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b-EQUIVALENT MULTILATTICES

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The aim of this paper is to investigate the b-equivalence of multilattices. The b-equivalence is a generalization of the m-equivalence, investigated by M. Kolibiar [4] and also a generalization of the graphic isomorphism, studied by J. Jakubík [3]. The method of this paper is a modification of the methods used in [3] and [4]. The main result of the paper is the following theorem: Directed distributive multilattices M, M' are b-equivalent iff there exist multilattices M_1, M_2 such that M is isomorphic with $M_1 \times M_2$, and M' is isomorphic with $M_1 \times M_2^{\sim}$.

Basic concepts and properties

A multilattice [1] is a poset M in which the conditions (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$, and (b) $z \in M$ $z \leqq v, z \geqq a, z \geqq b$ implies $z = v$.

Analogously as in [1] denote by $(a \vee b)_h$ the set of all elements $v \in M$ from (i) and by $(a \wedge b)_d$ the set of all elements $u \in M$ from (ii) and define the sets:

$$a \vee b = \bigcup_{\substack{a \leqq h \\ b \leqq h}} (a \vee b)_h, \qquad a \wedge b = \bigcup_{\substack{d \leqq a \\ d \leqq b}} (a \wedge b)_d.$$

Let A and B be nonvoid subsets of M, then we define

$$A \vee B = \bigcup (a \vee b), \qquad A \wedge B = \bigcup (a \wedge b),$$

where $a \in A$ and $b \in B$. In the whole paper we denote $[(a \vee x) \wedge (b \vee x)]_x = x([(a \wedge x) \vee (b \wedge x)]_x = x)$ if $a, b, x \in M$ and $[(a \vee x) \wedge (b \vee x)]_x = \{x\}$ $([(a \wedge x) \vee (b \wedge x)]_x = \{x\})$.

A poset A is called upper (lower) directed if for each pair elements $a, b \in A$ there exists an element $h \in A$ $(d \in A)$ such that $a \leqq h, b \leqq h$ $(d \leqq a, d \leqq b)$. The upper and lower directed poset A is called directed.

A multilattice M is modular [1] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leqq a \leqq h$, $d \leqq b \leqq b' \leqq h$, $(a \vee b)_h = h$, $(a \wedge b')_d = d$ we have $b = b'$.

A multilattice M is distributive [1] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leqq a, b, b' \leqq h$, $(a \vee b)_h = (a \vee b')_h = h$, $(a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$.

Let M be a multilattice and N a nonvoid subset of M. N is called a submultilattice [1] of M iff $N \cap (a \vee b)_h \neq \emptyset$ and $N \cap (a \wedge b)_d \neq \emptyset$ for every $a, b, d, h \in N$ satisfying $a \leqq h$, $b \leqq h$, $a \geqq d$, $b \geqq d$. It is obvious that each interval is a submultilattice.

The following definition and results are in [4]:

The multilattices M and M' are said to be isomorphic (denoted as $M \sim M'$) if there exists a bijection f of M onto M' satisfying: $x \leqq y$ iff $f(x) \leqq f(y)$ $(x, y \in M)$.

Let M be a Cartesian product of two posets M_1, M_2. M is upper (lower) directed iff M_1 and M_2 is upper (lower) directed. M is a multilattice iff M_1 and M_2 are multilattices. Let x_1, x_2 $(x_i \in M_i)$ be Cartesian coordinates of any element $x \in M$. For all $a, b, h, v \in M$, $v \in (a \vee b)_h$ $(v \in (a \wedge b)_h)$ if and only if $v_i \in (a_i \vee b_i)_{h_i}$ $(v_i \in (a_i \wedge b_i)_{h_i})$ for $i = 1, 2$.

b-equivalence of multilattices

Let M be a directed multilattice and $a, b, x \in M$. We say that x is between a and b and write axb if

$$(b) \qquad [(a \wedge x) \vee (b \wedge x)]_x = x, \quad (a \wedge x) \wedge (b \wedge x) \subset a \wedge b.$$

Definition. *Directed multilattices M, M' are said to be b-equivalent if there exists a bijection f of M onto M' satisfying axb iff $f(a)f(x)f(b)$. The bijection f is called a b-equivalence.*

Let M, M' be directed b-equivalent multilattices and $x \in M$. An element $x' \in M'$ denotes the image of the element x under the given b- equivalence. We denote a partial ordering and multioperations in the multilattice M by $\leqq, \wedge, \vee$ and in M' by $\subseteq, \cap, \cup$.

In Lemma 1 and Lemma 2 M denotes a directed multilattice.

Lemma 1. *Let $a, b, x \in M$. If $a \leqq b$, then axb iff $a \leqq x \leqq b$.*

Proof. Evidently, from $a \leqq x \leqq b$ it follows that axb. Conversely, let $a \leqq b$, axb, $u \in a \wedge x$, $z \in (b \wedge x)_u$. From axb it follows that $(u \vee z)_x = x$, $u \wedge z \subset a \wedge b$. Since $u \vee z = z$, we get $z = x$, $x \in b \wedge x$ and $x \leqq b$. Since $u \wedge z = u$, we get $u \in a \wedge b = a$, hence $u = a$ and $a \leqq x$.

Lemma 2. *Let $a, b, x \in M$. If $x \leqq a, x \leqq b$ $(a \leqq x, b \leqq x)$, then axb iff $x \in a \wedge b$ $(x \in a \vee b)$.*

Proof. Evidently, from $x \in a \wedge b$ it follows that axb. Conversely, if $x \leqq a$, $x \leqq b$, axb, then from (b) it follows that $x = x \wedge x = (a \wedge x) \wedge (b \wedge x) \subset a \wedge b$, hence $x \in a \wedge b$. Next we show the validity of the dual assertion. Evidently, from $x \in a \vee b$ it follows that axb. Conversely if $a \leqq x$, $b \leqq x$, axb, then $a \wedge x = a$, $b \wedge x = b$. From (b) it follows that $x = [(a \wedge x) \vee (b \wedge x)]_x = (a \vee b)_x$, hence $x \in a \vee b$.

We say that the interval $\langle u, v\rangle$, $u \leqq v$, $u, v \in M$ is preserved (is reversed) [3] if $u' \subseteq v'(v' \subseteq u')$ in M'; the one-element interval $\{u\} = \langle u, u\rangle$ is preserved and reversed at the same time.

In Lemma 3, Lemma 4 and Lemma 5 M and M' denote directed b-equivalent multilattices.

Lemma 3. *Let $a, b, u, v \in M$. If $u \leqq a \leqq b \leqq v$ and the interval $\langle u, v\rangle$ is preserved (is reversed), then the interval $\langle a, b\rangle$ is preserved (is reversed).*

Proof. By Lemma 1 ubv, uab. Hence $u'b'v'$, $u'a'b'$ and by Lemma 1 $u' \subseteq b' \subseteq \subseteq v'$, $u' \subseteq a' \subseteq b'$. We have proved that the interval $\langle a, b\rangle$ is preserved. The assertion in the brackets can be proved analogously.

Lemma 4. *Let $a, x, b \in M$. Then (1) if $a \leqq b$ and $x' \in a' \cap b'(x' \in a' \cup b')$, then $a \leqq x \leqq b$; (2) if $a \leqq x \leqq b$,. $x' \subseteq a'$, $x' \subseteq b'$ $(a' \subseteq x', b' \subseteq x')$, then $x' \in a' \cap b'$ $(x' \in a' \cup b')$.*

Proof. (1) $a'x'b'$ follows from Lemma 2. Consequently axb. By Lemma 1 $a \leqq x \leqq b$. (2) We get axb by Lemma 1. Hence $a'x'b'$. The relation $x' \in a' \cap b'$ follows by Lemma 2.

The other statements follow by duality.

Lemma 5. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$. If the interval*
$a, v\rangle$ ($\langle u, b\rangle$) is preserved and the interval
$\langle b, v\rangle$ ($\langle u, a\rangle$) is reversed, then the interval
$u, b\rangle$ ($\langle a, v\rangle$) is preserved and the interval
$u, a\rangle$ ($\langle b, v\rangle$) is reversed.

Proof. If $\langle a, v\rangle$ is preserved and $\langle b, v\rangle$ is reversed, then we get $a' \subseteq v' \subseteq b'$. Since aub, $a'u'b'$, we get $a' \subseteq u' \subseteq b'$ by Lemma 1. Hence, the interval $\langle u, a\rangle$ is reversed and the interval $\langle u, b\rangle$ is preserved. The proof of the second part of Lemma 5 is analogous.

In Lemma 6, Lemma 7, Lemma 8 and Lemma 9 M and M' denote directed distributive b-equivalent multilattices.

Lemma 6. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$. If the intervals $\langle a, v\rangle$, $\langle b, v\rangle$ or the intervals $\langle u, a\rangle$, $\langle u, b\rangle$ are preserved (are reversed), then the interval $\langle u, v\rangle$ is preserved (is reversed).*

Proof. Let $\langle a, v\rangle$, $\langle b, v\rangle$ be preserved, $r' \in a' \cap u'$, $s' \in b' \cap u'$. By (1) of Lemma 4 $u \leqq r \leqq a$, $u \leqq s \leqq b$, hence $\langle u, r\rangle$, $\langle u, s\rangle$ are reversed and $\langle r, v\rangle$, $\langle s, v\rangle$ are preserved. Let

$$t \in (r \vee s)_v. \tag{3}$$

By Lemma 3 the intervals $\langle r, t\rangle$, $\langle s, t\rangle$ are preserved. Let $w' \in t' \cup u'$. We have $t'w'u'$, twu and by Lemma 1 $u \leqq w \leqq t$. Hence the interval $\langle u, w\rangle$ is preserved and the interval $\langle w, t\rangle$ is reversed. Since $u \in a \wedge b$, then

$$(4) \qquad u \in r \wedge s .$$

Since $r' \subseteq t' \subseteq u'$, $r \leqq t$, $w \leqq t$, $r' \subseteq u' \subseteq w'$, $u \leqq r$, $u \leqq w$, then by (2) of Lemma 4

$$(5) \qquad t \in r \vee w, \qquad u \in r \wedge w .$$

Since M and M' are distributive, from (3), (4), (5) we get $w = s$, hence $w' = s'$ and from $s' \subseteq u' \subseteq w'$ we get $s = u$. We have proved that the interval $\langle u, v\rangle$ is preserved. Analogously we can verify that if $\langle u, a\rangle$ and $\langle u, b\rangle$ are preserved, then $\langle u, v\rangle$ is preserved. The assertion in the brackets can be proved analogously (we replace M' by the dual multilattice).

Lemma 7. *Let $a, b \in M$. We define a relation $R_1(R_2)$ on M as follows: aR_1b (aR_2b) if and only if there exists an element $v \in M$, $v \in a \vee b$ such that the intervals $\langle a, v\rangle$, $\langle b, v\rangle$ are reversed (are preserved). The relations R_1 and R_2 are equivalences.*

Proof. Evidently R_1 is reflexive and symmetric. Thus it remains to prove the transitivity. Let aR_1b, bR_1c, hence there exist $r \in a \vee b$, $s \in b \vee c$ such that the intervals $\langle a, r\rangle$, $\langle b, r\rangle$, $\langle b, s\rangle$, $\langle c, s\rangle$ are reversed. Let $w \in r \vee s$, $u \in (r \wedge s)_b$. Since the intervals $\langle b, r\rangle$, $\langle b, s\rangle$ are reversed, then by Lemma 3 the intervals $\langle u, s\rangle$ and $\langle u, r\rangle$ are reversed too. By Lemma 3 and Lemma 6 the intervals $\langle r, w\rangle$ and $\langle s, w\rangle$ are reversed, hence $\langle a, w\rangle$, $\langle c, w\rangle$ are reversed too. Let $v \in (a \vee c)_w$. By Lemma 3 the intervals $\langle a, v\rangle$, $\langle c, v\rangle$ are reversed, hence aR_1c and the relation R_1 is transitive. Analogously it can be proved that the relation R_2 is an equivalence.

Lemma 8. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$, aR_1b (aR_2b). Then the interval $\langle u, v\rangle$ is reversed (is preserved).*

Proof. Let aR_1b, then there exists $v_1 \in a \vee b$ such that the intervals $\langle a, v_1\rangle$ $\langle b, v_1\rangle$ are reversed. Let $u \in a \wedge b$, then the interval $\langle u, v_1\rangle$ is reversed by Lemma 6. Hence by Lemma 3 the intervals $\langle u, a\rangle$, $\langle u, b\rangle$ are reversed. Let $v \in a \vee b$. The interval $\langle u, v\rangle$ is reversed by Lemma 6. The assertion in the brackets can be proved analogously.

Lemma 9. *Let R_1 and R_2 be the equivalences from Lemma 7 and 0 (I) denotes the least (the greatest) element of the lattice of all equivalence relations on M. Then*

(i) $R_1R_2 = R_2R_1$.

(ii) $R_1 \cup R_2 = I$, $R_1 \cap R_2 = O$.

(iii) *if* $a, b, c \in M$, $a \leqq c$, aR_1b, bR_2c, *then* $a \leqq b$, $b \leqq c$.

(iv) *if* $a, b, c, d \in M$, aR_1b, cR_1d, aR_2c, bR_2d, *then from* $a \leqq b$ *it follows that* $c \leqq d$ *and from* $a \leqq c$ *it follows that* $b \leqq d$.

Proof. (i) Let $a, b \in M$, aR_1R_2b, hence there exists an element $r \in M$ such that aR_1r and rR_2b. Hence there exist elements $u, v \in M$, with $u \in b \vee r$, $v \in a \vee r$ such that the intervals $\langle a, v\rangle$, $\langle r, v\rangle$ are reversed and the intervals $\langle b, u\rangle$, $\langle r, u\rangle$ are preserved. Let $w \in u \vee v$. Since $v' \subseteq r' \subseteq u'$, $r \leqq u$, $r \leqq v$, then $r \in u \wedge v$ by (2) of Lemma 4. By Lemma 5 the interval $\langle v, w\rangle$ is preserved and the interval $\langle u, w\rangle$ is reversed. Let $n' \in b' \cap w'$ and $m' \in a' \cup w'$. Since bnw and amw, we get $b \leqq n \leqq w$ and $a \leqq m \leqq w$ by Lemma 1. Because $n' \subseteq b' \subseteq u'$, $n' \subseteq w' \subseteq u'$, $v' \subseteq w' \subseteq m'$, $v' \subseteq a' \subseteq m'$, $v \leqq w$, $m \leqq w$, $n \leqq w$, $u \leq w$, $a \leqq v$, $a \leqq m$, $b \leqq n$, $b \leqq u$, then $b \in n \wedge u$, $a \in v \wedge m$, $w \in v \vee m$, $w \in n \vee u$ by (2) of Lemma 4. By Lemma 5 the intervals $\langle a, m\rangle$, $\langle n, w\rangle$ are preserved and the intervals $\langle b, n\rangle$, $\langle m, w\rangle$ are reversed. Let $s \in m \wedge n$. Since $n' \subseteq w' \subseteq m'$, $n \leqq w$, $m \leqq w$, then $w \in n \vee m$ and the interval $\langle s, n\rangle$ is reversed and the interval $\langle s, m\rangle$ is preserved by Lemma 5. Let $p \in (a \vee s)_m$, $q \in (b \vee s)_n$. Evidently the intervals $\langle a, p\rangle$, $\langle s, p\rangle$ are preserved and the intervals $\langle b, q\rangle$, $\langle s, q\rangle$ are reversed. Hence aR_2s, sR_1b and $R_1R_2 \subset R_2R_1$. The assertion $R_2R_1 \subset$ $\subset R_1R_2$ can be proved analogously.

(ii) Let $a, b \in M$, $aR_1 \cap R_2b$. Then aR_1b, aR_2b, hence there exist $u, v \in M$, $u \in a \vee b$, $v \in a \vee b$ such that the intervals $\langle a, u\rangle$, $\langle b, u\rangle$ are reversed and the intervals $\langle a, v\rangle$, $\langle b, v\rangle$ are preserved. Using Lemma 8 and Lemma 3 we get that the intervals $\langle a, u\rangle$, $\langle b, u\rangle$ are preserved and $\langle a, v\rangle$, $\langle b, v\rangle$ are reversed. It follows that $a = u = b$ and $R_1 \cap R_2 = 0$.

Let $a, b \in M$. We shall show that $aR_1 \cup R_2b$. Let $v \in a \vee b$, $u' \in a' \cup v'$, $w' \in b' \cup v'$. By Lemma 1 $a \leqq u \leqq v$ and $b \leqq w \leqq v$. Hence the intervals $\langle a, u\rangle$, $\langle b, w\rangle$ are preserved and the intervals $\langle u, v\rangle$, $\langle w, v\rangle$ are reserved. Hence aR_2u, bR_2w. Evidently $v \in u \vee w$ and uR_1w. From this it follows that $aR_1 \cup R_2b$ and $R_1 \cup R_2 = I$.

(iii) Let $w \in c \vee b$, $v \in (a \vee b)_w$. From cR_2b it follows that the interval $\langle b, w\rangle$ is preserved. Using Lemma 3 we get that the interval $\langle b, v\rangle$ is preserved too. From aR_1b it follows that the interval $\langle b, v\rangle$ is reversed. Hence $b = v$ and $a \leqq b$. The assertion $b \leqq c$ can be proved analogously.

(iv) First we prove the assertion: $a \leqq b$ implies $c \leqq d$. Let $v \in a \vee c$, $u \in b \vee v$, $t \in (b \wedge v)_a$. The interval $\langle a, b\rangle$ is reversed and the interval $\langle a, v\rangle$ is preserved and since the interval $\langle a, t\rangle$ is a part of these intervals, we get $a = t$. By Lemma 5 the interval $\langle b, u\rangle$ is preserved and the interval $\langle v, u\rangle$ is reversed. Let $r \in b \vee d$, $s \in u \vee r$, $n \in (u \wedge r)_b$. The intervals $\langle b, r\rangle$, $\langle b, u\rangle$

are preserved, hence the intervals $\langle n, r \rangle$, $\langle n, u \rangle$ are preserved too (Lemma 3). From this it follows, by Lemma 6 and Lemma 3, that the interval $\langle r, s \rangle$ is preserved. Since the intervals $\langle d, r\rangle$, $\langle r, s\rangle$ are preserved it follows that the interval $\langle d, s\rangle$ is preserved too. Let $w \in (c \wedge d)_s$. By Lemma 3 it follows that $\langle d, w\rangle$ is preserved. Since cR_1d it follows that $\langle d, w\rangle$ is reversed. This implies $w = d$ and $c \leqq d$. The validity of the assertion "$a \leqq c$ implies $b \leq d$" can be proved analogously.

In the following theorem we shall use the theorem [5, Thm, 3.4.2]:

Theorem K. *Let A be a quasiordered set. There exists a one-one correspondence between the non-trivial direct decompositions of the quasiordered set A into two factors and couples (R_1, R_2) of non-trivial equivalence relations on A satisfying the properties* (i), (ii), (iii), (iv) *from Lemma 9. To each couple (R_1, R_2) fulfilling these conditions there corresponds the direct decomposition $A \sim A/R_1 \times A/R_2$ and to each element $a \in A$ there corresponds the element (a_1, a_2), where a_i is the equivalence class under R_i $(i = 1, 2)$ containing a.*

Theorem 1. *Let M and M' be directed distributive multilattices. Let φ be a b-equivalence of M onto M'. Then there exist multilattices M_1, M_2 such that $M \sim M_1 \times M_2$, $M' \sim M_1 \times \tilde{M_2}$ and the image (x_1, x_2) of the element $x \in M$ under the first isomorphism is the same as the image of the element $x' \in M'$, $x' = \varphi(x)$ under the second isomorphism.*

Proof. Let R_1 and R_2 be the equivalences from Lemma 7. From Lemma 9 it follows that the equivalences R_1 and R_2 satisfy the conditions of the Theorem K. Let us denote $M/R_1 = M_1$, $M/R_2 = M_2$. By Theorem K there exists an isomorphism $\psi : M \sim M_1 \times M_2$ (M, M_1, M_2 are quasiordered sets). Since M is a multilattice, then $M_1 \times M_2$ is a multilattice and M_1, M_2 are multilattices too. Similarly there exists an isomorphism $\psi' : M' \sim M_1' \times M_2'$ ($M_i' = M'/R_i'$, where R_i' $(i = 1, 2)$ are equivalences defined on M' in the same way as R_i on M and clearly $a'R_i'b'$ iff aR_ib). Let $X = \psi'\varphi\psi^{-1}$. It is obvious that X is the b-equivalence of $M_1 \times M_2$ onto $M_1' \times M_2'$.

We shall show that M_1 and M_1' are isomorphic, M_2 and M_2' are anti-isomorphic.

Let $(m_1, m_2) \in M_1 \times M_2$. Let us denote $X(m_1, m_2) = (m_1', m_2')$. Let us construct $M_1 \times A_2$ $(M_1' \times A_2')$, where A_2 (A_2') is a multilattice with one and only one element m_2 (m_2'). It is obvious that $M_1 \times A_2$ $(M_1' \times A_2')$ is a sub multilattice of $M_1 \times M_2$ $(M_1' \times M_2')$ and the mapping $f : M_1 \times A_2 \to M_1$ $(f' : M_1' \times A_2' \to M_1')$, which maps a pair (a_1, m_2) $((a_1', m_2'))$ onto an element a_1 (a_1') is an isomorphism. The mappings

$$M_1 \xrightarrow{f^{-1}} M_1 \times A_2 \xrightarrow{X} M_1' \times A_2' \xrightarrow{f'} M_1'$$

give a b-equivalence $h = f'Xf^{-1} = f'\psi'\varphi\psi^{-1}f^{-1}$ of the multilattice M_1 onto the multilattice M'_1, which each $x_1 \in M_1$ maps onto $x'_1 \in M'_1$, where $\psi^{-1}f^{-1}(x_1) = x$, $x \in M$ and $x \in x_1$, $x \in m_2$, $\varphi(x) = x'$, $x' \in M'$, $\psi'(x') = (x'_1, m'_2)$, $x' \in x'_1$, $x' \in m'_2$ and $f'(x'_1, m'_2) = x'_1$. We shall prove that h is an isomorphism. Clearly h is a bijection. Let $a_1, b_1 \in M_1$, $a_1 \leqq b_1$. We have $\psi^{-1}f^{-1}(a_1) = a$ and $a \in m_2$, $\psi^{-1}f^{-1}(b_1) = b$ and $b \in m_2$. Since f and ψ are isomorphisms, then $a \leqq b$ holds. From $a \in m_2$, $b \in m_2$ it follows that aR_2b, hence the interval $\langle a, b\rangle$ is preserved and it implies $a' \subseteq b'$. Since ψ' and f' are isomorphism, then $a'_1 \subseteq b'_1$ holds. The assertion: "$a'_1 \subseteq b'_1$ implies $a_1 \leqq b_1$" —can be proved analogously. Hence the multilattices M_1, M'_1 are isomorphic.

Analogously we construct a mapping $k : M_2 \to M'_2$, $k = g'\psi'\varphi\psi^{-1}g^{-1}$, where $g: A_1 \times M_2 \to M_2$ and $g' : A'_1 \times M'_2 \to M'_2$ are isomorphisms ($A_1(A'_1)$ is a multilattice with one and only one element $m_1(m'_1)$). We shall show that k is an anti-isomorphism. Evidently k is a bijection. Let $c_2, d_2 \in M_2$, $c_2 \leqq d_2$. We have $\psi^{-1}g^{-1}(c_2) = c$ and $c \in m_1$, $\psi^{-1}g^{-1}(d_2) = d$ and $d \in m_1$. Since g and ψ are isomorphisms, then $c \leqq d$ holds. From $c \in m_1$, $d \in m_1$ it follows that cR_1d, hence the interval $\langle c, d\rangle$ is reversed therefore $d' \subseteq c'$. Since ψ' and g' are isomorphisms then $d'_2 \subseteq c'_2$ holds. The assertion: "$d'_2 \subseteq c'_2$ implies $c_2 \leqq d_2$", can be proved analogously. Hence the multilattices M_2, M'_2 are anti-isomorphic. Consequently

$$h^{-1} \times k^{-1} : M'_1 \times M'_2 \to M_1 \times M_2^{\sim}$$

is an isomorphism ($M_2^{\sim}$ is the dual multilattice of M_2) and we get

$$M \sim M_1 \times M_2, \qquad M' \sim M_1 \times M_2^{\sim}.$$

From the construction of h and k it follows that $\psi(x) = (x_1, x_2) = (h^{-1} \times k^{-1})\psi'(x')$, where $x \in M$, $x_1 \in M_1$, $x_2 \in M_2$, $x' \in M'$, $x' = \varphi(x)$.

In Lemma 10, Lemma 11, Lemma 12 and Lemma 13 M denotes a distributive multilattice.

Lemma 10. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$, then there exist isomorphisms:*
f: $\langle u, a\rangle \to \langle b, v\rangle$ *with* $f(x) = (b \vee x)_v$ *for* $x \in \langle u, a\rangle$;
g: $\langle b, v\rangle \to \langle u, a\rangle$ *with* $g(y) = (a \wedge y)_u$ *for* $y \in \langle b, v\rangle$;
h: $\langle u, b\rangle \to \langle a, v\rangle$ *with* $h(r) = (r \vee a)_v$ *for* $r \in \langle u, b\rangle$;
k: $\langle a, v\rangle \to \langle u, b\rangle$ *with* $k(s) = (b \wedge s)_u$ *for* $s \in \langle a, v\rangle$.

The proof of Lemma 10 follows from 6.4, §6 of paper [1].

Lemma 11. *Let $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$, then there exist isomorphisms:*
m: $\langle u, v\rangle \to \langle a, v\rangle \times \langle b, v\rangle$ *with* $m(x) = ((a \vee x)_v, (b \vee x)_v)$
for $x \in \langle u, v\rangle$;
n: $\langle a, v\rangle \times \langle b, v\rangle \to \langle u, v\rangle$ *with* $n(x_1, x_2) = (x_1 \wedge x_2)_u$
for $x_1 \in \langle a, v\rangle$ *and* $x_2 \in \langle b, v\rangle$.

This Lemma is a corollary of 3.2, 3.4, 3.7 of paper [2].

Lemma 12. *If* $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$, $u \leqq x \leqq v$, $x_1 \in (a \vee x)_v$, $y \in (x_1 \wedge b)_u$, *then* $y \leqq x \leqq x_1$.

Proof. Let us denote $x_2 \in (x \vee b)_v$. By Lemma 11 $x = (x_1 \wedge x_2)_u$, where $y \leqq x_1$, $y \leqq x_2$, $u \leqq y$, hence $y \leqq x$.

Lemma 13. *Let* $a, b, c, d, e, f \in M$. *If* $f \in e \vee d$, $c \in e \wedge d$, $d \in c \vee b$, $a \in e \wedge b$, $a \leqq c$, *then* $f \in e \vee b$.

Proof. Let $r \in (b \vee e)_f$, $s \in (b \vee c)_r$. From the isomorphism of the intervals $\langle a, e\rangle$, $\langle b, r\rangle$ (Lemma 10) it follows that $(s \wedge e)_a = c$, hence

$$c \in s \wedge e . \tag{6}$$

Let us choose $w \in (r \wedge d)_c$. From the isomorphism of the intervals $\langle c, d\rangle$ $\langle e, f\rangle$ it follows that $(w \vee e)_f = r$, hence

$$r \in w \vee e . \tag{7}$$

By Lemma 12 we get $w \leqq s \leqq r$. Hence there holds

$$c \leqq w \leqq s \leqq r . \tag{8}$$

From (6), (7), (8) and from the modularity it follows that $w = s$, hence $c \leqq \leqq s \leqq d$. Since $d \in b \vee c$, we get $d = s$. Because $e \leqq r \leqq f$ and $d \leqq r$, by (8) we get $f = r$. Hence $f \in e \vee b$.

Lemma 14. *Let M be a directed distributive multilattice, $a, b, x \in M$. Then the following conditions are equivalent*:

$$(b) \qquad [(a \wedge x) \vee (b \wedge x)]_x = x , \qquad (a \wedge x) \wedge (b \wedge x) \subset a \wedge b ,$$
$$(b') \qquad [(a \vee x) \wedge (b \vee x)]_x = x , \qquad (a \vee x) \vee (b \vee x) \subset a \vee b .$$

Proof. Let $x_1 \in a \wedge x$, $x_2 \in b \wedge x$, $u \in x_1 \wedge x_2$. Let (b) be valid. Let $y_1 \in a \vee x$, $y_2 \in b \vee x$, $y \in (y_1 \wedge y_2)_x$, $v \in y_1 \vee y_2$. It is obvious, that $u \in x_1 \wedge b$. By Lemma 13 we get from this

$$y_2 \in x_1 \vee b . \tag{9}$$

Let us choose $r \in (a \wedge y_2)_{x_1}$. There holds $u \in r \wedge b$. From this and from (9) it follows that $r = x_1$. Hence

$$x_1 \in a \wedge y_2 \tag{10}$$

and $x_1 \in a \wedge y$ too. From this and from $y_1 \in a \vee x$ we get $x = y$ by modularity. Hence we have proved that $[(a \vee x) \wedge (b \vee x)]_x = x$. By Lemma 13 it follows from (10) that $v \in a \vee y_2$. From this and from (9), (10), $u \in a \wedge b$ by Lemma 13 we get $v \in a \vee b$. Thus we have obtained $(a \vee x) \vee (b \vee x) \subset a \vee b$, too. Hence we have proved that (b) implies (b'). The implication $(b') \Rightarrow (b)$ can be obtained by duality.

Lemma 15. *Let M, M_1, M_2 be directed multilattices and let φ be an isomorphism of M onto $M_1 \times M_2$. For $x \in M$ we denote $\varphi(x) = (x_1, x_2)$. Let $a, b, x \in M$. Then the elements a, b, x satisfy the condition (b) iff $a_i, b_i, x_i \in M_i$ $(i = 1, 2)$ satisfy this condition.*

The proof of this assertion follows from the isomorphism.

Lemma 16. *Let M be a distributive directed multilattice and let $M^{\sim}$ be the dual of M. The elements $a, b, x \in M$ satisfy the condition (b) iff they satisfy this condition in $M^{\sim}$.*

Proof. It suffices to use Lemma 14.

Theorem 2. *Let M, M' be directed distributive multilattices and $M \sim M_1 \times M_2$, $M' \sim M_1 \times M_2^{\sim}$. Then M and M' are b-equivalent.*

Proof. Let f be an isomorphism of M onto $M_1 \times M_2$ and let g be an isomorphism of $M_1 \times M_2^{\sim}$ onto M'. Further let $h\colon M_1 \times M_2 \to M_1 \times M_2^{\sim}$ be the identical mapping. Hence $\varphi = ghf$ is a bijection. Let $a, b, x \in M$. We shall show that axb iff $\varphi(a)\varphi(x)\varphi(b)$. Using Lemma 15 and Lemma 16 we get: axb iff $f(a)f(x)f(b)$, $f(a)f(x)f(b)$ iff $h(f(a))\,h(f(x))\,h(f(b))$, $h(f(a))\,h(f(x))\,h(f(b))$ iff $g[h(f(a))]\,g[h(f(x))]\,g[h(f(b))]$. Consequently axb iff $\varphi(a)\,\varphi(x)\,\varphi(b)$.

The following theorem is a corrollary of Theorem 1 and Theorem 2.

Theorem 3. *Let M, M' be directed distributive multilattices. M, M' are b-equivalent if and only if there exist multilattices M_1, M_2 such that $M \sim M_1 \times M_2$ and $M' \sim M_1 \times M_2^{\sim}$.*

In paper [4] the notion of the m-equivalence is defined as follows: The metric multilattices M, M' are m-equivalent if there exists a bijection φ of M onto M' such that for each $a, b, x \in M$, the following conditions are equivalent:

(i) $\varrho(a, x) + \varrho(x, b) = \varrho(a, b)$

(ii) $\varrho(\varphi(a), \varphi(x)) + \varrho(\varphi(x), \varphi(b)) = \varrho(\varphi(a), \varphi(b))$.

Lemma 17. *Let M, M' be directed distributive metric multilattices. M, M' are b-equivalent if and only if M, M' are m-equivalent.*

The proof of this Lemma follows from 2.2 [4].

Using Lemma 17 and Theorem 3 we get:

Theorem 4. (Thm. 3.3.2 [4]). *Directed distributive metric multilattices M, M' are m-equivalent if and only if there exist multilattices A_1, A_2 such that $M \sim A_1 \times A_2$, $M' \sim A_1 \times A_2^{\sim}$.*

Kolibiar [4] has shown that Thm. 4 fails to hold if we omit the assumption that M and M' are distributive, or the assumption that M and M' are directed; hence also Thm. 3 fails to be valid if we omit some of these assumptions.

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