scl}_{\mathrm{FO}(X,T)} U$ for each $U \subset X$ ([8]), then $\mathrm{scl}_T A = \mathrm{scl}_{\mathrm{FO}(X,T)} A$. Since $A \in \mathrm{FO}(X,T)$, then $\mathrm{Cl}_T(A) = \mathrm{Cl}_{\mathrm{FO}(X,T)}(A)$ ([16]), and $\mathrm{Ext}_T(A) = X \setminus \mathrm{Cl}_T(A) = X \setminus \mathrm{Cl}_{\mathrm{FO}(X,T)}(A) = \mathrm{Ext}_{\mathrm{FO}(X,T)}(A)$. □

Thus the characterization of $RO(X,T)$ given in Corollary 2.1 is not a new characterization of $RO(X,T)$, and no new characterizations of $RO(X,T)$ or $\mathrm{SR}(X,T)$ can be obtained using the topology $\mathrm{FO}(X,T)$.

3. Extremally disconnectedness

The investigation of semi open sets has led to numerous characterizations of extremally disconnectedness. Below many more characterizations are given for extremally disconnectedness.

Theorem 3.1. *Let (X,T) be a space and let Tn be the topology on X obtained by repeating the semi open set generated topology process n times starting with (X,T), where $n \in \mathbb{N}$, the set of natural numbers. Then the following are equivalent:*

(a) *(X,T) is extremally disconnected.*
(b) *(X,Tn) is extremally disconnected, and $Tn = SO(X,T)$ for each $n \in \mathbb{N}$.*
(c) *$Tn \subset SO(X,T)$ for each $n \in \mathbb{N}$.*
(d) *$TSO \subset SO(X,T)$.*
(e) *(X,Tn) is extremally disconnected, and $Tn = SO(X,T)$ for some $n \in \mathbb{N}$.*
(f) *$Tn = SO(X,T)$ for some $n \in \mathbb{N}$.*
(g) *$Tn \subset SO(X,T)$ for some $n \in \mathbb{N}$.*

Proof.

(a) *implies* (b): Let $n \in \mathbb{N}$. Then $Tn = SO(X,T)$ ([7]) and $SO(X,T) = Tn \subset SO(X,Tn) \subset T(n+1) = SO(X,T)$ ([7]), which implies $SO(X,Tn) = Tn$ is a topology on X, and thus (X,Tn) is extremally disconnected ([20]).

Clearly, (b) implies (c) and (c) implies (d).

(d) *implies* (e): Since $SO(X,T) \subset TSO$, then $SO(X,T) = TSO$ is a topology on X, which implies (X,T) is extremally disconnected. Then (e) follows from the argument above.

Clearly, (e) implies (f) and (f) implies (g).

(g) *implies* (a): For each $n \in \mathbb{N}$, let $S(n)$ be the statement $SO(X,T) \subset Tn$. Since $SO(X,T) \subset TSO = T1$, then $S(1)$ is true. Assume $S(k)$ is true. Then $SO(X,T) \subset Tk \subset SO(X,Tk) \subset TkSO = T(k+1)$. Thus $S(k+1)$ is true. Hence by mathematical induction $SO(X,T) \subset Tn$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $Tn \subset SO(X,T)$. Then $SO(X,T) = Tn$ is a topology on X, which implies (X,T) is extremally disconnected. □

THEOREM 3.2. *Let (X,T) be a space and let TRC be the topology on X with subbase $\mathrm{RC}(X,T)$. Then the following are equivalent:*

(a) *(X,T) is extremally disconnected.*
(b) *$\mathrm{RC}(X,T)$ is a base for T_s.*
(c) *$TRC = T_s$.*
(d) *$TRC \subset T_s$.*
(e) *$\mathrm{RC}(X,T) \subset T_s$.*
(f) *$TSR = T_s$.*
(g) *$TSR \subset T_s$.*
(h) *$\mathrm{SR}(X,T) = \mathrm{RC}(X,T)$.*
(i) *$\mathrm{SR}(X,T) \subset \mathrm{RC}(X,T)$.*

Proof.

(a) *implies* (b): Since (X,T) is extremally disconnected, then $RO(X,T) = \mathrm{RC}(X,T)$ ([13]), which implies $\mathrm{RC}(X,T)$ is a base for T_s.

Clearly, (b) implies (c), (c) implies (d), and (d) implies (e).

(e) *implies* (f): Let $O \in \mathrm{RC}(X,T)$. Then $O = \mathrm{Cl}(\mathrm{Int}(O))$, and since $O \in T_s \subset T$, then $O = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(O))) \in RO(X,T)$. Thus $\mathrm{RC}(X,T) \subset RO(X,T)$, which implies (X,T) is extremally disconnected ([13]). Then $RO(X,T) = \mathrm{RSO}(X,T) \overset{[13]}{=} \mathrm{SR}(X,T)$, which implies $TSR = T_s$.

Clearly, (f) implies (g).

(g) *implies* (h): Since $T_s \subset TSR$ ([7]), then $TSR = T_s$. Let $O \in \mathrm{RC}(X,T)$. Then $O = \mathrm{Cl}(\mathrm{Int}(O)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(O)))) \in TSR = T_s$. Thus $\mathrm{RC}(X,T) \subset T_s$, and, by the argument above, (X,T) is extremally disconnected. Then $\mathrm{SR}(X,T) = \mathrm{RSO}(X,T) = RO(X,T) = \mathrm{RC}(X,T)$ ([13]).

Clearly, (h) implies (i).

(i) *implies* (a): Since $RO(X,T) \subset \mathrm{SR}(X,T) \subset \mathrm{RC}(X,T)$, then (X,T) is extremally disconnected ([13]). □

THEOREM 3.3. *Let (X,T) be a space and, for each $n \in \mathbb{N}$, let TSn be the topology on X obtained by repeating the semi-regular set generated topology process n times starting with (X,T). Then the following are equivalent:*

(a) *(X,T) is extremally disconnected.*
(b) *For each $n \in \mathbb{N}$, (X,TSn) is extremally disconnected, and $TSn = T_s$.*
(c) *For each $n \in \mathbb{N}$, $TSn = T_s$.*
(d) *For each $n \in \mathbb{N}$, $TSn \subset T_s$.*
(e) *$TSn \subset T_s$ for some $n \in \mathbb{N}$.*
(f) *$TSn = T_s$ and (X,TSn) is extremally disconnected for some $n \in \mathbb{N}$.*
(g) *$TSn = T_s$ for some $n \in \mathbb{N}$.*

Proof.

(a) *implies* (b): The proof is by mathematical induction. Since (X,T) is extremally disconnected, then $TS1 = TSR = T_s$ and $TSR = TSR_s = TSRSR$ ([7]), which implies $TS1 = TS1_s = TS1SR$, and thus $(X,TS1)$ is extremally disconnected. Assume the statement is true for $n = k$. Then (X,TSk) is extremally disconnected, which implies $TS(k+1) = TSkSR = TSk_s = TSk = T_s$ and $\big(X,TS(k+1)\big) = \big(X,(TSk)S1\big)$ is extremally disconnected. Thus the statement is true for $n = k+1$. Hence, by mathematical induction, the statement is true for each $n \in \mathbb{N}$.

Clearly, (b) implies (c), (c) implies (d), and (d) implies (e).

(e) *implies* (f): For each $n \in \mathbb{N}$, $n \geq 2$, let $S(n)$ be the statement $TS1 \subset TSn$. Since $TSR = TS1 \subset TSRSR \overset{[7]}{=} TS2$, then $S(2)$ is true. Assume the statement is true for $n = k \geq 2$. Then $TS1 \subset TSk = TS(k-1)SR \subset TS(k-1)SRSR = TS(k+1)$. Hence $S(k+1)$ is true. Thus, by mathematical induction, $TS1 \subset TSn$ for each $n \geq 2$. Let $n \in \mathbb{N}$ such that $TSn \subset T_s$. Since $T_s \subset TSR = TS1 \overset{[7]}{\subset} TSn \subset T_s$, then $TSR = T_s$, which implies (X,T) is extremally disconnected. Then (f) follows from the argument above.

Clearly, (f) implies (g).

(g) *implies* (a): Since $TSn = T_s$ for some $n \in \mathbb{N}$, then $TSn \subset T_s$ for some $n \in \mathbb{N}$, and, by the argument above, (X,T) is extremally disconnected. □

THEOREM 3.4. *Let (X,T) be a space and, for each $n \in \mathbb{N}$, let TRn be the topology on X obtained by repeating the regular closed set generated topology process n times starting with (X,T). Then the following are equivalent:*

(a) *(X,T) is extremally disconnected.*

(b) (X, TRn) *is extremally disconnected, and* $TRn = TSn = T_s$ *for each* $n \in N$.
(c) $TRn = T_s$ *and* (X, TRn) *is extremally disconnected for each* $n \in \mathbb{N}$.
(d) $TRn = T_s$ *for each* $n \in \mathbb{N}$.
(e) $TRn \subset T_s$ *for all* $n \in \mathbb{N}$.

Proof.

(a) *implies* (b): The proof is by mathematical induction. Since (X, T) is extremally disconnected, then $TSR = TS1 = T_s$ and $TRC = TR1 = T_s$ and $(X, TS1) = (X, TR1)$ is extremally disconnected. Thus the statement is true for $n = 1$. Assume the statement is true for $n = k$. Then (X, TRk) is extremally disconnected and $TRk = TSk = T_s$, which implies $\big(X, TS(k+1)\big)$ is extremally disconnected and $TS(k+1) = TSkSR = TSk_s = (T_s)_s = T_s \overset{[2]}{=} TRk_s = TRkRC = TR(k+1)$. Thus the statement is true for $n = k+1$. Hence, by mathematical induction, the statement is true for each $n \in \mathbb{N}$.

Clearly, (b) implies (c), (c) implies (d), and (d) implies (e).

(e) *implies* (a): Since $TRn \subset T_s$ for all $n \in \mathbb{N}$, then $TRC = TR1 \subset T_s$, which implies (X, T) is extremally disconnected. □

COROLLARY 3.1. *If* (X, T) *is extremally disconnected, then* (X, Tn), (X, TSn), *and* (X, TRn) *are extremally disconnected for each* $n \in \mathbb{N}$.

The following example shows that the converse of Corollary 3.1 is false.

Example 3.1. Let $X = \{a, b, c\}$ and $T = \big\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\big\}$. Then for each $n \in \mathbb{N}$, (X, Tn), (X, TSn), and (X, TRn) are extremally disconnected, but (X, T) is not extremally disconnected.

THEOREM 3.5. *Let* (X, T) *be a space. Then the following are equivalent:*

(a) (X, T) *is extremally disconnected.*
(b) $\mathrm{scl}_{\mathrm{FO}(X,T)}\, O = \mathrm{Cl}_{\mathrm{FO}(X,T)}(O)$ *for each* $O \in \mathrm{FO}(X, T)$.
(c) $\mathrm{scl}_T\, O = \mathrm{Cl}_T(O)$ *for each* $O \in \mathrm{FO}(X, T)$.
(d) $\mathrm{scl}\, O = \mathrm{Cl}(O)$ *for each* $O \in T$.
(e) $\mathrm{Int}(C) = C$ *for each* $C \in \mathrm{RC}(X, T)$.
(f) $\mathrm{Cl}\big(\mathrm{Ext}(O)\big) = \mathrm{Ext}(O)$ *for each* $O \in T$.
(g) $\mathrm{scl}\, O = \mathrm{Cl}(O)$ *for each* $O \in RO(X, T)$.
(h) $\mathrm{scl}\, O = \mathrm{Cl}(O)$ *for each* $O \in \mathrm{SR}(X, T)$.
(i) $\mathrm{Cl}(O) = \mathrm{scl}\, O = \mathrm{Int}\big(\mathrm{Cl}(O)\big) \in RO(X, T)$ *for each* $O \in \mathrm{SR}(X, T)$.

Proof. Since (X, T) is extremally disconnected, then $\big(X, \mathrm{FO}(X, T)\big)$ is extremally disconnected ([20]) and $\mathrm{scl}_{\mathrm{FO}(X,T)}\, U = \mathrm{Cl}_{\mathrm{FO}(X,T)}(U)$ for each $U \in SO\big(X, \mathrm{FO}(X, T)\big)$ ([5]). Since $\mathrm{FO}(X, T) \subset SO\big(X, \mathrm{FO}(X, T)\big)$, then for each $O \in \mathrm{FO}(X, T)$, $\mathrm{scl}_{\mathrm{FO}(X,T)}\, O = \mathrm{Cl}_{\mathrm{FO}(X,T)}(O)$.

By Theorem 2.3, (b) implies (c), and since $T \subset \mathrm{FO}(X,T)$, then (c) implies (d).

(d) *implies* (e): Let $C \in \mathrm{RC}(X,T)$. Then $C = \mathrm{Cl}(\mathrm{Int}(C)) = \mathrm{scl}(\mathrm{Int}(C)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(C))) = \mathrm{Int}(C)$.

(e) *implies* (f): Let $O \in T$. Then $\mathrm{Int}(\mathrm{Cl}(O)) \in RO(X,T)$ and $X \backslash \mathrm{Int}(\mathrm{Cl}(O)) = \mathrm{Cl}(\mathrm{Ext}(O))$ is regular closed. Thus $\mathrm{Cl}(\mathrm{Ext}(O)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Ext}(O))) = \mathrm{Ext}(O)$.

(f) *implies* (g): Let $O \in RO(X,T)$. Let $U \in T$ such that $O = \mathrm{Ext}(U)$. Then $\mathrm{Cl}(O) = \mathrm{Cl}(\mathrm{Ext}(U)) = \mathrm{Ext}(U) = O$. Since $O \subset \mathrm{scl}\, O \subset \mathrm{Cl}(O) = O$, then $\mathrm{scl}\, O = \mathrm{Cl}(O)$.

(g) *implies* (h): Let $O \in \mathrm{SR}(X,T)$. Let $U \in RO(X,T)$ such that $U \subset O \subset \mathrm{Cl}(U)$. Then $\mathrm{scl}\, U \subset \mathrm{scl}\, O \subset \mathrm{Cl}(O) \subset \mathrm{Cl}(U) = \mathrm{scl}\, U$, which implies $\mathrm{scl}\, O = \mathrm{Cl}(O)$.

(h) *implies* (i): Let $O \in \mathrm{SR}(X,T)$. Then $\mathrm{Cl}(O) = \mathrm{scl}\, O$. Let $A \in T$ such that $\mathrm{Int}(\mathrm{Cl}(A)) \subset O \subset \mathrm{Cl}(A)$. Since $\mathrm{Int}(\mathrm{Cl}(A)) \in RO(X,T) \subset T \subset \mathrm{FO}(X,T)$ and $RO(X,T) \subset \mathrm{SR}(X,T)$, then $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(\mathrm{Cl}((\mathrm{Int}(A)))) = \mathrm{scl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Cl}(A)$. Then $\mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(O) \subset \mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$ and $\mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{Int}(\mathrm{Cl}(O)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A))$, which implies $\mathrm{scl}\, O = \mathrm{Cl}(O) = \mathrm{Int}(\mathrm{Cl}(O)) = \mathrm{Int}(\mathrm{Cl}(A)) \in RO(X,T)$.

(i) *implies* (a): Since $\mathrm{RC}(X,T) \subset \mathrm{SR}(X,T)$, then, for each $O \in \mathrm{RC}(X,T)$, $\mathrm{Cl}(O) = O \in RO(X,T)$. Thus $\mathrm{RC}(X,T) \subset RO(X,T)$, which implies (X,T) is extremally disconnected. □

THEOREM 3.6. *Let (X,T) be a space and let $U \in SO(X,T_s)$. Then $\mathrm{scl}_T\, U = \mathrm{scl}_{T_s} U$ and $\mathrm{Cl}_T(U) = \mathrm{Cl}_{T_s}(U)$.*

Proof. Since $SO(X,T_s) \subset SO(X,T)$ ([11]), then $U \in SO(X,T)$ and $\mathrm{scl}_T\, U = \mathrm{scl}_{T_s} U$ ([17]). Let $O \in T_s$ such that $O \subset U \subset \mathrm{Cl}_{T_s}(O)$. Then $O \in T$ and $\mathrm{Cl}_{T_s}(O) = \mathrm{Cl}_T(O)$ ([2]). Thus $\mathrm{Cl}_{T_s}(U) = \mathrm{Cl}_{T_s}(O) = \mathrm{Cl}_T(O)$ and $\mathrm{Cl}_T(U) = \mathrm{Cl}_T(O)$, which implies $\mathrm{Cl}_{T_s}(U) = \mathrm{Cl}_T(U)$. □

Combining the results above with the fact that, for a space (X,T), $\mathrm{SRO}(X,T) = \mathrm{SRO}(X,T_s)$ [12] gives the last result in the paper.

COROLLARY 3.2. *Let (X,T) be a space. Then the following are equivalent:*

(a) *(X,T) is extremally disconnected.*
(b) *$\mathrm{Cl}_T(\mathrm{Ext}_T(O)) = \mathrm{Ext}_T(O)$ for each $O \in \mathrm{FO}(X,T)$.*
(c) *$\mathrm{scl}_{T_s} U = \mathrm{Cl}_{T_s}(U)$ for each $U \in SO(X,T_s)$.*
(d) *$\mathrm{scl}_T\, U = \mathrm{Cl}_T(U)$ for each $U \in SO(X,T_s)$.*
(e) *$\mathrm{scl}_{\mathrm{FO}(X,T_S)} O = \mathrm{Cl}_{\mathrm{FO}(X,T_s)}(O)$ for each $O \in \mathrm{FO}(X,T_s)$.*
(f) *$\mathrm{scl}_{T_s} O = \mathrm{Cl}_{T_s}(O)$ for each $O \in \mathrm{FO}(X,T_s)$.*
(g) *$\mathrm{scl}_T\, O = \mathrm{Cl}_T(O)$ for each $O \in \mathrm{FO}(X,T_s)$.*
(h) *$\mathrm{scl}_T\, O = \mathrm{Cl}_T(O)$ for each $O \in T_s$.*

(i) $\mathrm{Cl}_T(\mathrm{Ext}_T(O)) = \mathrm{Ext}_T(O)$ *for each* $O \in T_s$.
(j) $\mathrm{scl}_{T_s} O = \mathrm{Cl}_{T_s}(O)$ *for each* $O \in \mathrm{SR}(X,T)$.
(k) $\mathrm{Cl}_{T_s}(O) = \mathrm{scl}_{T_s} O = \mathrm{Int}_T(\mathrm{Cl}_T(O)) \in RO(X,T)$ *for each* $O \in \mathrm{SR}(X,T)$.
(l) $(X, \mathrm{FO}(X,T_s))$ *is extremally disconnected.*
(m) $SO(X,T_s)$ *is a topology on* X.
(n) $T_sSR = T_s$.
(o) $T_sRC = T_s$.
(p) $T_sSO = SO(X,T_s)$.

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