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Mathematica Slovaca, Vol. 31 (1981), No. 3, 291--296

Persistent URL: http://dml.cz/dmlcz/128848

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SOME ANALOGUES FOR HIGHER MONOTONICITY OF THE SONIN—BUTLEWSKI—POLYA THEOREM

MILOŠ HÁČIK

1. Sonin's theorem (see [5] pg. 168) states that if $z(x)$ is a solution of $y''+f(x)y=0$, where $f(x)$ is positive continuous function, then the successive maxima of $[z(x)]^2$ form a decreasing or increasing sequence according to whether $f(x)$ is increasing or decreasing. An extension, due independently to Butlewski ([2] Théorème I, pg. 42) and to G. Polya ([6] footnote, pg. 166) concerns the more general equation

$$(gy')'+fy=0, \tag{1.1}$$

with $f(x)$ and $g(x)$ continuous. Their result says that if $z(x)$ is a solution of (1.1), the relative maxima of $[z(x)]^2$ form an increasing or decreasing sequence according to whether $f(x)g(x)$ is decreasing or increasing when $f(x)>0$, $g(x)>0$.

In [4] L. Lorch, M. E. Muldoon and P. Szego give a partial extension to a higher monotonicity corresponding to the hypothesis of $f(x)g(x)$ increasing (Theorems 4.1 and 4.3) and to the assumption that $f(x)g(x)$ is decreasing (Theorem 4.2).

In the present paper there is given a partial generalization of ([4] Theorems 4.1 and 4.2) by means of the well-known Kummer's transformation (see e.g. [8]) and results approached by the author [3] and R. Blaško [1].

2. Definitions and notations

A function $\varphi(x)$ is said to be n-times monotonic (or monotonic of order n) on an interval I if

$$(-1)^i\varphi^{(i)}(x)\geqq 0 \quad i=0, 1, \ldots, n; \quad x\in I. \tag{2.1}$$

For such a function we write $\varphi(x)\in M_n(I)$ or $\varphi(x)\in M_n(a, b)$ in the case when I is an open interval (a, b). In the case when the strict inequality holds throughout

(2.1) we write $\varphi(x)\in M_n^*(I)$ or $\varphi(x)\in M_n^*(a, b)$. We say that $\varphi(x)$ is completely monotonic on I if (2 1) holds for $n=\infty$.

A sequence $\{\mu_k\}_{k=1}^{\infty}$, denoted simply by $\{\mu_k\}$, is said to be n-times monotonic if

$$(2.2)\qquad (-1)^i\Delta^i\mu_k \geqq 0 \quad i=0, 1, \ldots, n;\ k=1, 2, \ldots.$$

Here $\Delta\mu_k=\mu_{k+1}-\mu_k$, $\Delta^2\mu_k=\Delta(\Delta\mu_k)$ etc. For such a sequence we write $\{\mu_k\}\in M_n$. In the case when the strict inequality holds throughout (2.2) we write $\{\mu_k\}\in M_n^*$. $\{\mu_k\}$ is called completely monotonic if (2.2) holds for $n=\infty$.

As usual, we write $[a, b)$ to denote the interval $\{x|a\leqq x<b\}$. $\varphi(x)\in C_n(I)$ means that $\varphi(x)$ has continuous derivatives including the n-th order.

$D_x[\varphi(x)]$ denotes the first derivatives $\dfrac{d\varphi(x)}{dx}$.

3. New results

Consider a differential equation (1.1) with $f(x)$ and $g(x)$ continuous, $g(x)>0$ for $a<x<\infty$.

Lemma 3.1. *Let $z(x)$ be a solution of (1.1) for $x\in(a, \infty)$. Suppose that $z(x)$ has consecutive zeros at $x_1, x_2, \ldots$ Let $f(x)$ and $g(x)$ be differentiable and $D_x[((g\psi')'+f\psi)\psi^3 g]$ integrable on (a, ∞) for a covenient function $\psi(x)>0$, $\psi(x)\in C_2(a, \infty)$. Then*

$$(3.1)\qquad [g(x_{k+1})\psi(x_{k+1})z'(x_{k+1})]^2-[g(x_k)\psi(x_k)z'(x_k)]^2= \\ =\int_{x_k}^{x_{k+1}}\left[\frac{z(x)}{\psi(x)}\right]^2 D_x[((g\psi')'+f\psi)\psi^3 g]\,dx.$$

Proof. The change of the variable

$$(3.2)\qquad \xi=\int_a^x\frac{du}{g(u)\psi^2(u)};\ \psi(x)>0,\quad \psi(x)\in C_2(a, \infty),$$

where the integral is assumed convergent, transforms (1.1) into

$$(3.3)\qquad \frac{d^2\eta}{d\xi^2}+\varphi(\xi)\eta=0,$$

where $\eta(\xi)=\dfrac{y(x)}{\psi(x)}$ and $\varphi(\xi)=((g\psi')'+f\psi)\psi^3 g$ (see e.g. [8] pg. 597). For this equation (3.3) there holds Wiman's formula ([9] pg. 125):

$$(3.4)\qquad [\eta'(\xi_{k+1})]^2-[\eta'(\xi_k)]^2=\int_{\xi_k}^{\xi_{k+1}}[\eta(\xi)]^2 D_\xi[\varphi(\xi)]\,d\xi,$$

where $\xi_1, \xi_2, \ldots$ are zeros of $\eta(\xi)=\frac{z(x)}{\psi(x)}$ corresponding to zeros $x_1, x_2, \ldots$ of $z(x)$.
Now

$$\eta'(\xi)=\frac{d\eta}{d\xi}=\frac{d\eta}{dx}\frac{dx}{d\xi}=[z'(x)\psi(x)-z(x)\psi'(x)]g(x)$$

and

$$D_\xi[\varphi(\xi)]=D_x[\varphi(\xi)]\frac{dx}{d\xi}.$$

Since $x_1, x_2, \ldots$ are zeros of $z(x)$, the assertion of lemma is obvious.

Theorem 3.1. *Let $y(x)$ and $z(x)$ be linearly independent solutions of (1.1) on (a, ∞). Let there hold on (a, ∞)*

$$0<\lim_{x\to\infty}[((g(x)\psi'(x))'+f(x)\psi(x))\psi^3(x)g(x)]\leqq\infty, \tag{3.5}$$

$x_1^{+}>a$, $x_1\geqq a$ for some $n\geqq 0$ and a convenient function $\psi(x)>0$, $\psi(x)\in C_2(a, \infty)$. Furthermore let there hold that $g(x)\psi^2(x)$ and $D_x[((g\psi')'+f\psi)\psi^3 g]$ are positive and belong to $M_n(a, \infty)$. If $x_1=a$, let the hypotheses of Lemma 3.1 hold on $[a, x_2)$. Then,

$$\{[g(x_{k+1})\psi(x_{k+1})z'(x_{k+1})]^2-[g(x_k)\psi(x_k)z'(x_k)]^2\}\in M_n^*, \tag{3.6}$$

and if $y(x)$ is continuous at x_1^{+}, then

$$\left\{\left[\frac{\psi(x_{k+1})}{y(x_{k+1})}\right]^2-\left[\frac{\psi(x_k)}{y(x_k)}\right]^2\right\}\in M_n^*. \tag{3.7}$$

Proof. For $n\geqq 1$, Lemma 3.1 asserts that (3.1) holds. Hence, (3.6) follows from ([3] Theorem 2.1) with $y(x)=z(x)$, $\lambda=2$ and $W(x) = g(x)\psi^2(x)D_x[((g\psi')' + f\psi)\psi^3 g]$. Abel's formula for the Wronskian shows that

$$y(x)z'(x)-y'(x)z(x)=\frac{c}{g(x)}, \tag{3.8}$$

where c is a non-zero constant. Multiplying (3.8) by $\psi(x)$ and remembering that $z(x_k)=0$ for $k=1, 2, \ldots$, we obtain that

$$[g(x_k)\psi(x_k)z'(x_k)]^2=c^2\left[\frac{\psi(x_k)}{y(x_k)}\right]^2$$

and (3.7) follows from (3.6).

If $n=0$, the result can be reduced to Makai's ([5], pg. 168) or Watson's versions ([7], pg. 518) of Sonin's theorem by applying a transformation of the type (3.2) to the equation (3.11).

Remark 1. Results (4.4) and (4.5) of ([4] Theorem 4.1) can be obtained if we choose in Theorem 3.1 $\psi(x) \equiv 1$.

Example 1. The Bessel function $y = \mathscr{C}_\nu(x)$ satisfies the differential equation

$$(xy')' + (x^2 - \nu^2)\frac{1}{x}y = 0 \quad x \in (0, \infty). \tag{3.9}$$

This equation does not fulfil the hypotheses of ([4] Theorem 4.1). But for $\psi(x) = \frac{1}{\sqrt{x}}$ the hypotheses of Theorem 3.1 of the present paper are fulfilled for $|\nu| \geqq \frac{1}{2}$ and $n = \infty$. Therefore

$$\{[\sqrt{c_{\nu k+1}}\,\mathscr{C}'_\nu(c_{\nu_1 k+1})]^2 - [\sqrt{c_{\nu k}}\,\mathscr{C}'_\nu(c_{\nu k})]^2\} \in M^*_\infty,$$

where $c_{\nu_1}, c_{\nu_2}, \ldots$ are consecutive zeros of $\mathscr{C}_\nu(x)$ and

$$\left\{\left[\frac{1}{\sqrt{c_{\nu k+1}}\,\mathscr{D}_\nu(c_{\nu_1 k+1})}\right]^2 - \left[\frac{1}{\sqrt{x_{\nu k}}\,\mathscr{C}_\nu(c_{\nu k})}\right]^2\right\} \in M^*_\infty,$$

where $\mathscr{D}_\nu(x)$ is a solution of (3.9) linearly independent on $\mathscr{C}_\nu(x)$.

Theorem 3.1 is useful even for the differential equation of the Jacobi type as follows.

Example 2. Consider a differential equation

$$y'' + (e^{a^2x} - \nu^2)y = 0 \quad a \neq 0.$$

We have not had any information about higher monotonicity properties of its solution for the time being. On choosing $\psi(x) = \exp\left(-\frac{1}{4}a^2x\right)$ we find that the hypotheses of Theorem 3.1 are fulfilled for $|\nu| > \frac{a^2}{4}$ on $(-\infty, \infty)$ and $n = \infty$.

Lemma 3.2. *Let $z(x)$ be a solution of (1.1) for $x \in (a, \infty)$. Suppose that $z'(x)$ has consecutive zeros at $x'_1, x'_2, \ldots$. Let both $f(x)$ and $g(x)$ be positive and differentiable for $x \in (a, \infty)$ and*

$$D_x\left[\left(\left(\frac{\psi'}{f}\right)' + \frac{\psi}{g}\right)\frac{\psi'^3}{f}\right]$$

integrable for $x \in (a, \infty)$. Then

$$\begin{aligned}&[z(x'_{k+1})\psi(x'_{k+1})]^2 - [z(x'_k)\psi(x'_k)]^2 = \\ &= \int_{x'_k}^{x'_{k+1}} \left[\frac{g(x)z'(x)}{\psi(x)}\right]^2 D_x\left[\left(\left(\frac{\psi'}{f}\right)' + \frac{\psi}{g}\right)\frac{\psi'^3}{f}\right]\mathrm{d}x.\end{aligned} \tag{3.10}$$

Proof. By a direct calculation one can obtain that if $y(x)$ and $z(x)$ are solutions of (1.1) with $f(x)>0$, $g(x)>0$, $f(x)$, $g(x)$ continuous on (a, ∞), then $g(x)y'(x)$, $g(x)z'(x)$ are solutions of the differential equation

$$\left(\frac{1}{f(x)}u'\right)'+\frac{1}{g(x)}u=0. \tag{3.11}$$

If we apply the proof of Lemma 3.1 to the differential equation (3.11) we obtain (3.10).

Theorem 3.2. *Let $y(x)$, $z(x)$ be linearly independent solutions of* (1.1) *on* (a, ∞), *where*

$$0<\lim_{x\to\infty}\left[\left(\left(\frac{\psi'}{f}\right)'+\frac{\psi}{g}\right)\frac{\psi^3}{f}\right]\leqq\infty.$$

Let

$$\frac{\psi^2(x)}{f(x)} \quad \text{and} \quad D_x\left[\left(\left(\frac{\psi'}{f}\right)'+\frac{\psi}{g}\right)\frac{\psi^3}{f}\right]$$

be positive and belong to $M_n(a, \infty)$, for some $n\geqq 0$ and function $\psi(x)>0$, $\psi(x)\in C_2(a, \infty)$, $x_1>a$ and $x_1'>a$.

Then

$$\{[z(x'_{k+1})\psi(x'_{k+1})]^2-[z(x'_k)\psi(x'_k)]^2\}\in M_n^*. \tag{3.12}$$

Proof of this theorem is similar to the one of Theorem 3.1 by using Lemma 3.2 and applying [1] Theorem 2.1.

Remark 2. If we choose $\psi(x)\equiv 1$, we obtain the result (4.9) of [4] Theorem 4.2.

Example 3. Consider a differential equation

$$\left(\frac{1}{x^2-v^2}y'\right)'+x^3y=0.$$

If we choose $\psi(x)=\sqrt[4]{x}$, then $\dfrac{\psi^2}{f}=\dfrac{1}{\sqrt{x^5}}\in M_\infty^*(0, \infty)$,

$$\varphi(\xi)=-\frac{11}{16}\frac{1}{x^7}-\frac{v^2}{x^2}+1\Rightarrow\varphi(\infty)=1$$

and

$$D_x(\varphi(\xi))=\frac{77}{16}\frac{1}{x^8}+\frac{2v^2}{x^3}\in M_\infty^*(0, \infty).$$

Thus, (3.12) holds for any real v and $n=\infty$.

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Received April 20, 1979

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НЕКОТОРЫЕ АНАЛОГИИ СОНИН–БУТЛЕВСКИ–ПОЛЯ ТЕОРЕМЫ ДЛЯ ВЫСШЕЙ МОНОТОННОСТИ

Милош Гачик

Резюме

В этой статье дедуцированы достаточные условия для того, чтобы последовательности

$$\{\Delta[g(x_k)\psi(x_k)z'(x_k)]^2\}_1^\infty;$$

$$\{\Delta[z(x_k')\psi(x_k')]^2\}_1^\infty; \quad \left\{\Delta\left[\frac{\psi(x_k)}{y(x_k)}\right]^2\right\}_1^\infty,$$

где $z(x)$, $y(x)$ линейно независимые решения дифференциального уравнения

$$(gy')' + fy = 0,$$

$x_1, x_2, \ldots$ – последовательность нулевых точек решения $z(x)$,
$x_1', x_2', \ldots$ – последовательность нулевых точек функции $z'(x)$
и Δ – первая дифференция, обладали свойством монотонности высшего порядка. Исследование было сделано при помощи трансформации Куммера для линейных дифференциальных уравнений второго порядка.