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ON THE SUM OF OBSERVABLES IN A LOGIC

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In this paper the sum of observables, including also the case of unbounded observables, is studied and some results regarding the full and quite full systems of states are proved.

1. Introduction

Let L be a poset with the first and the last element 0 and 1, resceptively, with the orthocomplementation $\perp: L \to L$, for which we have (i) $(a^\perp)^\perp = a$ for all $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee a^\perp = 1$ for all $a \in L$. If $a < b$, then a, b are said to be orthogonal and we write $a \perp b$. Further we assume that if $a_i \perp a_j$, $i \neq j$, then $\bigvee_i a_i$ exists in L; and if $a < b$, then there is $c \perp a$ such that $b = a \vee c$. A poset L satisfying the above axioms is called a logic [9].

An observable is a map x: $B(R_1) \to L$ such that (i) $x(R_1) = 1$; (ii) if $E \cap F = \emptyset$, then $x(E) \perp x(F)$; (iii) $x\left(\bigcup_i E_i\right) = \bigvee_i x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$. If f is a Borel function and x an observable, then $f \circ x$: $E \mapsto x(f^{-1}(E))$, $E \in B(R_1)$ is an observable. We denote by $\sigma(x)$ the smallest closed set $C \subset R_1$ such that $x(C) = 1$, and x is called bounded if $\sigma(x)$ is a compact set.

A state is a map m: $L \to \langle 0, 1 \rangle$ such that (i) $m(1) = 1$; (ii) $m\left(\bigvee_i a_i\right) = \sum_i m(a_i)$ if $a_i \perp a_j$, $i \neq j$. An element $a \in L$ is a carrier of a state m if $m(b) = 0$ iff $b \perp a$. If a carrier of m exists, then it is unique. A system $\mathcal{M}$ of states of L is called (i) quite full if the statement $m(b) = 1$, whenever $m(a) = 1$, $m \in \mathcal{M}$ implies $a < b$; (ii) full if $a < b$ iff $m(a) \leq m(b)$ for all $m \in \mathcal{M}$. Gudder [6] showed that if $\mathcal{M}$ is quite full, then (i) $\mathcal{M}$ is full; (ii) if $a \neq 0$, then there is $m \in \mathcal{M}$ such that $m(a) = 1$.

Let m be a state and x an observable; then $m(x) = \int t \, \mathrm{d}m_x(t)$, where m_x: $E \mapsto m(x(E))$, $E \in B(R_1)$ is called the mean of x in the state m, provided that the integral exists. Analogically there is defined $m(x^2) = \int t^2 \, \mathrm{d}m_x(t)$.

A state m is pure if the statement $m = cm_1 + (1-c)m_2$, $0 < c < 1$ implies

$m = m_1 = m_2$. For a system $\mathcal{M}$ of states we denote by Co $(\mathcal{M}) = \left\{\sum_i c_i m_i,\ c_i > 0,\ \sum_i c_i = 1,\ m_i \in \mathcal{M},\ i \in I \subset \{1, 2, \ldots\}\right\}$, that is, Co $(\mathcal{M})$ is the set of all s-convex combinations of the states of $\mathcal{M}$.

Lemma 1.1. (i) *A system $\mathcal{M}$ of states is quite full (full) iff* Co$(\mathcal{M})$ *is quite full (full).*

(ii) *If $\mathcal{M}_\alpha$, $\alpha \in A$ are quite full (full) systems, then* $= \bigcup_{\alpha \in A} \mathcal{M}_\alpha$ *is quite full (full).*

(iii) *If $\mathcal{M}$ is quite full (full) and $\mathcal{M}_0$ is a system of states, then $\mathcal{M} \cup \mathcal{M}_0$ is quite full (full).*

The proof of this lemma is obvious and is omitted.

2. Systems of states

One of the most important examples of logics is a logic $L(H)$ of a Hilbert space over D (D is a real or complex field), that is, $L(H)$ is a complete lattice of all closed subspaces of H.

Since there is a one-to-one correspondence between the closed subspaces M of H and their projectors P^M, we shall write M for a subspace as well as for its projector. If $u \in H$ is a unit vector, then the system $\mathcal{M}_v$ of all vector states $m_u: M \mapsto (Mu, u)$, $M \in L(H)$ is a quite full system of states of a logic $L(H)$, and, moreover, $m_u = m_v$ iff there is $\alpha \in D$, $|\alpha| = 1$, such that $u = \alpha v$. The excellent Gleason theorem [10] says that if H is a separable Hilbert space of dimension at least three, then Co$(\mathcal{M}_v)$ is the set of all states of $L(H)$, or, equivalently, for any state m on $L(H)$ there is a unique von Neumann operator T such that $m(M) = tr(TM)$, $M \in L(H)$.

Due to the spectral theorem there is a one-to-one correspondence $x \leftrightarrow A_x$ between the set of observables on $L(H)$ and the set of all self-adjoint operators on H. An observable x is bounded iff A_x is a bounded operator.

Lemma 2.1. *Let m_u be a vector state; then $m_u(x^2) < \infty$ iff $u \in \mathscr{D}(A_x)$($\mathscr{D}(A_x)$ is the domain of the linear operator A_x); in this case*

$$m_u(x) = (A_x u, u),\quad m_u(x^2) = \|A_x u\|^2.$$

Proof. Since $u \in \mathscr{D}(A)$ *iff* $\int \lambda^2\, d(P^{A_x}(\lambda)u, u)$, where $P^{A_x}(E) = x(E)$, $E \in B(R_1)$ is a spectral measure of A_x, we have $m_u(x^2) = \int t^2\, dm_{u,x}(t) = \int \lambda^2\, d(P^{A_x}(\lambda)u, u) = \|A_x u\|^2$. Analogically we obtain $m_u(x) = (A_x u, u)$. Q.E.D.

Theorem 2.2. *A system $\mathcal{M} \subset \mathcal{M}_v$ of a logic $L(H)$ (H is of an arbitrary dimension) is quite full iff $\mathcal{M} = \mathcal{M}_v$.*

Proof. If $\mathcal{M} \neq \mathcal{M}_v$, then there is a unit vector $v \in H$ such that $m_v \notin \mathcal{M}$. But for the subspace P_v generated by v there is no $m_u \in \mathcal{M}$ such that $m_u(P_v) = 1$. Actually, if there would be $m_u \in \mathcal{M}$, $m_u(P_v) = 1$, then $\|P_v u\|^2 = 1$. Hence there is $\alpha \in D$ such that $u = \alpha v$ and therefore $|\alpha| = 1$, which implies $m_v = m_u \in \mathcal{M}$. Q.E.D.

Corollary 2.3. *If A is a self-adjoint operator, then the system $\mathcal{M}(A)$ of all vector states generated by unit vectors from $\mathcal{D}(A)$ is quite full iff A is a bounded operator.*

Proof. An operator A is bounded iff $\mathcal{D}(A) = H$. If $\mathcal{M}(A)$ is quite full and $\mathcal{D}(A) \neq H$, then there is a vector $u \neq 0$, $u \notin \mathcal{D}(A)$. A unit vector $u_0 = u/\|u\|$ determines a vector state m_{u_0} which does not belong to $\mathcal{M}(A)$. Hence, by Theorem 2.2, $\mathcal{M}(A)$ is not quite full. Q.E.D.

Theorem 2.4. *A system of states $\mathcal{M} \subset \mathrm{Co}\,(\mathcal{M}_v)$ is quite full iff $\mathcal{M}_v \subset \mathcal{M}$.*

Proof. Let P_f be a one-dimensional subspace generated by f, $\|f\| = 1$. Since $\mathcal{M}$ is quite full, there is $m = \sum_i c_i m_{v_i} \in \mathcal{M}$ such that $m(P_f) = 1$. Hence $m_{v_i}(P_f) = 1$ for any i and $v_i \in P_f$. This implies $m_f = m_{v_i}$ for any i and $m = m_f \in \mathcal{M}$, that is, $\mathcal{M}_v \subset \mathcal{M}$. Q.E.D.

Theorem 2.5. *Let H_0 be a linear manifold dense in H. Then $\mathcal{M}(H_0) = \{m_u, \|u\| = 1, u \in H_0\}$ is a full system of states.*

Conversely, if $\mathcal{M}(K) = \{m_u, u \in K, \|u\| = 1\}$ is a full system of states, then the linear manifold $H(K)$ generated by K is dense in H.

Proof. If $m_u(M) \leqslant m_u(N)$ for any unit vector $u \in H_0$, then due to the density of H_0 we have $m_u(M) \leqslant m_u(N)$ for any $u \in H$, that is, $M \subset N$; consequently $\mathcal{M}(H_0)$ is full.

Conversely, let $\mathcal{M}(K)$ be a full system of states. Then $H_K \equiv \overline{H(K)} \in L(H)$. For any $m_u \in \mathcal{M}(K)$ we have $m_u(H_K) = 1$, which implies $m_u(H_K) = m_u(H)$ for any m_u and therefore $H_K = H$. The density of $H(K)$ is proved. Q.E.D.

Now let $W = W(H)$ be a von Neumann algebra of bounded operators of a Hilbert space H (real or complex) and let $L(W, H)$ be a sublogic of $L(H)$ constituted by projectors belonging to W. We denote by W' the commutant of W, is the set of all bounded operators B in H such that $AB = BA$ for all $A \in W$. Then we may formulate the next assertion.

Theorem 2.6. *Let $W(H)$ be a von Neumann algebra and $K \subset H$ be a set of unit vectors. If $\mathcal{M}(K) = \{m_u, u \in K\}$ is full, then $W'K$ is dense in H.*

Proof. Let $\mathcal{M}(K)$ be full. It will be shown that K is a separator of W, that is, if for $A \in W$ we have $Au = 0$ for all $u \in K$, then $A = 0$. Indeed, if $Au = 0$, then $A^*Au = 0$. An operator $B = A^*A$ is Hermitian and for the corresponding observable x_B we have

$$m_u(x_B) = \int t \, \mathrm{d}m_{u, x_B}(t) = (Bu, u) = 0.$$

Since $m_u(x_B(\{0\}))=1=m_u(H)$, we have $x_B(\{0\})=H$ and thus $B=0$, $A=0$.

Due to [3, p. 6] K is a separator of W iff $W'K$ is dense in H. Q.E.D.

According to Bugajska, Bugajski [1] we introduce the next axioms:

Axiom 1. *L is a separable logic,* that is, every subset of mutually orthogonal elements from L is at most countable.

Axiom 2. *The system $\mathcal{M}_p$ of all pure states of L is quite full.*

Axiom 3. *If for $m \in \mathcal{M}_p$ $m(a_t)=1$, $t \in T$ then $\bigwedge_{t \in T} a_t$ exists in L and $m\left(\bigwedge_{t \in T} a_t\right)=1$.*

In [1] it is shown that the above axioms imply that (i) any state $m \in \mathcal{M} = \mathrm{Co}\,(\mathcal{M}_p)$ has a carrier; (ii) for any $a \in L$, $a \neq 0$, there is $m \in \mathcal{M}$ such that a is its carrier; (iii) L is a lattice. Zierler [11] showed, moreover, that L is a complete lattice.

Let $m \in \mathcal{M}_p$ and let I_m be its carrier. According to Deliyannis [2] the following axioms are supposed, in addition:

Axiom 4. *For any n, $m \in \mathcal{M}_p$ $n(I_m)=m(I_n)$.*

Axiom 5. *If $n(I_m)=1$, then $n=m$.*

The corollaries of the axioms 1—5 are (i) for any $m \in \mathcal{M}_p$ I_m is an atom of L; (ii) for any atom $a \in L$ there is a unique pure state $m \in \mathcal{M}_p$ such that $I_m=a$; (iii) any $a \in L$, $a \neq 0$, is a join of mutually orthogonal atoms.

Theorem 2.7. *$\mathcal{M} \subset \mathrm{Co}\,(\mathcal{M}_p)$ is quite full iff $\mathcal{M} \supset \mathcal{M}_p$.*

Proof. If $\mathcal{M} \supset \mathcal{M}_p$, then $\mathcal{M}$ is quite full (Lemma 1.1). Conversely, let $\mathcal{M}$ be quite full. If a is an atom, then there is $m \in \mathcal{M}$, $m(a)=1$. Hence m is of the form $m=\sum_i c_i m_i$, $c_i>0$, $\sum_i c_i=1$, $m_i \in \mathcal{M}_p$ and $m_i(a)=1$ for all i. This implies $I_{m_i}<a$. Therefore $I_{m_i}=a$ and the pure state corresponding to a is equal to $m \in \mathcal{M}$, that is, $\mathcal{M}_p \subset \mathcal{M}$. Q.E.D.

Theorem 2.8. *Let a system of pure states $\mathcal{M}=\{m_{a_\alpha}, \alpha \in A\}$ (a_α, $\alpha \in A$, is an atom) be full; then*

$$\bigvee_{\alpha \in A} a_\alpha = 1\,.$$

Proof. Let $a=\bigvee_{\alpha \in A} a_\alpha$; then for any m_{a_α}, $\alpha \in A$ we have $m_{a_\alpha}(a)=1$. Due to the fullnees of $\mathcal{M}$ we have $a=1$. Q.E.D.

3. Sum of observables

The sum of bounded observables has been studied by Gudder [6, 7], Dvurečenskij [4]. In [7, p. 331] there is given the definition of the sum of unbounded observables: *We say that the sum of x, y exists if there is a quite full system $\mathcal{M}$ of states and an observable z such that $m(x)$, $m(y)$ exist and are finite, and $m(z) = m(x) + m(y)$ for all $m \in \mathcal{M}$.*

But this definition does not include the important case of a logic $L(H)$, $3 \leq \dim H \leq \aleph_0$.

In more detail: Let A_x, A_y be two unbounded positive self-adjoint operators with $\mathcal{D}(A_x) \cap \mathcal{D}(A_y)$ dense in H. Let x and y be observables corresponding to A_x, A_y, respectively. Then $m_u(x)$, $m_u(y)$ exist and are finite iff $u \in \mathcal{D} = \mathcal{D}(A_x^{1/2}) \cap \mathcal{D}(A_y^{1/2})$ (Lemma 2.1). If the system $\mathcal{M}$ of vector states generated by unit vectors from $\mathcal{D}$ were quite full, then, by Theorems 2.2 and 2.5, $\mathcal{D} = H$. Hence $\mathcal{D} \subset \mathcal{D}(A_x^{1/2})$, $\mathcal{D}(A_y^{1/2})$ and $A_x^{1/2}$, $A_y^{1/2}$ are bounded operators [8]. Consequently A_x, A_y are bounded, which contradicts to our assumption.

On the other hand, $A_x + A_y$ is a self-adjoint operator and it is reasonable to consider the corresponding observable z for the sum of x and y.

By Theorem 2.5 it is evident that the above $\mathcal{M}$ is only full. For this reason we accept the following definitions.

Let us suppose that on a logic L a quite full system $\mathcal{M}$ of states, $\mathcal{M} = \mathrm{Co}(\mathcal{M})$, is given. The pair $(L, \mathcal{M})$ is called a quantum logic.

Definition 3.1. *We shall say that on a quantum logic $(L, \mathcal{M})$ the observables $x_1, \ldots, x_n$ are summable if*

(i) *$\mathcal{M}(x, \ldots, x_n) = \{m \in \mathcal{M} : m(x_i^2) < \infty,\ i = 1, \ldots, n\}$ is a full system;*

(ii) *there is an observable z such that $\mathcal{M}(z) \supset \mathcal{M}(x_1, \ldots, x_n)$ and $m(z) = m(x_1) + \ldots + m(x_n)$ for all $m \in \mathcal{M}(x, \ldots, x_n)$.*

In this case z is called the sum of $x_1, \ldots, x_n$ and is written $z = x_1 + \ldots + x_n$.

Definition 3.2. *We shall say that a quantum logic $(L, \mathcal{M})$ is a sum logic if there holds: for every finite system of observables $x_1, \ldots, x_n$ for which $\mathcal{M}(x_1, \ldots, x_n)$ is full there is a unique sum $z = x_1 + \ldots + x_n$.*

In the following we assume that $(L, \mathcal{M})$ is a sum logic.

Proposition 3.3. *On a sum logic the sum of any two bounded observables x and y exists and is a bounded observable.*

Proof. Since $\mathcal{M}(x, y) = \mathcal{M}$, x and y are summable. For $z = x + y$ we have that $m(z)$ is finite for every $m \in \mathcal{M}$ and, by [5, Theorem 6.3] this is the necessary and sufficient condition for z to be bounded.

Thus, by this proposition, the case of bounded observables from [5] is included in Definition 3.2.

Proposition 3.4. *Let $x_1, \ldots, x_n$ be summable. Then*

(i) *$x_{i_1}, \ldots, x_{i_n}$ are summable for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and $x_1 + \ldots + x_n = x_{i_1} + \ldots + x_{i_n}$;*

(ii) *for any $\alpha_1, \ldots, \alpha_n \in R_1$ $\alpha_1 x_1, \ldots, \alpha_n x_n$ are summable, especially, $\alpha(x_1 + \ldots + x_n) = \alpha x_1 + \ldots + \alpha x_n$ for $\alpha \in R_1$.*

(iii) *any subsystem $x_{i_1}, \ldots, x_{i_k}$, $1 \leq k \leqq n$ is summable, especially $z_1 = x_1 + \ldots + x_k$ and $z_2 = x_{k+1} + \ldots + x_n$ are summable and $z_1 + z_2 = x_1 + \ldots + x_n$.*

Proof. (i) Since $\mathcal{M}(x_1, \ldots, x_n) = \mathcal{M}(x_{i_1}, \ldots, x_{i_n})$ for $z = x_1 + \ldots + x_n$, we have $m(z) = m(x_1) + \ldots + m(x_n) = m(x_{i_1}) + \ldots + m(x_{i_n})$, $m \in \mathcal{M}(x_1, \ldots, x_n)$. Analogically we prove (ii).

(iii) There holds

$$\mathcal{M}_1 = \mathcal{M}(x_1, \ldots, x_k) \supset \mathcal{M}(x_1, \ldots, x_n), \mathcal{M}_2 = \mathcal{M}(x_{k+1}, \ldots, x_n) \supset \mathcal{M}(x_1, \ldots, x_n).$$

Then there are unique observables z_1, z_2 such that $\mathcal{M}(z_1) \supset \mathcal{M}_1$, $\mathcal{M}(z_2) \supset \mathcal{M}_2$ and $z_1 = x_1 + \ldots + x_k$, $z_2 = x_{k+1} + \ldots + x_n$. Since $\mathcal{M}(z_1, z_2) = \mathcal{M}(z_1) \cap \mathcal{M}(z_2) \supset \mathcal{M}(x_1, \ldots, x_n)$, z_1 are z_2 are summable and there is a unique z' such that $\mathcal{M}(z') \supset \mathcal{M}(z_1, z_2)$, $m(z') = m(z_1) + m(z_2)$, $m \in \mathcal{M}(z_1, z_2)$. For any $m \in \mathcal{M}(x_1, \ldots, x_n)$ we have $m(z') = m(z_1) + m(z_2) = m(x_1) + \ldots + m(x_k) + m(x_{k+1}) + \ldots + m(x_n) = m(z)$. From the uniqueness of the sum $z = x_1 + \ldots + x_n$ we have $z = z'$. Q.E.D.

Proposition 3.5. *If $x_1, \ldots, x_n$ are summable and $x_i = f_i \circ u$ for some Borel functions f_i, $i = 1, \ldots, n$ and an observable u, then $x_1 + \ldots + x_n = (f_1 + \ldots + f_n) \circ u$.*

Proof. If $m \in \mathcal{M}(x_1, \ldots, x_n)$, then $f_i \in L_2(R_1, B(R_1), m_u)$ and there holds

$$m(z) = m(x_1) + \ldots + m(x_n) = \int f_1 \, dm_u + \ldots + \int f_n \, dm_u =$$
$$= \int (f_1 + \ldots + f_n) \, dm_u = m((f_1 + \ldots + f_n) \circ u). \qquad \text{(Q.E.D.}$$

Proposition 3.6. *If $(K, \mathcal{M})$ is a sum logic, then L is a lattice.*

The proof of this proposition is the same as that of Lemma 6.2 [6].

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О СУММЕ НАБЛЮДАЕМЫХ В ЛОГИКЕ

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Резюме

В работе исследуется понятие суммы наблюдаемых в логике, заключающее в себе тоже случай неограниченных наблюдаемых. Доказаны некоторы результаты о системах сострояний. В работе введено понятие суммируемых наблюдаемых. В частности исследуется случай логики всех проекторов в пространстве Гильберта.