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ON THE EXISTENCE OF A SOLUTION OF $F(x) = 0$ IN SOME COMPACT SETS

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0. Introduction

In this paper we consider the problem of the existence of a solution of a system of n equations in n real variables

$$F(x) = 0 \tag{1}$$

($F\colon \mathrm{cl}\, K \to \mathscr{R}^n$ continuous) in the closure $\mathrm{cl}\, K$ of an open, bounded subset K of the real n-dimensional space $\mathscr{R}^n$.

We use the homotopy approach to prove a theorem asserting the existence of a solution $\bar{x}$ of (1) such that $\bar{x} \in \mathrm{cl}\, K$. The proof is constructive for twice continuously differentiable maps on $U \subset \mathscr{R}^n$ ($\mathrm{cl}\, K \subset U$, U open) and it is based on a special form of the set K (described in Section 1). Further, we give an example where the assumptions of our existence theorem (Theorem 2) are weaker in comparison with the following commonly used

Theorem 1 [5, Theorem 6.3.4]. *Let K be an open bounded set in $\mathscr{R}^n$ and assume that $F\colon \mathrm{cl}\, K \to \mathscr{R}^n$ is continuous and satisfies $\langle F(x), x - x^0 \rangle \geqslant 0$ for some $x^0 \in K$ and all $x \in \partial K$ (where $\partial K = \mathrm{cl}\, K \setminus K$ denotes the boundary of K and $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ the scalar product in $\mathscr{R}^n$). Then $F(x) = 0$ has a solution in $\mathrm{cl}\, K$.*

1. Regular sets

We introduce here a class $\mathscr{K}$ of sets — we call them regular — which are given by finitely many inequalities and satisfy a regularity condition.

By $\mathscr{C}^k$ we denote the class of k-times continuously differentiable maps.

Definition 1. *An open, nonempty set of the form*

$$K = \{x \in \mathscr{R}^n \mid g_i(x) > 0 \ (i = 1, \ldots, m)\} \tag{2}$$

(where $g_i\colon \mathscr{R}^n \to \mathscr{R}$ are $\mathscr{C}^3$ for $i = 1, \ldots, m$) will be called regular iff

$$\mathrm{cl}\, K \text{ is compact} \tag{3}$$

and, moreover, the following regularity condition holds: for each point $x \in \partial K$ *there exists a direction* $z \in \mathscr{R}^n$ *such that*

$$\langle \nabla g_i(x), z \rangle > 0 \quad \textit{for } i \in J(x) \tag{4}$$

(*where* $\nabla g_i(x)$ *is the column vector of partial derivatives of* g_i *at* x *and* $J(x) = \{i \mid g_i(x) = 0\}$; *thus if* $x \in \partial K$, *then* $J(x) \neq 0$).

It is clear that $\mathscr{K}$ contains some convex sets (e.g. the interior of a unit ball $K = \{x \in \mathscr{R}^n \mid 1 - \|x\|^2 > 0\}$) and also some nonconvex sets (e.g. $K = \{x \in \mathscr{R}^2 \mid 4 - x_1^2 - (x_2 - x_1^2)^2 > 0\}$). The regularity condition (4) is in fact the Mangasarian – Fromovitz constraint qualification used in mathematical programming.

2. Barrier homotopy

Theorem 1 is usually proved using the degree theory (especially the homotopy invariance theorem for the Brouwer degree and the Brouwer fixed-point theorem [5]). We shall, however, pursue another approach based on the parametrized Sard's Theorem and the differential topology [2]. In our approach we use a special homotopy map (called barrier homotopy), which was originally used in [1] to construct numerically implementable homotopy methods for finding the Kuhn – Tucker points of mathematical programming problems with inequality constraints.

Definition 2. *Let* $K \in \mathscr{K}$ *and* $F: U \subset \mathscr{R}^n \to \mathscr{R}^n$ *be* $\mathscr{C}^2$, *U open*, $\operatorname{cl} K \subset U$ *and let* $P \subset \mathscr{R}^n$ *be open and nonempty. By the barrier homotopy we understand a map* $H: K \times [0, 1] \times P \to \mathscr{R}^n$, *where*

$$H(x, t, a) = (1 - t) \cdot Q(x, a) + t \cdot F(x) + t(1 - t) \cdot \sum_{i=1}^{m} \beta'(g_i(x)) \cdot \nabla g_i(x), \tag{5}$$

$\beta: \mathscr{R}^+ \to \mathscr{R}$ *is* $\mathscr{C}^3$ ($\mathscr{R}^+ = \{r \in \mathscr{R} \mid r > 0\}$), β' *is its first derivative, which we suppose to satisfy*

$$\lim_{s \downarrow 0} \beta'(s) = -\infty \tag{6}$$

$$\beta'(s) < 0 \quad \textit{for all } s > 0 \tag{7}$$

and $Q: \mathscr{R}^n \times P \to \mathscr{R}^n$ *is* $\mathscr{C}^2$ *satisfying for each* $a \in P$ *the following three conditions:*

$$\textit{there exists exactly one } x_a \in K \textit{ such that } Q(x_a, a) = 0, \tag{8}$$

$$\textit{the matrix } \mathrm{D}_x Q(x_a, a) \textit{ is regular}, \tag{9}$$

$$\textit{for each } x \in K \textit{ the matrix } \mathrm{D}_a Q(x, a) \textit{ is regular}, \tag{10}$$

($D_x Q$, $D_a Q$ *denote the Jacobi matrices of the partial differentials of* Q *with respect to* x, a, *respectively*).

The variables t, a are called the homotopy variable and the homotopy parameter, respectively.

Remark 1. Functions β satisfying (6), (7) are for example: $\beta(s) = -\ln s$, $\beta(s) = -\sqrt{s}$, $\beta(s) = s^{-1}$. Each of these functions can be used in Definition 1. The map Q can be chosen for any $K \in \mathscr{K}$, e.g. as follows

$$Q(x, a) = x - a, \quad P = K. \tag{11}$$

There may be, however, other and more suitable choices of Q for some sets K.

The following lemma gives the crucial technical result for our approach. It characterizes the limit points of the zero set $H_a^{-1}(0)$ of the barrier homotopy H_a (the value of the homotopy parameter is fixed). By a limit point of a set S a point from $\operatorname{cl} S \backslash S$ is understood.

Lemma 1. *Let* F *be a* $\mathscr{C}^2$ *map,* $K \in \mathscr{K}$ *and let* H *be the barrier homotopy. Then there is a dense subset* $\bar{P}$ *of* P *such that* $P \backslash \bar{P}$ *is of Lebesgue measure zero in* $\mathscr{R}^n$ *and for all* $a \in \bar{P}$ *there holds*:

(a) *The set* $H_a^{-1}(0)|_{K \times I} = \{(x, t) \in K \times I \mid H(x, t, a) = 0\}$ *is a differentiable submanifold of* $K \times I$ *of dimension* 1 (*where* I *denotes the open interval* (0, 1)),

(b) *any limit point* $(\bar{x}, \bar{t})$ *of the set* $H_a^{-1}(0)|_{K \times I}$ *satisfies one of the following two sets of properties*:

(b_0) $\bar{t} = 0$ *and there exists an* $u \in \mathscr{R}^m$ *such that*

$$\left.\begin{aligned} u_i &\geq 0 \\ g_i(\bar{x}) &\geq 0 \\ u_i \cdot g_i(\bar{x}) &= 0 \end{aligned}\right\} \quad i = 1, \ldots, m \qquad \begin{aligned} &(12.a) \\ &(12.b) \\ &(12.c) \end{aligned}$$

$$Q(\bar{x}, a) - \sum_{i=1}^{m} u_i \cdot \nabla g_i(\bar{x}) = 0, \tag{13}$$

(b_1) $\bar{t} = 1$ *and there exists an* $u \in \mathscr{R}^m$ *such that* (12) *and*

$$F(\bar{x}) - \sum_{i=1}^{m} u_i \cdot \nabla g_i(\bar{x}) = 0. \tag{14}$$

In the proof of this lemma we shall need

The Parametrized Sard's Theorem. *Let* $M \subset \mathscr{R}^m$, $P \subset \mathscr{R}^p$, $N \subset \mathscr{R}^n$ *be open and* $f\colon P \times M \to N$ *be* $\mathscr{C}^r$, *where* $r > \max(0, m - n)$. *If* $y \in N$ *is a regular value of* f (*i.e.* $Df(a, x)$ *is surjective at any* $(a, x) \in f^{-1}(y)$) *then there is a residual subset* $\bar{P} \subset P$ *such that* $P \backslash \bar{P}$ *is of Lebesgue measure zero and for each* $a \in \bar{P}$ *the value* y *is regular for* $f_a\colon M \to N$.

In most books on differential topology only a nonparametrized version is given:

Sard's Theorem [2, Theorem 3.1.3]. *Let* M *be a manifold of dimension*

m, $N \subset \mathscr{R}^n$ open and f: $M \to N$ be a $\mathscr{C}^r$ map, where $r > \max(0, m - n)$. *Then the set of critical values $y \in N$ of f (i.e. those y for which* $Df(x)$ *is not surjective for at least one $x \in f^{-1}(y)$) has the Lebesgue measure zero and the set of regular values $y \in N$ is residual and hence dense in N.*

We note that a residual set is a countable intersection of open dense sets and that a residual subset of a complete metric space is also dense.

The Parametrized Sard's Theorem can be obtained simply from the proof of the more general parametric transversality theorem (e.g. [2, Theorem 3.2.7]). This theorem, however, is usually formulated in such a way that it asserts only that $\bar{P}$ is residual. Because of the probability aspect of the constructive procedure based on this idea (where a random choice of a point from P is made), the conclusion on the zero measure of $P \setminus \bar{P}$ may be interesting. So we give here the proof of the Parametrized Sard's Theorem using the above (nonparametric) Sard's Theorem.

Proof. Let π: $f^{-1}(y) \subset P \times M \to P$ be the natural projection map, i.e. $\pi(a, x) = a$ for all $(a, x) \in f^{-1}(y)$. As y is a regular value of f the set $f^{-1}(y)$ is a differentiable submanifold of $P \times M$ and rank $Df = n$ for all $(a, x) \in f^{-1}(y)$. At each $(a, x) \in f^{-1}(y)$ the manifold $f^{-1}(y)$ can be locally parametrized by $(a^1, x^1) \in \mathscr{R}^{p+m-n}$ provided the square submatrix $(D_{a^2}f \; D_{x^2}f)$ of $(D_{a^1}f \; D_{a^2}f \; D_{x^1}f \; D_{x^2}f)$ is regular at $(a, x) = (a^1, a^2, x^1, x^2)$. In this case we can write

$$U \cap f^{-1}(y) = (a^1, \varphi_a(a^1, x^1), x^1, \varphi_x(a^1, x^1)),$$

where (φ_a, φ_x): $U^1 \to \mathscr{R}^n$ is $\mathscr{C}^r$ and U, U^1 are neighbourhoods of (a, x), (a^1, x^1), respectively. Consequently

$$\pi(a, x) = \begin{pmatrix} a^1 \\ \varphi_a(a^1, x^1) \end{pmatrix}$$

for $(a, x) \in U$, $(a^1, x^1) \in U^1$.

Now we prove that the set of regular values of π is exactly the set $\bar{P}$ of those $a \in P$ for which y is a regular value of f_a: $M \to N$. Then the Sard's Theorem applied to π implies the assertion of the Parametrized Sard's Theorem.

Let y be a regular value of $f_{\bar{a}}$, i.e. $D_x f$ has full rank n at any $(\bar{a}, x) \in f^{-1}(y)$. This implies that we can choose at such points $(\bar{a}, x)$ the local parametrization with $a^1 = a$. Then we have $\pi(a, x) = a$ and hence $\bar{a}$ is a regular value of π.

Let y be a critical value of $f_{\bar{a}}$, i.e. for at least one $(\bar{a}, \bar{x}) \in f^{-1}(y)$ any regular submatrix of $Df(\bar{a}, \bar{x})$ has to contain at least one column of $D_a f(\bar{a}, \bar{x})$. Let $(D_{a^2}f \; D_{x^2}f)$ be such submatrix. Moreover, let all columns of $D_{x^1}f$ be linear combinations of columns of $D_{x^2}f$. By the formula for computation of differentials we obtain for a component x_k of x^1:

$$D_{x_k}f(\bar{a}, \bar{x}) + D_{x^2}f(\bar{a}, \bar{x})\, D_{x_k}\varphi_x(\bar{a}^1, \bar{x}^1) + D_{a^2}f(\bar{a}, \bar{x})\, D_{x_k}\varphi_a(\bar{a}^1, \bar{x}^1) = 0.$$

As $(D_{x^2} f\, D_{a^2} f)$ is regular and $D_{x_k} f$ is in the range of $D_{x^2} f$ we have: $D_{x_k} \varphi_a \cdot$ $\cdot(\bar{a}^1, \bar{x}^1)$ is zero (for each component x_k of x^1). Thus $D_{x^1} \varphi_a(\bar{a}^1, \bar{x}^1)$ is a zero matrix, so

$$D\pi = \begin{pmatrix} E & 0 \\ D_{a^1}\varphi_a & D_{x^1}\varphi_a \end{pmatrix}$$

has not full rank at $(\bar{a}, \bar{x})$, i.e. $\bar{a}$ is a critical value of π. ∎

Proof of Lemma 1. From (10) it follows that $0 \in \mathscr{R}^n$ is a regular value of the barrier homotopy H. As H is $\mathscr{C}^2$ we can apply the Parametrized Sard's Theorem to H and in this way we obtain that there is a dense subset $\bar{P} \subset P$ with $P \setminus \bar{P}$ of measure zero such that 0 is a regular value of H_a: $K \times I \to \mathscr{R}^n$ for each $a \in \bar{P}$. By [2, Theorems 1.3.2, 1.3.3] the part (a) of this lemma is valid.

Let $a \in \bar{P}$, $(x^k, t^k) \xrightarrow[k \to \infty]{} (\bar{x}, \bar{t})$, where $H_a(x^k, t^k) = 0$ for each k. As the set $H_a^{-1}(0)$ is closed in $K \times I$ each its limit point $(\bar{x}, \bar{t})$ belongs to the boundary $\partial(K \times I)$. First we prove that $(\bar{x}, \bar{t}) \notin \partial K \times I$, which implies $\bar{x} \in \operatorname{cl} K$ and either $\bar{t} = 0$ or $\bar{t} = 1$. Then the properties (12), (13) or (12), (14) will be proved to hold at $\bar{x}$.

The first step $((\bar{x}, \bar{t}) \notin \partial K \times I)$ will be proved by contradiction. Let $(x^k, t^k) \to$ $\to (\bar{x}, \bar{t})$ and $\bar{x} \in \partial K$, $\bar{t} \in I$. Then $J(\bar{x}) \neq \emptyset$ and for $i \in J(\bar{x})$ we have $\lim_{k \to \infty} \beta'(g_i(x^k)) =$ $= -\infty$. Let $v^k = (v_1^k, \ldots, v_m^k)$, where $v_i^k = \beta'(g_i(x^k)) < 0$. Dividing $H_a(x^k, t^k) = 0$ by $\|v^k\|$ and passing to the limit for a subsequence of $k \to \infty$ we obtain that there exist finite nonpositive numbers $\bar{v}_i$ ($\|\bar{v}\| = 1$, i.e. $\bar{v}_i$ are not all zero) such that

$$\sum_{i \in J(\bar{x})} \bar{v}_i \nabla g_i(\bar{x}) = 0.$$

Taking the scalar product of the above equation with a vector z from the regularity property (4) we obtain

$$\sum_{i \in J(\bar{x})} \bar{v}_i \langle \nabla g_i(\bar{x}), z \rangle = 0,$$

which contradicts (4).

It remains to prove that if $(\bar{x}, \bar{t})$ is a limit point of $H_a^{-1}(0)|_{K \times I}$, then there exists $u \in \mathscr{R}^m$ such that either $\bar{t} = 0$, (12), (13) or $\bar{t} = 1$, (12), (14) are satisfied. Both cases can be treated in the same way, hence we do this only for the case $\bar{t} = 0$.

For each k there holds $H_a(x^k, t^k) = 0$. Passing to the limit for $k \to \infty$ (for a subsequence if necessary) we obtain

$$Q(\bar{x}, a) + \sum_{i=1}^{m} \nabla g_i(\bar{x}) \cdot \lim_{k \to \infty} (t^k(1 - t^k) \cdot \beta'(g_i(x^k))) = 0, \tag{15}$$

where the limits exist (nonpositive or $-\infty$) and for $i \notin J(\bar{x})$ there holds

$$\lim_{k \to \infty} (t^k(1 - t^k) \cdot \beta'(g_i(x^k))) = 0. \tag{16}$$

We prove now by contradiction that these limits are finite for $i \in J(\bar{x})$ as well. Let $u_i^k = -t^k(1 - t^k) \cdot \beta'(g_i(x^k))$ and $\|u^k\| \to \infty$. Dividing $H_a(x^k, t^k) = 0$ by $\|u^k\|$ and passing to the limit for $k \to \infty$ we obtain that nonnegative $u_i = \lim_{k \to \infty} u_i^k \|u^k\|^{-1}$ exist ($\|u\| = 1$) such that

$$\sum_{i=1}^{m} -u_i \nabla g_i(\bar{x}) = 0.$$

Analogously to the proof of part (a), this leads to a contradiction with the regularity property of K.

Now we can assume that a subsequence $\{j\}$ of $\{k\}$ was chosen such that $\lim_{j \to \infty} u_i^j = u_i \geqslant 0$ exists for each $i = 1, \ldots, m$. Clearly (12.a) is valid and also (12.b) because $\bar{x} \in \operatorname{cl} K$ implies $g_i(x) \geqslant 0$ for all $i = 1, \ldots, m$. For the subsequence $\{j\}$ we obtain from (15) the relation (13) and from (16)

$$g_i(\bar{x}) > 0 \;\Rightarrow\; u_i = 0.$$

The last implication is equivalent to (12.c). ■

3. Main Result

In this section the results of previous sections are used to prove the existence theorem:

Theorem 2. *Let $K \in \mathscr{K}$ and F: $\operatorname{cl} K \subset \mathscr{R}^n \to \mathscr{R}^n$ be a continuous map on $\operatorname{cl} K$. Let us suppose:*

(a) *there is a $\mathscr{C}^2$ map Q satisfying the conditions* (8—10) *of Definition* 2,

(b) *for each $a \in P$ and the map Q from (a) the conditions* (12), (13) *are satisfied only for the point $(\bar{x}, u) = (x_a, 0)$,*

(c) *if* (12), (14) *are satisfied for $(\bar{x}, u)$, then $u = 0$.*

Then $F(x) = 0$ has at least one solution in $\operatorname{cl} K$.

Proof. Let us first suppose that F is $\mathscr{C}^2$ on an open set containing $\operatorname{cl} K$. Then we can define a barrier homotopy H using the map Q satisfying (a), (b). By Lemma 1(a) for $a \in \bar{P}$ the set $H_a^{-1}(0)|_{K \times I}$ is a differentiable submanifold of $K \times I$ with $(x_a, 0)$ as one of its limit points. We call the connected component of this set, which has $(x_a, 0)$ as its limit point, the homotopy path. Because of (8), (9) and the implicit function theorem the homotopy path is in the neighbourhood of $(x_a, 0)$ a curve parametrizable by t. Hence the homotopy path is homeomorphic to an open interval with at least one limit point in $\partial(K \times I)$ different from $(x_a, 0)$. Due to Lemma 1(b) and assumption (b) of this theorem we have that all other limit points $(\bar{x}, \bar{t}) \neq (x_a, 0)$ satisfy $\bar{t} = 1$ and (12), (14). By (c) we obtain that $F(\bar{x}) = 0$.

Now let us suppose F to be only continuous on $\mathrm{cl}\, K$. The set $\mathrm{cl}\, K$ is compact, so we can approximate F uniformly on $\mathrm{cl}\, K$ with arbitrary small tolerance $\varepsilon_k > 0$ by a $\mathscr{C}^2$ map $F^k: \mathscr{R}^n \to \mathscr{R}^n$ [1, Theorem 6.2] such that

$$\max_{x \in \mathrm{cl}\, K} \|F(x) - F^k(x)\| \leqslant \varepsilon_k. \tag{17}$$

Hence there is a sequence $\{F^k\}_{k=1}^{\infty}$ of maps approximating F in the sense (17) such that $\varepsilon_k \to 0$. In an analogous way to the proof of this theorem for a $\mathscr{C}^2$ map F we can assert the existence of a limit point $(x^k, 1)$ of a homotopy path of the barrier homotopy for F^k. By Lemma 1(b) $u^k \in \mathscr{R}^m$ exists such that

$$\left.\begin{aligned} u_i^k &\geqslant 0 \\ g_i(x^k) &\geqslant 0 \\ u_i^k \cdot g_i(x^k) &= 0 \end{aligned}\right\} \quad i = 1, \ldots, m \qquad \begin{aligned} &(12'.\mathrm{a}) \\ &(12'.\mathrm{b}) \\ &(12'.\mathrm{c}) \end{aligned}$$

$$F^k(x^k) - \sum_{i=1}^{m} u_i^k \cdot \nabla g_i(x^k) = 0. \tag{14'}$$

By compactness of $\mathrm{cl}\, K$ we can choose a subsequence of $\{k\}$ such that $x^k \to \bar{x} \in$ $\in \mathrm{cl}\, K$. By the approximation property (17) and $\varepsilon_k \to 0$ we have

$$\lim_{k \to \infty} F^k(x^k) = F(\bar{x}). \tag{18}$$

We show by contradiction that $\{u_i^k\}$ is bounded for each $i = 1, \ldots, m$. If it is not so, i.e. if $|u_i^k| \xrightarrow[k \to \infty]{} \infty$ for some i, then $\|u^k\| \to \infty$. From (14′) divided by $\|u^k\|$ we obtain for $k \to \infty$ that a unit vector $\bar{u} \geqslant 0$ exists such that

$$\sum_{i=1}^{m} \bar{u}_i \cdot \nabla g_i(\bar{x}) = 0.$$

This, however, contradicts the regularity property (4). As $\{u^k\}$ is bounded we can choose a convergent subsequence such that $u^k \xrightarrow[k \to \infty]{} u$, $x^k \xrightarrow[k \to \infty]{} \bar{x}$. By (18) and the continuity of ∇g_i $(i = 1, \ldots, m)$ we obtain from (12′), (14′) that (12), (14) is valid. By (c) this implies $u = 0$, which implies $F(\bar{x}) = 0$. ■

4. Discussion

The proof of Theorem 2 is constructive with probability one for two times continuously differentiable maps F and $K \in \mathscr{K}$ provided a suitable map Q is known. Namely, having a suitable map Q satisfying (a), (b) of Theorem 2 we can define the barrier homotopy H. Let $a \in P$ be chosen at random. As $P \setminus \bar{P}$ has measure zero, with probability one we have $a \in \bar{P}$ and hence the homotopy path in $H_a^{-1}(0)$ will lead to the solution of $F(x) = 0$. Using a numerical path-following method we can compute a sufficiently good approximation of the solution to $F(x) = 0$.

To illustrate the application of Theorem 2 we give here two corollaries.

Corollary 1. *Let $K \in \mathscr{K}$ be a convex subset of $\mathscr{R}^n$. If the continuous map $F\colon \mathrm{cl}\, K \to \mathscr{R}^n$ satisfies*

$$(12), (14) \;\Rightarrow\; u = 0, \tag{19}$$

then there exists at least one point $\bar{x} \in \mathrm{cl}\, K$ such that $F(x) = 0$.

Proof. Let $Q(x, a) = x - a$ and $P = K$. For this choice (8—10) are obviously satisfied. Because of the convexity of K it holds that for each $a \in K$ there is no Kuhn—Tucker point of the mathematical programming problem

$$\mathrm{Min}\left\{\frac{1}{2}\|x - a\|^2 \,|\, x \in \mathrm{cl}\, K\right\}$$

on the boundary ∂K, i.e. no vectors $u \in \mathscr{R}^m$, $\bar{x} \in \partial K$ satisfying (12), (13) exist. Hence due to (8) the assumption (b) of Theorem 2 is also satisfied. As (19) is exactly the assumption (c), it is clear that Corollary 1 is a special case of Theorem 2. ■

Corollary 2. *Let $K = \{x \in \mathscr{R}^n \,|\, 1 - \|x\|^2 > 0\}$. If a continuous map $F\colon \mathrm{cl}\, K \to \mathscr{R}^n$ satisfies at any boundary point $x \in \partial K = \{x \in \mathscr{R}^n \,|\, \|x\| = 1\}$ the property*

$$F(x) = \lambda x \;\Rightarrow\; \langle F(x), x\rangle \geqslant 0 \tag{20}$$

(where $\lambda \in \mathscr{R}$), then there exists at least one $x \in \mathrm{cl}\, K$ such that $F(x) = 0$.

Proof. As K is convex, all we need to prove is that (19) is equivalent to (20). In our case (K is an interior of a unit ball) the assumption (19) has the form

$$\left.\begin{aligned} F(x) + 2ux &= 0 \\ u &\geqslant 0 \\ 1 - \|x\|^2 &\geqslant 0 \\ u(1 - \|x\|^2) &= 0 \end{aligned}\right\} \;\Rightarrow\; u = 0.$$

This implication holds trivially at any interior point. At a boundary point (19) is reduced to

$$\left.\begin{aligned} F(x) + 2ux &= 0 \\ u &\geqslant 0 \end{aligned}\right\} \;\Rightarrow\; u = 0.$$

This is equivalent to the fact that there is no $u > 0$ such that $F(x) = -2ux$. The last statement can be formulated as follows

$$F(x) = \lambda x \;\Rightarrow\; \lambda \geqslant 0,$$

which is clearly equivalent to (20). ■

R e m a r k 2. The assumption (20) is weaker than the assumption

$$\langle F(x), x \rangle \geqslant 0 \quad \text{for all } x \in \partial K \tag{21}$$

of the lemma [3, p. 53]. Namely, according to (20) $\langle F(x), x \rangle \geqslant 0$ need to be verified only at points for which $F(x) = \lambda x$.

The following example demonstrates that (20) is actually weaker than (21), i.e. there are F and K such that (21) is not satisfied and (20) is satisfied.

E x a m p l e.

$$F(x) = \begin{pmatrix} x_2 x_1 + x_2 \\ x_2 x_2 - x_1 \end{pmatrix}, \quad K = \{x \in \mathcal{R}^2 \mid \|x\|^2 < 1\}.$$

Let us look closer at the above example. As $\langle F(0, -1), (0, -1) \rangle = -1$ and $\langle F(0, 1), (0, 1) \rangle = 1$, so (21) is not satisfied. As no point $\|x\| = 1$ exists such that $F(x) = \lambda x$ for some $\lambda \in \mathcal{R}$ the implication (20) is satisfied.

We show now that in the above example even the assumptions of Theorem 1 are not satisfied (i.e. there is no $x_0 \in K$ such that $\langle F(x), x - x^0 \rangle \geqslant 0$ for all $x \in \partial K$).

It can be easily verified that $\langle F(x), x \rangle = x_2 \|x\|^2$ holds for each $x \in \operatorname{cl} K$. Hence the choice $x^0 = 0$ is not feasible.

Let $0 < \|x^0\| < 1$ be fixed and denote x^1, x^2 the two points of intersection of the line through x^0 and (0, 0) with the sphere $\|x\| = 1$. There holds $x^0 = \alpha_i x^i$ $(i = 1, 2)$, where $0 < \alpha_1 < 1$, $-1 < \alpha_2 < 0$. At these points there holds

$$\langle F(x^i), x^i - x^0 \rangle = (1 - \alpha_i) \langle F(x^i), x^i \rangle = (1 - \alpha_i) x_2^i \|x^i\|^2,$$

where $1 - \alpha_i > 0$.

If $x_2^0 \neq 0$, then x_2^2 and x_2^1 have different signs.

If $x^0 = (x_1^0, 0)$, $0 < \|x^0\| < 1$, then for $\|x\| = 1$ there holds

$$\langle F(x), x - x^0 \rangle = x_2(1 - x_1^0(x_1 + 1)). \tag{22}$$

For each $1 > |x_1^0| > 0$ fixed a positive $\bar{x}_1 < 1$ can be found such that $(1 - x_1^0 \cdot \cdot (\bar{x}_1 + 1)) > 0$. Hence the scalar product (22) has the opposite sign at the points $(\bar{x}_1, x_2)$, $(\bar{x}_1, -x_2)$ on the sphere $\|x\| = 1$.

R e m a r k 3. For the example of a nonconvex regular set given in Section 1 an analogous existence theorem to Corollary 1 can be proved. In this case one can take $Q(x, a) = -\nabla g(x - a)$, $P = \{a \in \mathcal{R}^2 \mid \|a\| < 0.25\}$, where $g(x) = = 4 - x_1^2 + (x_2 - x_1^2)^2$.

R e m a r k 4. In [1] the following statement is formulated in Problem 2.9: Let $K = \{x \in \mathcal{R}^n \mid \|x\| < 1\}$, F continuous on $\operatorname{cl} K$. If $F(x) \neq 0$ for all $x \in \operatorname{cl} K$, then there exist two points $x^i \in \operatorname{cl} K$ and constants $\lambda^i \in \mathcal{R}$ such that

$$F(x^i) = \lambda^i x^i, \quad \text{where} \quad \lambda^1 > 0,\ \lambda^2 < 0. \tag{23}$$

Following the hint in [1], the proof of this statement is by contradiction. As the assumption (23) is in contradiction with (20) the above statement from [1] is equivalent to Corollary 2. It is interesting that so far we have not seen this statement in literature in the form of an existence theorem.

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О СУЩЕСТВОВАНИИ РЕШЕНИЯ $F(x) = 0$ НА НЕКОТОРЫХ КОМПАКТНЫХ МНОЖЕСТВАХ

Pavol Meravý

Резюме

В статье изучается вопрос о существовании решения уравнения $F(x) = 0$ (F: $\mathrm{cl}\, K \to \mathscr{R}^n$ непрерывное отображение) на замыкании регулярного множества $K \subset \mathscr{R}^n$. В статье введены понятия регулярного множества и специального гомотопического отображения — барьерной гомотопии — используемого при доказательстве теоремы о существовании решения (Теорема 2). Доказательство Теоремы 2 является конструктивным для случая два раза непрерывно дифференцируемого отображения F. Приводится также пример показывающий, что для специального множества K условия Теоремы 2 слабее условий Теоремы 1 доказанной раньше на пример в [3].