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ON TWO COMBINATORIAL IDENTITIES

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I.

When investigating Bessel functions, the third author discovered a combinatorical identity which, after a small modification, can be written in the following form

$$\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\binom{k+j}{2r}\binom{k+2r-j}{2r}=(-1)^r\binom{k}{r}\binom{k+r}{r}, \tag{1}$$

where r and k are arbitrary non-negative integers. Special cases of (1) can be found in an implicit form in the last part of [3].

The proof presented in part II is due to L. Carlitz. In the last part J. Kaucký uses Vosmanský's identity (1) to give a new simple proof of the well-known Dixon's combinatorial identity

$$\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}^3=(-1)^r\frac{(3r)!}{(r!)^3}. \tag{2}$$

The different proofs of (2) can be found, e.g., in [2] § 6.3 or in [1] § 5.4.

II.

Put

$$R(n, k, r)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{n+j}{r}\binom{k+r-j}{r}.$$

Then

$$\sum_{n,k=0}^{\infty}R(n, k, r)x^ny^k=$$

$$=\sum_{j=0}^{r}(-1)^j\binom{r}{j}x^{r-j}(1-x)^{-r-1}y^j(1-y)^{-r-1}=$$

$$=(1-x)^{-r-1}(1-y)^{-r-1}\sum_{j=0}^{r}(-1)^j\binom{r}{j}x^{r-j}y^j$$

so that

$$\sum_{n,k=0}^{\infty} R(n, k, r)x^n y^k = (x-y)^r (1-x)^{-r-1}(1-y)^{-r-1} \tag{3}$$

Clearly it follows from (3) that

$$F(x, y, z) \equiv \sum_{r=0}^{\infty} z^r \sum_{n,k=0}^{\infty} R(n, k, r)x^n y^k =$$

$$=(1-x)^{-1}(1-y)^{-1} \sum_{0}^{\infty} z^r \frac{(x-y)^r}{(1-x)^r(1-y)^r} =$$

$$=(1-x)^{-1}(1-y)^{-1}\left\{1-\frac{(x-y)z}{(1-x)(1-y)}\right\}^{-1}$$

so that

$$F(x, y, z) = \{(1-x)(1-y)-(x-y)z\}^{-1}. \tag{4}$$

In the right-hand side of (4) replace y by $x^{-1}y$ and we get

$$\{(1-x)(1-x^{-1}y)-(x-x^{-1}y)z\}^{-1} =$$

$$=\{(1+y)-(1+z)x-(1-z)x^{-1}y\}^{-1} =$$

$$=(1+y)^{-1}\left\{1-\frac{(1+z)x}{1+y}-\frac{(1-z)x^{-1}y}{1+y}\right\}^{-1} =$$

$$=(1+y)^{-1}\sum_{s,t=0}^{\infty}\binom{s+t}{s}\frac{(1+z)^s(1-z)^t}{(1+y)^{s+t}} y^t x^{s-t}.$$

We retain only those terms that are independent on x, that is, those in which $s=t$. This gives

$$(1+y)^{-1}\sum_{s=0}^{\infty}\binom{2s}{s}\frac{(1-z^2)^s}{(1+y)^{2s}} y^s =$$

$$=(1+y)^{-1}\left\{1-4y\frac{1-z^2}{(1+y)^2}\right\}^{-1/2} = \{(1-y)^2+4yz^2\}^{-1/2} =$$

$$=(1-y)^{-1}\left\{1+\frac{4yz^2}{(1-y)^2}\right\}^{-1/2} =$$

$$=\sum_{r=0}^{\infty}(-1)^r\binom{2r}{r}\frac{y^r z^{2r}}{(1-y)^{2r+1}}.$$

Thus we have proved that

$$\sum_{r=0}^{\infty} z^r \sum_{k=0}^{\infty} R(k, k, r) y^k = \sum_{r=0}^{\infty}(-1)^r\binom{2r}{r}\frac{y^r z^{2r}}{(1-y)^{2r+1}}.$$

Hence

$$R(k, k, 2r+1) = 0$$

(which of course is clear from the definition) and

$$\sum_{k=0}^{\infty} R(k, k, 2r) y^k = (-1)^r \binom{2r}{r} \frac{y^r}{(1-y)^{2r+1}} =$$

$$= (-1)^r \binom{2r}{r} y^r \sum_{s=0}^{\infty} \binom{2r+s}{s} y^s = (-1)^r \binom{2r}{r} \sum_{k=r}^{\infty} \binom{k+r}{2r} y^k.$$

Hence finally

$$R(k, k, 2r) = (-1)^r \binom{2r}{r} \binom{k+r}{2r} = (-1)^r \binom{k}{r} \binom{k+r}{r}$$

in agreement with the asserted result.

III.

In well-known formula (see e.g. [2] chapter 6)

$$\sum_{j=0}^{p} \binom{p}{j} \binom{q}{j} \alpha^{p-j} \beta^j =$$

$$= \sum_{j=0}^{p} \binom{p}{j} \binom{q+j}{j} (\alpha - \beta)^{p-j} \beta^j, \quad p \leq q$$

we replace p, q, α, β by $p = q = 2r$, $\alpha = \Delta$, $\beta = -1$. We obtain

$$\sum_{j=0}^{2r} (-1)^j \binom{2r}{j}^2 \Delta^{2r-j} =$$

$$= \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} \binom{2r+j}{j} E^{2r-j},$$

where $E = \alpha - \beta = \Delta + 1$. Now E and Δ will be considered as operators. Then for the function

$$f(n) = \binom{2r+n}{2r}$$

we have

$$\Delta f(n) = \binom{2r+n+1}{2r} - \binom{2r+n}{2r} = \binom{2r+n}{2r-1}$$

so that

$$\Delta^{2r-j} f(n) = \binom{2r+n}{j}$$

and because

$$E^{2r} f(n) = \binom{4r-j+n}{2r}$$

we have

$$\sum_{j=0}^{2r}(-1)^j \binom{2r}{j}^2 \binom{2r+n}{j} =$$

$$= \sum_{j=0}^{r}(-1)^j \binom{2r}{j}\binom{2r+j}{2r}\binom{4r-j+n}{2r}.$$

Finally for $n=0$ and using Vosmanský's identity (1) with $k=2r$, we have

$$\sum_{j=0}^{2r}(-1)^j \binom{2r}{j}^3 = (-1)^r \binom{2r}{r}\binom{3r}{r} = (-1)^r \frac{(3r)!}{(r!)^3},$$

the stated Dixon formula.

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