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Mathematica Slovaca, Vol. 26 (1976), No. 2, 119--129

Persistent URL: http://dml.cz/dmlcz/128857

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CHARACTERIZATION OF DISTRIBUTIVE MULTILATTICES BY A BETWEENNESS RELATION

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Some authors have studied the following betweenness relation:

(1) $$(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b).$$

In the metric lattices this relation is equivalent to

(2) $$\varrho(a, x) + \varrho(x, b) = \varrho(a, b).$$

A characterization of lattices by the relation (1) is given in paper [3]. In the present paper an analogous characterization of distributive directed multilattices is given (Thm. 2). Following [4] we take the ternary relation defined by

(b) $$[(a \wedge x) \vee (x \wedge b)]_x = x, \quad (a \wedge x) \wedge (x \wedge b) \subset a \wedge b$$

as the starting point. In metric directed multilattices (b) is equivalent to (2) In distributive multilattices (b) holds iff the relation

(r) $$[(a \wedge x) \vee (x \wedge b)]_x = x = [(a \vee x) \wedge (x \vee b)]_x$$

is satisfied (see Thm. 1 and [6, Lemma 14]). In lattices (r) reduces to (1).

The author was stimulated by conversations with M. Kolibiar in developing this approach to the problem.

Basic concepts and properties

A multilattice [1] is a poset M in which the conditions (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$, and (b) $z \in M$, $z \geqq a$, $z \geqq b$, $z \leqq v$ implies $z = v$.

Analogously as in [1] denote by $(a \vee b)_h$ the set of all elements $v \in M$ from (i) and by $(a \wedge b)_d$ the set of all elements $u \in M$ from (ii) and define the sets:

$$a \vee b = \bigcup_{\substack{a \leqq h \\ b \leqq h}} (a \vee b)_h, \qquad a \wedge b = \bigcup_{\substack{d \leqq a \\ d \leqq b}} (a \wedge b)_d.$$

Let A and B be nonvoid subsets of M, then we define

$$A \vee B = \bigcup (a \vee b), \qquad A \wedge B = \bigcup (a \wedge b),$$

where $a \in A$ and $b \in B$. Troughout the paper we denote $[(a \vee x) \wedge (b \vee x)]_x = x$ $([(a \wedge x) \vee (b \wedge x)]_x = x)$, if $a, b, x \in M$ and $[(a \vee x) \wedge (b \vee x)]_x = \{x\}$ $([(a \wedge x) \vee (b \wedge x)]_x = \{x\})$.

A poset A is called upper (lower) directed if for each pair of elements $a, b \in A$ there exists an element $h \in A$ ($d \in A$) such that $a \leqq h$, $b \leqq h$ ($d \leqq a$, $d \leqq b$). The upper and lower directed poset A is called directed.

A multilattice M is modular [1] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leqq a \leqq h$, $d \leqq b \leqq b' \leqq h$, $(a \vee b)_h = h$ $(a \wedge b')_d = d$ we have $b = b'$.

A multilattice M is distributive [1] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leqq a, b, b' \leqq h$, $(a \vee b)_h = (a \vee b')_h = h$, $(a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$.

Let M be a multilattice and N a nonvoid subset of M. N is called a submultilattice [1] of M iff $N \cap (a \vee b)_h \neq \emptyset$ and $N \cap (a \wedge b)_d \neq \emptyset$ for every a, b, d, $h \in N$ satisfying $a \leqq h$, $b \leqq h$, $a \geqq d$, $b \geqq d$. It is obvious that each interval is a submultilattice.

The following definition and results are in [4]:

The multilattices M and M' are said to be isomorphic (denoted as $M \sim M'$) if the partially ordered set M is isomorphic with the partially ordered set M'.

Let M be a cardinal product of two posets M_1, M_2. M is upper (lower) directed iff M_1 and M_2 is upper (lower) directed. M is a multilattice iff M_1 and M_2 are multilattices. Let x_1, x_2 ($x_i \in M_i$) be Cartesian coordinates of any element $x \in M$. For all $a, b, h, v \in M$ $v \in (a \vee b)_h$ ($v \in (a \wedge b)_h$) iff $v_i \in (a_i \vee b_i)_{h_i}$ ($v_i \in (a_i \wedge b_i)_{h_i}$) for $i = 1, 2$.

§ 1.

Lemma 1. *If M is a distributive multilattice $a, b, u\ v \in M$, $u \in a \wedge b$, $v \in a \quad b$, then a mapping $f : \langle u, a\rangle \to \langle b, v\rangle$ with $f(x) = (b \vee x)_v$ for $x \in \langle u, a\rangle$ ($g : \langle b, v\rangle \to \langle u, a\rangle$ with $g(y) = (a \wedge y)_u$ for $y \in \langle b, v\rangle$) is a isomorphism of $\langle u, a\rangle$ ($\langle b, v\ \rangle$) onto $\langle b, v\rangle$ ($\langle u, a\rangle$).*

The proof of the Lemma 1 follows from 6.4, § 6 of paper [1].

Lemma 2. *If M is a distributive multilattice, $a, b, u, v \in M$, $u \in a \wedge b$, $v \in a \vee b$, then a mapping $m : \langle u, v\rangle \to \langle a, v\rangle \times \langle b, v\rangle$ with $m(x) = ((a \vee x)_v, (b \vee x)_v)$ for $x \in \langle u, v\rangle$ ($n : \langle a, v\rangle \times \langle b, v\rangle \to \langle u, v\rangle$ with $n(x_1, x_2) = (x_1 \wedge x_2)_u$ for $x_1 \in \langle a, v\rangle$ and $x_2 \in \langle b, v\rangle$) is a isomorphism of $\langle u, v\rangle$ ($\langle a, v\rangle \times b, v\rangle$) onto $\langle a, v\rangle \times \langle b, v\rangle$ ($\langle u, v\rangle$).*

This Lemma is a corollary of 3.2, 3.4, 3.7 of paper [2].

Remark. Evidently the dual assertion with respect to Lemma 2 is valid too. Throughout the paper we consider one of the isomorphisms from Lemma 1 (Lemma 2) if we have the isomorphism of any interval onto another interval (of any interval onto a direct product of two intervals).

Lemma 3. *Let M be a distributive multilattice, $a, b, u, v, x, x_1, y \in M$, $u \in a \wedge \wedge b$, $v \in a \vee b$, $u \leqq x \leqq v$, $x_1 \in (a \wedge x)_u$, $y \in (x_1 \vee b)_v$, then $x_1 \leqq x \leqq y$.*

Lemma 3 is dual to Lemma 12 from [5].

Lemma 4. *Let M be a distributive multilattice $a, b, p, q, r, x \in M$, $r \in a \vee x$, $r \in b \vee x$, $p \in a \wedge x$, $p \in a \wedge x$, $q \in b \wedge x$, $p \leqq q$, then $a \leqq b$.*

Proof. It is obvious that the intervals $\langle a, r\rangle$ and $\langle p, x\rangle$ are isomorphic. Denote by $s \in \langle a, r\rangle$ the image of the element $q \in \langle p, x\rangle$ in this isomorphism. There hold $(a \vee q)_r = s$ and $(s \wedge x)_p = q$. Evidently $r \in s \vee x$ and

$$(s \wedge x)_q = q = (x \wedge b)_q, \quad (s \vee x)_r = r = (x \vee b)_r.$$

By distributivity $s = b$ and consequently $a \leqq b$.

Lemma 5 ([5, Lemma 13]). *Let M be a distributive multilattice, $a, b, c, d, e, f \in M$. If $f \in e \vee d$, $c \in e \wedge d$, $d \in c \vee b$, $a \in e \wedge b$, $a \leqq c$, then $f \in e \vee b$.*

Theorem 1. *Let M be a directed distributive multilattice, $a, b, x \in M$. Then the following conditions are equivalent.*

$$(r) \qquad [(a \wedge x) \vee (b \wedge x)]_x = x = [(a \vee x) \wedge (b \vee x)]_x.$$

$$(s) \qquad (a \wedge x) \wedge (b \wedge x) \subset a \wedge b, \ (a \vee x) \vee (b \vee x) \subset a \vee b.$$

Proof. Let us choose $x_1 \in a \wedge x$, $x_2 \in b \wedge x$, $x_1' \in a \vee x$, $x_2' \in b \vee x$, $u \in x_1 \wedge x_2$, $v \in x_1' \vee x_2'$. First we prove that (r) implies (s). It is sufficient to show that $u \in a \wedge b$ (the proof of the assertion $v \in a \vee b$ is dual). First we show

$$(3) \qquad u \in a \wedge x_2, \ u \in b \wedge x_1, \ v \in a \vee x_2', \ v \in b \vee x_1'$$

$$(4) \qquad x_1' \in a \vee x_2, \ x_2' \in b \vee x_1, \ x_1 \in a \wedge x_2', \ x_2 \in b \wedge x_1'.$$

Choose $f \in (a \wedge x_2)_u$ and $g \in (a \wedge x)_f$. By (r)

$$(5) \qquad x \in g \vee x_2.$$

Next let us choose $h \in (x_1 \vee f)_x$. From the isomorphism of the intervals $\langle u, x_2\rangle$, $\langle x_1, x\rangle$ it follows that $(h \wedge x_2)_u = f$, hence

$$(6) \qquad f \in h \wedge x_2.$$

Since $f \in a \wedge x_2$, $f \leqq g \leqq a$, we get

$$(7) \qquad f \in g \wedge x_2.$$

From $x \in x_1 \vee x_2$ it follows that

$$(8) \qquad x \in h \vee x_2.$$

By distributivity and using (5), (6), (7), (8) we get $g = h$, hence $g = x_1$.

Consequently $f \leqq x_1$ and $f = u$. We have proved that $u \in a \quad x_2$. By symmetry and duality we get the other assertions from (3). The assertions in (4) can be proved by Lemma 5 and its dual.

Next we prove $u \in a \wedge b$. Let $r \in (a \wedge b)_u$, $s \in (a \vee b)_v$, $a_1 \in (x_1 \vee r)_a$, $a_2 \in \in (a_1 \vee x)_{x'_1}$, $c \in (r \vee x_2)_{a_2}$. From (3), (4) and the dual of Lemma 2 we get

$$\langle u, x_1 \rangle \sim \langle u, a \rangle \times \langle u, x_2 \rangle, \tag{9}$$

where $a \mapsto (a, u)$, $c \mapsto (r, x_2)$, $x \mapsto (x_1, x_2)$, $a_2 \mapsto (a_1, x_2)$. (We use the isomorphism of the intervals $\langle x_1, a \rangle$, $\langle x, x_1' \rangle$ and the isomorphism of the intervals $\langle x_2, x_1'$, $\langle u, a \rangle$, where $(a_1 \vee x)_{x_1'} = a_2$ and $(a_2 \wedge a)_{x_1} = a_1 = (a_2 \wedge a)_u$. Because $c \in \in (r \vee x_2)_{a_2}$ it follows that $c \in (r \vee x_2)_{x_1'}$ and we get $r \in (a \quad c)_u$.) Now we prove

$$a_2 \in c \vee x, \ x_2 \in c \wedge x. \tag{10}$$

Let $z \in (x \vee c)_{a_2}$. Evidently $z \in \langle u, x_1' \rangle$. In the isomorphism (9) $z \mapsto (z_1, z_2)$, where $z_1 \in (x_1 \vee r)_{a_1}$ and $z_2 \in (x_2 \vee x_2)_{x_2} = x_2$. Since $(x_1 \vee r)_{a_1} = a_1$, we get $z_1 = a_1$, $z_2 = x_2$. Since (a_1, x_2) corresponds to the element a_2 in theisomorphism (9), it follows $z = a_2$. The assertion $x_2 \in c \wedge x$ can be proved analogously.

Now we shall show that the assertion $u \in a \wedge b$ follows from

$$c \leqq s. \tag{11}$$

Indeed, if (11) holds from $c \in \langle r, s \rangle$, $r \in (a \wedge c)_u$ by Lemma 3 it follows that $r \leqq c \leqq b$. Hence we get $x_2 \leqq c \leqq b$, $x_2 \leqq c \leqq x_1'$. Since $x_2 \in x_1' \wedge b$, we get $c = x_2$ and therefore $r \leqq x_2$. Since $u \leqq r \leqq a$, $u \in a \wedge x_2$, we have $r = u$. This gives $u \in a \wedge b$.

It remains to prove (11). Let $a_3 = (a_2 \vee x_2')_v$. By Lemma 2

$$\langle x_2, v \rangle \sim \langle x_1', v \rangle \times \langle b, v \rangle. \tag{12}$$

In this isomorphism $x_1' \mapsto (x_1', v)$, $x \mapsto (x_1', x_2')$, $a_2 \mapsto (x_1', a_3)$, $s \mapsto (v, s)$ and $x_2 \mapsto (x_1', b)$. Let $b_2' \in (s \wedge x_2')_b$ and $w \in (s \wedge a_3)_{b'_2}$. It is obvious that $b_2' \in w \wedge x_2'$. As $v \in s \vee x_2'$, $b_2' \in s \wedge x_2'$, the intervals $\langle b_2', s \rangle$, $\langle x_2', v \rangle$ are isomorphic and from $w = (s \wedge a_3)_{b_2'}$ we get $a_3 = (w \vee x_2')_v$. Denote $d = (x_1' \wedge w)_{x_2}$. In the isomorphism (12) $d \mapsto (x_1', w)$. We shall prove that $d \in (a_2 \wedge s)_{x_2}$. Let $k \in (a_2 \wedge s)_{x_2}$. The element k corresponds to an element (k_1, k_2), where $k_1 \in (x_1' \quad v)_{x_1'}$ and $k_2 \in \in (a_3 \wedge s)_b$. Since $(x_1' \wedge v)_{x_1'} = x_1'$ and $(a_3 \wedge s)_b = w$, we have $k_1 = x_1'$ and $k_2 = w$. To the element (x_1', w) there corresponds the element d under the isomorphism (12), hence $k = d$ and

$$d \in a_2 \wedge s. \tag{13}$$

Next we denote $y = (x_1' \wedge b_2')_{x_2}$, then $y \mapsto (x_1', b_2')$ under the isomorphism (12) We shall show that

$$y \in (x \quad d)_{x_2}, \ a_2 \in (x \quad d)_{x_1'}. \tag{14}$$

Let $n \in (x \wedge d)_{x_2}$. The element n corresponds to an element (n_1, n_2) under the isomorphism (12) and $n_1 \in (x_1' \wedge x_1')_{x_1'} = x_1'$ $n_2 \in (x_2' \wedge w)_b = b_2'$. Since in (12) $y \mapsto (x_1', b_2')$, we get $n = y$ and consequently $y \in (x \wedge d)_{x_2}$. The assertion $a_2 \in \in (x \vee d)_{x_1'}$ can be proved analogously. From (10), (14) by Lemma 4 we get $c \leqq d$. This and (13) imply (11). We have proved that (r) implies (s).

By Lemma 2 and its dual (s) implies (r).

Let M be a multilattice, $a, b, c \in M$. We shall write abc, iff (r) and (s) is valid. From Theorem 1 it follows that in a directed distributive multilattice M we have abc iff (r) holds. Analogously as in [3] denote by $B(a, b)$ the set of all elements $x \in M$ for which axb holds.

Lemma 6. *If M is a multilattice, $a, b \in M$, then $B(a, b) = B(b, a)$ and $a, b \in B(a, b)$.*

Proof. The assertion follows directly from (r) and (s).

Lemma 7. *Let M be a multilattice, $a, b, x \in M$. If $a \leqq b$, then $x \in B(a, b)$ iff $a \leqq x \leqq b$, consequently $B(a, b) = \langle a, b \rangle$.*

Proof. Evidently from $a \leqq x \leqq b$ it follows that axb, hence $x \in B(a, b)$. Conversely, let $x \in B(a, b)$, $u \in a \wedge x$, $u' \in (b \wedge x)_u$. Then $x = (u \vee u')_x = u'$, hence $x \in b \wedge x$ and $x \leqq b$. The proof of the assertion $a \leqq x$ is dual.

Lemma 8. *Let M be a multilattice, $a, x, b \in M$. If $x \leqq a$ and $x \leqq b$, then $x \in B(a, b)$ iff $x \in a \wedge b$.*

Proof. Evidently from $x \in a \wedge b$ it follows that $x \in B(a, b)$. Conversely, let $x \in B(a, b)$. Since $a \vee x = a$, $b \vee x = b$, we get $x = [(a \vee x) \wedge (b \vee x)]_x = (a \wedge \wedge b)_x$, hence $x \in a \wedge b$.

Lemma 9. *Let M be a distributive directed multilattice. Then $B(a, b)$ is an interval iff $a \wedge b$ and $a \vee b$ are one-element sets.*

Proof. Let $B(a, b) = \langle u, v \rangle$. By Lemma 8 and its dual we get $u \in a \wedge b$ and $v \in a \vee b$. Let $u_1 \in a \wedge b$. By Lemma 8 it follows that $u_1 \in B(a, b)$, hence $u \leqq u_1$, consequently $u = u_1$. The proof of the assertion $a \vee b = \{v\}$ is dual.

Conversely, let $a \wedge b$ and $a \vee b$ be sets with exactly one element. Denote $a \wedge b = \{u\}$, $a \vee b = \{v\}$. We prove $B(a, b) = \langle u, v \rangle$. First we show $B(a, b) \subset \subset \langle u, v \rangle$. Let $x \in B(a, b)$. Bv theorem 1 we get

$$(a \wedge x) \wedge (b \wedge x) = u, \ (a \vee x) \vee (b \vee x) = v,$$

which implies $u \leqq x \leqq v$. Next we prove $\langle u, v \rangle \subset B(a, b)$. Let $x \in \langle u, v \rangle$, we show that ($r$) holds. First we prove

$$[(a \wedge x) \vee (b \wedge x)]_x = x.$$

Denote $x_1 \in (a \wedge x)_u$, $x_2 \in (b \wedge x)_u$. From the dual of Lemma 2 we get

$$\langle u, v\rangle \sim \langle u, a\rangle \times \langle u, b\rangle,$$

where $a \mapsto (a, u)$, $b \mapsto (u, b)$, $x \mapsto (x_1, x_2)$. Evidently $[(a \wedge x) \vee (b \wedge x)]_x = x$ iff

$$[\{(a, u) \wedge (x_1, x_2)\} \vee \{(u, b) \wedge (x_1, x_2)\}]_{(x_1, x_2)} = (x_1, x_2).$$

Since

$$[\{(a, u) \wedge (x_1, x_2)\} \vee \{(u, b) \wedge (x_1, x_2)\}]_{(x_1, x_2)} =$$

$$= [(a \wedge x_1, u \wedge x_2) \vee (u \wedge x_1, b \wedge x_2)]_{(x_1, x_2)} =$$

$$= [(x_1, u) \vee (u, x_2)]_{(x_1, x_2)} =$$

$$= (x_1 \vee u, u \vee x_2)_{(x_1, x_2)} = (x_1, x_2),$$

we get $[(a \wedge x) \vee (b \wedge x)]_x = x$. The assertion $[(a \vee x) \wedge (b \vee x)]_x = x$ follows by duality. Hence $\langle u, v\rangle \subset B(a, b)$.

Lemma 10. *Let the elements a, b, x of a distributive directed multilattice satisfy the condition*
(m) there exist elements $x_1 \in a \wedge x$, $x_2 \in b \wedge x$ and $u \in x_1 \wedge x_2$ such that $x \in x_1 \vee x_2$ and $u \in a \wedge b$.
Then axb.

Proof. 1. First we prove that (m) implies

$$[(a \vee x) \wedge (b \vee x)]_x = x, \ (a \vee x) \vee (b \vee x) \subset a \vee b.$$

Choose $y_1 \in a \vee x$, $y_2 \in b \vee x$, $y \in (y_1 \wedge y_2)_x$, $v \in y_1 \vee y_2$. We show that $y = x$. Clearly $u \in x_1 \wedge b$. By Lemma 5 we get

$$(15) \qquad y_2 \in x_1 \vee b.$$

Choose $r \in (a \wedge y_2)_{x_1}$. Then $u \in r \wedge b$. It implies (by (15) using modularity) $r = x_1$. Hence

$$(16) \qquad x_1 \in a \wedge y_2$$

and $x_1 \in a \wedge y$. From this and from $y_1 \in a \vee x$ we get $x = y$. Consequently (m) implies $[(a \vee x) \wedge (b \vee x)]_x = x$. Next we prove that $v \in a \vee b$. By Lemma 5 from (16) we get $v \in a \vee y_2$. From this and from (15), (16) and $u \in a \wedge b$ we have by Lemma 5 $v \in a \vee b$. Hence (m) implies $(a \vee x) \vee (b \vee x) \subset a \vee b$.

2. By the first part of the proof, (m) implies the dual condition of (m). Hence we get

$$(a \wedge x) \wedge (b \wedge x) \subset a \wedge b, \ [(a \wedge x) \vee (b \wedge x)]_x = x$$

by duality.

Lemma 11. *A directed multilattice M is distributive iff $B(u, v) = \langle u, v\rangle \subset B(a, b)$ for each $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$.*

Proof. Let M be a directed distributive multilattice. By Lemma 7 $B(u, v) = \langle u, v\rangle$. We prove that $\langle u, v\rangle \subset B(a, b)$. Let $x \in \langle u, v\rangle$, $x_1 \in (a \wedge x)_u$, $x_2 \in (b \wedge x)_u$. By the dual Lemma of Lemma 2 we get $(x_1 \vee x_2)_x = x$. Hence the assertion (m) holds, consequently $x \in B(a, b)$. It remains to prove the second part of Lemma 11. Let M be a non-distributive directed multilattice. Then M contains a submultilattice M_5 or N_5 of Figures 1 and 2. In M_5 and N_5 $x \in \langle u, v\rangle$ and $x \notin B(a, b)$. Hence if M is non-distributive, then $B(u, v) \subset B(a, b)$ do not hoid.

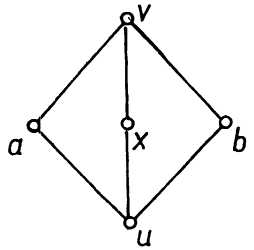

Fig. 1

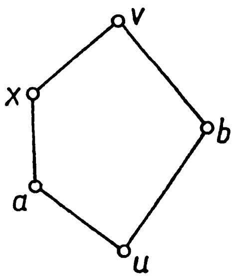

Fig. 2

Lemma 12. *Let M be a distributive directed multilattice, $a, b \in M$. Then*

$$B(a, b) = \bigcup_{\substack{u \in a \wedge b \\ v \in a \vee b}} \langle u, v\rangle.$$

Proof. By Lemma 11 we get

$$\bigcup_{\substack{u \in a \wedge b \\ v \in a \vee b}} \langle u, v\rangle \subset B(a, b).$$

We prove the converse inclusion. Let $x \in B(a, b)$. Denote $x_1 \in a \wedge x$, $x_2 \in b \wedge x$ $y_1 \in a \vee x$, $y_2 \in b \vee x$. By Theorem 1 $y_1 \vee y_2 \subset a \vee b$ and $x_1 \wedge x_2 \subset a \wedge b$. Let $u \in x_1 \wedge x_2$, $v \in y_1 \vee y_2$, then $u \in a \wedge b$, $v \in a \vee b$. Hence there exist $u \in a \wedge b$, $v \in a \vee b$ such that $x \in \langle u, v\rangle$.

Lemma 13. *Let M be a directed distributive multilattice, $a, b, x \in M$. $x \in B(a, b)$ iff $B(a, x) \cap B(b, x) = \{x\}$.*

Proof. Let $x \in B(a, b)$ and $y \in B(a, x) \cap B(b, x)$. Obviously $y \in B(a, x)$ and by Lemma 12 there exist $x_1 \in a \wedge x$ and $x_1' \in a \vee x$ such that

$$x_1 \leqq y \leqq x_1'. \tag{17}$$

Similarly $y \in B(b, x)$ and there exist $x_2 \in b \wedge x$, $x_2' \in b \vee x$ such that

$$x_2 \leqq y \leqq x_2'. \tag{18}$$

Choose $u \in x_1 \wedge x_2$, $v \in x_1' \vee x_2'$. Since $x \in B(a, b)$ by Theorem 1 $u \in a \wedge b$ and $v \in a \vee b$. By the dual assertion with respect to Lemma 2 we have

$$\langle u, v\rangle \sim \langle u, a\rangle \times \langle u, b\rangle, \tag{19}$$

where $x \mapsto (x_1, x_2)$, $x_1 \mapsto (x_1, u)$, $x_2 \mapsto (u, x_2)$, $x_1' \mapsto (a, x_2)$, $x_2' \mapsto (x_1, b)$ and $y \mapsto (y_1, y_2)$. From (17), (18), (19) it follows

$$(x_1, u) \leqq (y_1, y_2) \leqq (a, x_2),$$
$$(u, x_2) \leqq (y_1, y_2) \leqq (x_1, b).$$

From this we get $x_1 \leqq y_1$, $y_2 \leqq x_2$, $x_2 \leqq y_2$, $y_1 \leqq x_1$, consequently $x_1 = y_1$, $x_2 = y_2$ and $x = y$. We have proved that $x \in B(a, b)$ implies

$$B(a, x) \cap B(b, x) = \{x\}. \tag{20}$$

Conversely, let (20) hold. Choose $x_1 \in a \wedge x$, $x_2 \in b \wedge x$, $x_1' \in a \vee x$, $x_2' \in b \vee x$, $t \in (x_1 \vee x_2)_x$. Clearly $t \in \langle x_1, x_1'\rangle \subset B(a, x)$ and $t \in \langle x_2, x_2'\rangle \subset B(b, x)$. From (20) we get $t = x$. The assertion $x = (x_1' \wedge x_2')_x$ follows by duality. Consequently (20) implies (r), hence $x \in B(a, b)$.

Lemma 14. *Let M be a distributive directed multilattice, a, b, $c \in M$. Then abc and acb iff $b = c$.*

Proof. If abc and acb, then $b \in B(a, c)$ and $c \in B(a, b)$. By Lemma 13 $B(a, b) \cap B(b, c) = \{b\}$. Since $c \in B(a, b)$ and $c \in B(b, c)$ we get $c \in B(a, b) \cap B(b, c) = \{b\}$, consequently $c = b$. The converse assertion is obvious.

Lemma 15. *Let M be a distributive directed multilattice, a, b, c, $d \in M$. If abc and acd, then bcd.*

Proof. Let abc and acd, hence $b \in B(a, c)$ and $c \in B(a, d)$. Then we have

$$[(a \wedge b) \vee (b \wedge c)]_b = b = [(a \vee b) \wedge (b \vee c)]_b, \tag{21}$$
$$[(a \wedge c) \vee (c \wedge d)]_c = c = [(a \vee c) \wedge (c \vee d)]_c. \tag{22}$$

Choose $x_1 \in b \wedge c$, $x_2 \in c \wedge d$, $y_1 \in a \wedge b$, $u \in x_1 \wedge y_1$. From (21) we get by Theorem 1 $u \in a \wedge c$. Hence if $x_1 \in b \wedge c$, then there exists $u \in a \wedge c$ such that $u \leqq x_1$. From (22) it follows that $(u \vee x_2)_c = c$. Consequently we have

$$(x_1 \vee x_2)_c = c. \tag{23}$$

Let $x_1' \in b \vee c$, $x_2' \in c \vee d$. By duality we get

$$(x_1' \wedge x_2')_c = c. \tag{24}$$

(23) and (24) implies $c \in B(b, d)$, hence bcd.

§ 2.

Let A be a set with a ternary relation axb and with a specified element $o \in A$ such that the next conditions hold:

(i) $B(a, b) = B(b, a)$;

(ii) abc and acb iff $b = c$;

(iii) from abc and acd it follows that bcd;

(iv) for each two elements $a, b \in A$ there exist sets $\{u_i \mid i \in I\}$, $\{v_j \mid j \in J\}$ contained in $B(a, b)$ such that:

1. oav_j, obv_j, ou_ia, ou_ib for all $i \in I$ and $j \in J$;
2. for each $c \in B(a, b)$ there exist $i \in I$, $j \in J$ such that ou_ic, ocv_j;
3. if $d \in A$, oad, obd (oda, odb), then there exists $j \in J$ ($i \in I$) such that ov_jd (odu_i);
4. if $z \in A$, oaz, obz and ozv_j (oza, ozb and ou_iz) for some $j \in J$ ($i \in I$, then $z = v_j$ ($z = u_i$).

(v) if for $x \in A$ there exist $u_i, v_j \in B(a, b)$ such that ou_ix, oxv_j, then $x \in B(a, b)$.

Lemma 16. *Let A be a set with a ternary relation axb which satisfies* (i), (ii) *and* (iii). *If $a, b, x \in A$, $x \in B(a, b)$, then*

$$B(a, x) \cap B(x, b) = \{x\}.$$

Proof. Let $y \in B(a, x) \cap B(x, b)$. Clearly ayx, byx and we suppose axb. By (*iii*) from ayx and axb we get yxb. By (i) and (ii) from byx and yxb it follows that $y — x$.

Theorem 2. *Let A be a set with a specified element o and with a ternary relation axb such that* (i), (ii), (iii), (iv), (v) *are satisfied. Then there is a directed distributive multilattice on A with the least element o in which axb iff* (r) *is valid. Conversely, if in a directed distributive multilattice we define axb by* (r), *then the conditions* (i), (ii), (iii), (iv), (v) *are satisfied.*

Proof. Assume that (i) — (v) hold. First we prove that A is a poset. We define $a \leqq b$ iff oab, hence $a \in B(o, b)$. From (i) and (ii) it follows that $a, b \in B(a, b)$. Consequently oaa and the relation $\leqq$ is reflexive. Suppose $a \leqq b$ and $b \leqq a$, hence oab and oba. By (ii) $a = b$ and the relation $a \leqq b$ is antisymmetric. Let $a \leqq b$ and $b \leqq c$, hence oab and obc. By (iii) abc, therefore $b \in B(a, c)$. By (iv) for $b \in B(a, c)$ there exists $v_j \in B(a, c)$ such that oav_j, obv_j, ocv_j. Now by (iii) from oab, obv_j we get

$$abv_j, \tag{25}$$

from obc, ocv_j we get

$$bcv_j, \tag{26}$$

and finally (25) and av_jc imply

$$bv_jc. \tag{27}$$

From (26), (27) and (ii) it follows $c = v_j$. Since oav_j we get oac, hence $a \leqq c$ and the relation $\leqq$ is transitive. We proved that A is a poset. Since $o \in B(o, x)$ for each element $x \in A$, o is the least element of A.

The condition 1 of (iv) implies that A is a directed set.

Now we shall show that A is a multilattice. The property (a) from the definition of the multilattice follows from 1 and 3 of (iv). The property (b) from the definition of the multiiattice follows from 4 of (iv). Consequently

$$a \vee b = \{v_j \mid v_j \in B(a, b), j \in J\},$$

$$a \wedge b = \{u_i \mid u_i \in B(a, b), i \in I\}.$$

Next we suppose that $a, x, b \in A$ and axb, hence $x \in B(a, b)$. We shall show that (r) holds. Let $u_i \in a \wedge x$, $u_n \in b \wedge x$, $v_j \in a \vee x$, $v_k \in b \vee x$ where $u_i, v_j \in B(a, x)$ and $u_n, v_k \in B(b, x)$. We shall prove

$$(u_i \vee u_n)_x = x, \ (v_j \wedge v_k)_x = x.$$

Let $(u_i \vee u_n)_x = z$. Clearly $z \leqq x$, $u_i \leqq z$, $u_n \leqq z$, $x \leqq v_j$, $x \leqq v_k$. Hence $z \in \langle u_i, v_j \rangle$ and $z \in \langle u_n, v_k \rangle$. By (v) $z \in B(a, x)$ and $z \in B(b, x)$, consequently $z \in B(a, x) \cap B(b, x)$ and by Lemma 16 from $x \in B(a, b)$ we get $z = x$. The assertion $(v_j \vee v_k)_x = x$ follows by duality. Hence axb implies (r).

Now we shall show that A is a distributive multilattice. Let $a, b, b', u, v \in A$ and $u \leqq a \leqq v$, $u \leqq b \leqq v$, $u \leqq b' \leqq v$,

$$(a \vee b)_v = (a \vee b')_v = v, \ (a \wedge b)_u = (a \wedge b')_u = u.$$

Obviously $u, v \in B(a, b)$. By (v) $b' \in B(a, b)$ and (r) implies

$$[(a \wedge b') \vee (b' \wedge b)]_{b'} = b' \tag{28}$$

Let $t \in (b \wedge b')_u$. Since $(a \wedge b')_u = u$, from (28) we get $b' = (u \vee t)_{b'} \quad t$, hence $b' \leqq b$. Analogously we obtain $b \leqq b'$. We have proved that A is a dis tributive multilattice.

It remains to prove that (r) implies axb. Let (r) hold. By Lemma 12 $x \in B(a, b)$, hence axb.

The converse assertion follows from Lemma 6, Lemma 12, Lemma 14 and Lemma 15.

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Received November 28, 1974

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