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GENERALIZATION OF ŽELEZNÍK'S THEOREM ON EMBEDDINGS OF TENSOR PRODUCTS OF GRAPHS[1]

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ABSTRACT. If a simple graph G has a diagonalizable quadrilateral embedding, then given an arbitrary simple graph H, the tensor product $G \otimes H$ also has a diagonalizable quadrilateral embedding. Using a (generalized) rotation scheme for a diagonalizable quadrilateral embedding of the graph G and a rotation scheme for an embedding of the graph H, a (generalized) rotation scheme which determines a diagonalizable quadrilateral embedding of $G \otimes H$ is constructed.

Introduction

Throughout the paper, we will consider only simple finite graphs.

Let G be a connected graph, and let S be a closed surface. We say that a quadrilateral embedding $G \hookrightarrow S$ is *diagonalizable* if there exists an embedding $G' \hookrightarrow S$ with a 1-factor $F \subseteq G'$, such that $G = G' \setminus F$ and $G \hookrightarrow S$ is obtained from $G' \hookrightarrow S$ by removing F. Elements of F are called the *diagonals*.

In [12], Železník proved a theorem which says that, if a bipartite graph G has a diagonalizable quadrilateral embedding (DQE), orientable or nonorientable, then for an arbitrary graph H the tensor product $G \otimes H$ also has a diagonalizable quadrilateral embedding, orientable or nonorientable, respectively. Here we will give a constructive proof of this theorem and we will generalize it by allowing G to be non-bipartite.

We will adopt the notations of Železník [12], except that we will use the term *tensor product of graphs* instead of *conjunction of graphs* as it is now a more common usage. Thus, if G and H are graphs, the *tensor product* of graphs H and G is the graph $H \otimes G$ whose vertex set $V(H \otimes G)$ is the Cartesian

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product of the vertex sets $V(H)$ and $V(G)$, and whose edge set is $E(H \otimes G) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(H) \text{ and } v_1v_2 \in E(G)\}$. For more on the tensor product graphs see [9] and [8].

The graph embeddings in this paper will be described by (generalized) rotation schemes. Since we only consider simple graphs, the schemes will be in vertex form, i.e., they will be viewed as lists $Q = (q_1, \ldots, q_n)$, where each q_v is a cyclic permutation of the neighbours of the vertex v, and, if $q_v = (v_1v_2 \ldots v_k)$, then q_v will be usually represented by a list of the form $v\colon v_1v_2 \ldots v_k$ or $q_v\colon v_1v_2 \ldots v_k$.

Note that, in this paper, in a generalized embedding scheme, edges receive labels 0 and 1 instead of the more usual labels $+$ and $-$, respectively. Thus, we will refer to type 0 and type 1 walks instead of positive and negative walks.

For some terms not defined here, the reader is referred to Gross and Tucker [7], White [11], or Beineke and Wilson [2].

Lately, a great interest in embeddings of tensor product graphs on surfaces has arisen. For recent results in this area, we refer the reader to [1], [3], [4], [5], [6] (overview), and [10].

Example

The following rotation scheme (denoted by Q) determines a quadrilateral diagonalizable embedding of a non-bipartite graph G on the torus.

$$\begin{aligned}
a&\colon d\,b\,e\,c\\
b&\colon e\,c\,f\,a\\
c&\colon f\,a\,d\,b\\
d&\colon a\,f\,c\,e\\
e&\colon b\,d\,a\,f\\
f&\colon c\,e\,b\,d
\end{aligned}$$

The regions of this embedding (with diagonals overlined) are:

$$\begin{array}{ll}
acde & \bar{a}d\overline{f}c\\
\bar{b}e\bar{d}a & aefb\\
bfdc & \bar{c}f\bar{e}b
\end{array}$$

Now consider a graph H whose embedding on the sphere is described by the following rotation scheme. This scheme will be denoted by R:

$$
\begin{aligned}
1&: 5\,2\\
2&: 5\,3\,1\\
3&: 5\,2\\
4&: 5\\
5&: 4\,3\,2\,1
\end{aligned}
$$

The following rotation scheme (denoted by Q_1) describes a diagonalizable quadrilateral embedding of $G \otimes H$ in an orientable surface. The symbol v_i denotes the ordered pair (v,i).

$a_1: d_5\, b_5\, e_5\, c_5\, d_2\, b_2\, e_2\, c_2$

$a_2: d_5\, b_5\, e_5\, c_5\, d_3\, b_3\, e_3\, c_3\, d_1\, b_1\, e_1\, c_1$

$a_3: d_5\, b_5\, e_5\, c_5\, d_2\, b_2\, e_2\, c_2$

$a_4: d_5\, b_5\, e_5\, c_5$

$a_5: d_4\, b_4\, e_4\, c_4\, d_3\, b_3\, e_3\, c_3\, d_2\, b_2\, e_2\, c_2\, d_1\, b_1\, e_1\, c_1$

$b_1: e_5\, c_5\, f_5\, a_5\, e_2\, c_2\, f_2\, a_2$

$b_2: e_5\, c_5\, f_5\, a_5\, e_3\, c_3\, f_3\, a_3\, e_1\, c_1\, f_1\, a_1$

$b_3: e_5\, c_5\, f_5\, a_5\, e_2\, c_2\, f_2\, a_2$

$b_4: e_5\, c_5\, f_5\, a_5$

$b_5: e_4\, c_4\, f_4\, a_4\, e_3\, c_3\, f_3\, a_3\, e_2\, c_2\, f_2\, a_2 e_1\, c_1\, f_1\, a_1$

$c_1: f_5\, a_5\, d_5\, b_5\, f_2\, a_2\, d_2\, b_2$

$c_2: f_5\, a_5\, d_5\, b_5\, f_3\, a_3\, d_3\, b_3\, f_1\, a_1\, d_1\, b_1$

$c_3: f_5\, a_5\, d_5\, b_5\, f_2\, a_2\, d_2\, b_2$

$c_4: f_5\, a_5\, d_5\, b_5$

$c_5: f_4\, a_4\, d_4\, b_4\, f_3\, a_3\, d_3\, b_3\, f_2\, a_2\, d_2\, b_2\, f_1\, a_1\, d_1\, b_1$

$d_1: a_2\, f_2\, c_2\, e_2\, a_5\, f_5\, c_5\, e_5$

$d_2: a_1\, f_1\, c_1\, e_1\, a_3\, f_3\, c_3\, e_3\, a_5\, f_5\, c_5\, e_5$

$d_3: a_2\, f_2\, c_2\, e_2\, a_5\, f_5\, c_5\, e_5$

$d_4: a_5\, f_5\, c_5\, e_5$

$d_5: a_1\, f_1\, c_1\, e_1\, a_2\, f_2\, c_2\, e_2\, a_3\, f_3\, c_3\, e_3\, a_4\, f_4\, c_4\, e_4$

$e_1: b_2\, d_2\, a_2\, f_2\, b_5\, d_5\, a_5\, f_5$

$e_2: b_1\, d_1\, a_1\, f_1\, b_3\, d_3\, a_3\, f_3\, b_5\, d_5\, a_5\, f_5$

$e_3: b_2\, d_2\, a_2\, f_2\, b_5\, d_5\, a_5\, f_5$

$e_4: b_5\, d_5\, a_5\, f_5$

$e_5: b_1\, d_1\, a_1\, f_1\, b_2\, d_2\, a_2\, f_2\, b_3\, d_3\, a_3\, f_3\, b_4\, d_4\, a_4\, f_4$

$f_1: c_2\, e_2\, b_2\, d_2\, c_5\, e_5\, b_5\, d_5$

$f_2: c_1\, e_1\, b_1\, d_1\, c_3\, e_3\, b_3\, d_3\, c_5\, e_5\, b_5\, d_5$

$f_3: c_2\, e_2\, b_2\, d_2\, c_5\, e_5\, b_5\, d_5$

$f_4: c_5\, e_5\, b_5\, d_5$

$f_5: c_1\, e_1\, b_1\, d_1\, c_2\, e_2\, b_2\, d_2\, c_3\, e_3\, b_3\, d_3\, c_4\, e_4\, b_4\, d_4$

The regions of this embedding are the same, up to the erasure of subscripts, as the regions of G. For example, the regions which correspond to the region *adfc* are listed below on the left and the regions which correspond to the region *acde* are listed below on the right.

$$\begin{array}{ll} a_1d_5f_1c_2 & a_1c_5d_1e_5 \\ a_1d_2f_1c_5 & a_1c_2d_1e_2 \\ a_2d_5f_2c_1 & a_2c_5d_2e_5 \\ a_2d_3f_2c_5 & a_2c_3d_2e_3 \\ a_2d_1f_2c_3 & a_2c_1d_2e_1 \\ a_3d_5f_3c_2 & a_3c_5d_3e_5 \\ a_3d_2f_3c_5 & a_3c_2d_3e_2 \\ a_4d_5f_4c_5 & a_4c_5d_4e_5 \\ a_5d_4f_5c_1 & a_5c_4d_5e_4 \\ a_5d_3f_5c_4 & a_5c_3d_5e_3 \\ a_5d_2f_5c_3 & a_5c_2d_5e_2 \\ a_5d_1f_5c_2 & a_5c_1d_5e_1 \end{array}$$

Thus, the regions of the described embedding of $G \otimes H$ that correspond to the region *acde* are of the form $a_xc_yd_xe_y$, where x and y are the adjacent vertices in H. The regions of $G \otimes H$ that correspond to the region *adfc* are of the form $a_xd_yf_xc_z$, where the rotation at vertex x of H looks like

$$x\colon\ \ldots zy\ldots .$$

The difference arises (as we will show) because the region *acde* of G does not contain a diagonal while the region *adfc* does. Note that for every vertex x of H we can choose the diagonal $\{a_x, f_x\}$ inside any region of the form $a_xd_yf_xc_z$, so the diagonals of $G \otimes H$ can be chosen to be the same, up to the erasure of the subscripts, as the diagonals of G.

Now, let us take a look at the rotation scheme Q. It has the property that for each vertex v of G if $v\colon x_1\,x_2\ldots x_k$ is the local rotation at v, the diagonal incident with v lies between x_k and x_1.

We can assume that for each diagonal of G the sign $+$ is assigned to one of its ends, and the sign $-$ is assigned to the other end. In this example, we assigned the sign $+$ to the vertices a, b and c and the sign $-$ to the vertices f, d and e. We will describe how the rotation scheme Q_1 is obtained from rotation schemes Q and R. Let $v\colon v_1\,v_2\ldots v_n$ be the local rotation at a vertex v of G. If the rotation at a vertex x of H is $x\colon x_1\,x_2\ldots x_k$, and if v is assigned a $+$ sign, then the rotation at the vertex (v,x) of $G \otimes H$ looks like

$$(v,x)\colon (v_1,x_1)\,(v_2,x_1)\ldots(v_n,x_1)\,(v_1,x_2)\ldots(v_n,x_2)\ldots\ldots(v_1,x_k)\ldots(v_n,x_k).$$

If the vertex v is assigned a $-$ sign, then the local rotation at the vertex (v,x) looks like

$$(v,x)\colon (v_1,x_k)\,(v_2,x_k)\ldots(v_n,x_k)\ldots\ldots(v_1,x_2)\ldots(v_n,x_2)\,(v_1,x_1)\ldots(v_n,x_1).$$

The general construction is very similar.

Construction

Let a rotation scheme $Q = (q_1, q_2, \dots, q_n)$ describe a diagonalizable quadrilateral embedding of a graph G. Assume that the set of diagonals has been chosen (the set of diagonals is not necessarily uniquely determined by the quadrilateral embedding). In what follows, we shall always assume that the vertices are listed in the local rotations q_i, $i = 1, 2, \dots n$, in such a way that the following condition holds true:

If $q_v \colon x_1 x_2 \dots x_k$ is the local rotation at the vertex v of G, then the diagonal incident with v lies between x_k and x_1.

A (generalized) rotation scheme that describes a diagonalizable quadrilateral embedding written in this format will be called a (*generalized*) *DQE scheme.*

THEOREM 1. *Let $i \colon G \hookrightarrow S$ be a diagonalizable quadrilateral embedding of a graph G into some surface S and let H be a connected graph. Then there exists a diagonalizable quadrilateral embedding $j \colon G \otimes H \hookrightarrow S'$ of the tensor product $G \otimes H$ into some surface S'. Moreover, if i is an orientable embedding, then j is also an orientable embedding.*

Proof. Let $Q = (q_1, \dots, q_n)$ be a DQE rotation scheme for G, and let $P = (p_1, \dots, p_k)$ be any rotation scheme for H. Let us assign signs $+$ and $-$ to the vertices of G in such a way that the ends of each diagonal of G are assigned the opposite signs.

We now define a DQE rotation scheme Q_1 for $G \otimes H$. Let x be a vertex of G, and let the local rotation at x, determined by Q, be $x \colon x_1 x_2 \dots x_m$ (remember that the diagonal $\{x, y\}$ is contained inside the region $x_m x x_1 y$). Let a be a vertex of H, and let $a \colon a_1 a_2 \dots a_l$ be the local rotation at a determined by P. Then the rotation at a vertex (x, a) of $G \otimes H$ is

$$\begin{aligned}
(x,a)\colon\ & (x_1,a_1)(x_2,a_1)\dots(x_m,a_1)(x_1,a_2)(x_2,a_2)\dots(x_m,a_2)\dots \\
& \dots(x_1,a_l)(x_2,a_l)\dots(x_m,a_l) \qquad \text{if } x \text{ was assigned the } + \text{ sign,} \\
(x,a)\colon\ & (x_1,a_l)(x_2,a_l)\dots(x_m,a_l)\dots\dots(x_1,a_2)(x_2,a_2)\dots(x_m,a_2) \\
& (x_1,a_1)(x_2,a_1)\dots(x_m,a_1) \qquad \text{if } x \text{ was assigned the } - \text{ sign.}
\end{aligned}$$

We must now prove that the rotation scheme Q_1 obtained in this way determines a DQE of $G \otimes H$.

If $xyuv$ is a region of G that does not contain a diagonal, then for every vertex a of H and every vertex b incident with a, $(x,a)(y,b)(u,a)(v,b)$ is a region determined by Q_1, since Q_1 looks like

$$\begin{array}{ll} (x,a):\ \dots(v,b)(y,b)\dots, & \text{(since } yv \text{ is not a diagonal)} \\ (u,a):\ \dots(y,b)(v,b)\dots, & \\ (y,b):\ \dots(x,a)(u,a)\dots, & \\ (v,b):\ \dots(u,a)(x,a)\dots. & \end{array}$$

Let $\{x,u\}$ be a diagonal inside the region $xyuv$ in the embedding of G, and let $a\colon a_1a_2\dots a_l$ be the local rotation at a vertex a of H determined by P. Then, if we assume that the vertex x of G was assigned the $+$ sign, and the vertex u was assigned the $-$ sign, the rotations determined by Q_1 are

$$\begin{array}{ll} (x,a):\ (y,a_1)\dots(v,a_1)\,(y,a_2)\dots(v,a_2)\dots\dots(y,a_l)\dots(v,a_l), & \\ (u,a):\ (v,a_l)\dots(y,a_l)\dots\dots(v,a_2)\dots(y,a_2)\,(v,a_1)\dots(y,a_1), & \\ (y,a_i):\ \dots(x,a)\,(u,a)\dots \qquad \text{for} \quad i=1,2\dots k, & \\ (v,a_i):\ \dots(u,a)\,(x,a)\dots \qquad \text{for} \quad i=1,2\dots k. & \end{array}$$

Thus, the regions of $G\otimes H$ determined by Q_1 that correspond to the region $xyuv$ of G are of the form $(x,a)(y,a_i)(u,a)(v,a_{i-1})$, and it is clear that the diagonal $\{(x,a),(y,a)\}$ can be chosen inside any of these regions.

It is obvious that the embedding of $G\otimes H$ determined in this way is an orientable embedding and that it is diagonalizable. Note that this embedding depends on the way, $+$ and $-$ signs were assigned to the vertices.

If Q is a generalized DQE scheme for G, the proof is similar. The only difference is in the way, the $+$ and $-$ signs are assigned to the vertices of G. Let $xyuv$ be a region of G which contains a diagonal $\{x,u\}$. If the walk xyu is a type 0 walk, then the vertices x and u are assigned opposite signs. If the walk xyu is a type 1 walk, then the vertices x and u are assigned the same sign. Note that this assignment is well defined, since the boundary walk of a quadrilateral region of G must be of type 0. Also, if (u,v) is a type 1 edge in the embedding of G, then every edge of the form $(u,a)(v,b)$ in the embedding of $G\otimes H$ is defined to be a type 1 edge. □

Note that even if the generalized rotation scheme Q for G determines a nonorientable embedding of G, the generalized rotation scheme Q_1 can determine an orientable embedding of $G\otimes H$. The generalized rotation scheme Q_1 determines a nonorientable embedding of $G\otimes H$ if and only if there exists a positive integer n such that, in the embedding of G which is determined by the generalized DQE scheme Q, there is a type 1 closed walk of length n and there is a closed walk of length n in H.

This construction is based on the inductive proof of the above theorem for bipartite graphs H given by Železník. In his proof, he starts with an embedding of $G\otimes K_2$, where he considers K_2 to be an edge of H, and then he proceeds by induction, constructing a DQE for $G \otimes H_k$, where H_k are subgraphs of H such that $H_1 = K_2$, and H_{k+1} is obtained from H_k by adding one edge of H to H_k. The rotation scheme for $G \otimes H$ that looks like the one defined above can be obtained by choosing an appropriate set of diagonals at each inductive step. For example, in the above example, the diagonal $\{a_2, c_2\}$ can be chosen inside any of the regions $a_2d_5f_2c_1$, $a_2d_3f_2c_5$ and $a_2d_1f_2c_3$, but, if we want to add another edge $\{2,4\}$ to H so that the local rotations at 2 and 4 look like 2. 5 3 4 1 and 4: 5 2, and all the other local rotations remain the same, we must choose the diagonal $\{a_2, c_2\}$ inside the region $a_2d_1f_2c_3$ and similarly for every diagonal of the form $\{x_2, y_2\}$ or $\{x_4, y_4\}$.

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