Mathematica Slovaca 27, 1977, No. 1, 43—45

ON A PROBLEM OF R. HALIN CONCERNING SUBGRAPHS OF INFINITE GRAPHS

BOHDAN ZELINKA

In [1] R. Halin proposes the following problem:

Let G, A be graphs. For all positive integers n let G contain n disjoint copies of the graph A. Does then G necessarily contain infinitely many disjoint copies of A ?

Here we shall answer this question negatively.

Theorem. *There exist graphs G, A such that for each positive integer n the graph G contains n disjoint copies of the graph A, but it does not contain infinitely many disjoint copies of A.*

Proof. At first we construct an auxiliary graph H. Let $\{V_n\}_{n=1}^{\infty}$ be an infinite sequence of pairwise disjoint sets such that $|V_n| = n$ for each positive integer n. Let $V = \bigcup_{n=1}^{\infty} V_n$. The vertex set of the graph H is V. Two vertices u, v of H are adjacent in H if and only if $u \in V_i$, $v \in V_j$, $i \neq j$. It is easy to see that for an arbitrary positive integer n the graph H contains an independent set with n vertices, namely the set V_n. On the other hand, each independent set of H must be contained in some of the sets V_n ; these sets are finite, therefore H does not contain an infinite independent set.

In the graph H each vertex is incident with countably many edges. If u is a vertex of H, we choose an arbitrary one-to-one correspondence between the set of edges incident with u and the set of all integers. By $e_r(u)$ we denote the edge corresponding to the integer r. We make this for all vertices of H.

Now we construct the graph A. Its vertices are a, b_n, c_n, d_n for all integers n, its edges are ab_n, b_nb_{n+1}, b_nc_n, c_nd_n for all integers n (Fig. 1).

Take countably many disjoint copies of A and choose a one-to-one correspondence between these copies and the verticos of H. The copy of A corresponding to a vertex u of H will be denoted by $A(u)$. If φ_u is an isomorphic mapping of A onto $A(u)$, we denote $\varphi_u(a) = a(u)$, $\varphi_n(b_n) = b_n(u)$, $\varphi_n(c_n) = c_n(u)$, $\varphi_n(d_n) = d_n(u)$ for all integers n.

Let u, v be two adjacent vertices of H. The edge uv is denoted simultaneously by $e_r(u)$ and $e_s(v)$, where r, s are some integers. We identify the vertex $c_r(u)$ of

$A(u)$ with the vertex $d_s(v)$ of $A(v)$ and the vertex $d_r(u)$ of $A(u)$ with the vertex $c_s(v)$ of $A(v)$. We make this for all pairs of adjacent vertices of H. The graph obtained in this way will be denoted by G.

Let A' be a subgraph of $\mathcal{J}$ isomorphic to A, let φ be an isomorphic mapping of A onto A'. The vertex $\varphi(a)$ has an infinite degree in A', thus also in G and therefore it must be equal to some $a(u)$, because the vertices $a(u)$ are the unique

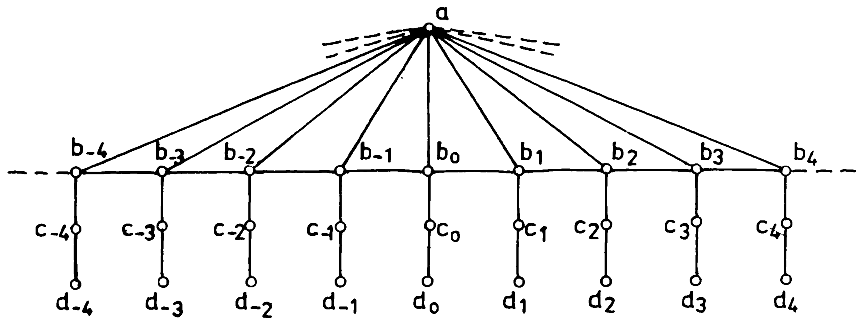

Fig. 1

vertices of infinite degrees in $\mathcal{J}$. Then the vertices $\varphi(b_n)$ are adjacent to $a(u)$, each of them has the degree four and they form a unique two-way infinite arc. The unique set of vertices in G with these properties in G is the set of all $b_n(u)$, therefore the set of all $\varphi(b_n)$ is equal to the set of all $b_n(u)$. Now it is easy to see that also the set of all vertices $\varphi(c_n)$ (or $\varphi(d_n)$) is equal to the set of all vertices $c_n(u)$ (or $d_n(u)$, respectively). We see that $A' = A(u)$ for some vertex u of H.

From the construction of $\mathcal{J}$ it follows that the graphs $A(u)$, $A(v)$ for $u \neq v$ have a common edge if and only if u and v are adjacent in H. If u and v are not adjacent in H, the graphs $A(u)$, $A(v)$ are disjoint (even vertex-disjoint). We have a one-to-one correspondence between independent sets of H and systems of pairwise disjoint copies of A. From the above mentioned assertion on independent sets of H the assertion of this theorem follows.

REFERENCE

[1] HALIN, R.: A Problem Concerning Infinite Graphs. In: Recent Advances in Graph Theory, Proc. Symp. Prague June 1974, Praha, Academia 1975.

Received June 13, 1975

Katedra matematiky
Vysoké školy strojní a textilní
Komenského 2
461 17 Liberec

ОБ ОДНОЙ ПРОБЛЕМЕ Р. ХАЛИНА КАСАЮЩЕЙСЯ ПОДГРАФОВ БЕСКОНЕЧНЫЦХ ГРАФОВ

Богдан Зелинка

Резюме

Р. Халин задал следующую проблему. Если G и A два графа таких, что G содержит n дизъюнктивных копий графа A для всякого натурального числа n, следует ли из этого, что G содержит бесконечно много дизъюнктивных копий графа A ? В настоящей статье на этот вопрос отвечается отрицательно.