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ON LATTICES OF GENERALIZED TOPOLOGIES

JOSEF ŠLAPAL

Generalized topologies obtained by replacing the Kuratowski axioms by some weaker ones occur in various branches of mathematics (for example in the theory of games as shown in [7]). In the present note we investigate some systems of these generalized topologies from the point of view of the theory of lattices.

Under a *topology* u on a non-empty set P we understand a mapping u: $\exp P \to \exp P$. These topologies (often called topologies without axioms or Koutský topologies) are studied in [9], [11] and [13]. We shall consider the following axioms for topologies on a given set $P \neq \emptyset$:

1. $u\emptyset = \emptyset$ — O-*axiom* ([5]),
2. $X \subseteq P \Rightarrow X \subseteq uX$ — I-*axiom* ([5]),
3. $X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$ — M-*axiom* ([5]),
4. $X, Y \subseteq P \Rightarrow u(X \cup Y) \subseteq uX \cup uY$ — A-*axiom* ([3]),
5. $\emptyset \neq X \subseteq P \Rightarrow uX \subseteq \bigcup_{x \in X} u\{x\}$ — S-*axiom* ([10]),
6. $X \subseteq P \Rightarrow uuX \subseteq uX$ — U-*axiom* ([8]).

If f is one of the listed axioms, i.e. $f \in \{O, I, M, A, S, U\}$, then a topology u on P is called an *f-topology* whenever it fulfils the f-axiom. If also $g \in \{O, I, M, A, S, U\}$ and u is both an f-topology and g-topology, then it is called an *fg-topology*, etc. Let us note that every MS-topology is an MA-topology, and provided that P is finite these two topologies even coincide. Many authors deal with topologies fulfilling some of the axioms above considered. Thus, OM-topologies occur in [7], IM-topologies are studied in [6], OI-topologies in [5], OIM-topologies in [3], [5], [8] and [11], OIMA-topologies in [4], OIMU-topologies in [12], OIMAU-topologies in [2], [4] and [9], and OISU-topologies in [4] and [10].

The system of all topologies on P is denoted by $\mathscr{P}$. By $\mathscr{P}_f$ we denote the system of all f-topologies on P, by $\mathscr{P}_{fg}$ the system of all fg-topologies on P, etc. The system $\mathscr{P}$ as well as every its subsystem will be considered as ordered by the relation $\leqq$ defined as usual: $u \leqq v \Leftrightarrow uX \subseteq vX$ for any subset $X \subseteq P$. If $u \leqq v$, then we say that u is *weaker* than v or that v is *stronger* than u. It is well known (see [9]) that $\mathscr{P}$ is a complete lattice and that for any non-empty system $\mathscr{T} \subseteq \mathscr{P}$ its join and meet in $\mathscr{P}$ are defined by $(\bigvee\mathscr{T})X = \bigcup_{u\in\mathscr{T}} uX$ and $(\bigwedge\mathscr{T})X = \bigcap_{u\in\mathscr{T}} uX$ for any subset $X \subseteq P$. (Moreover, $\mathscr{P}$ is a completely distributive complete Boolean algebra — see [11]). The least and the greatest elements in $\mathscr{P}$ will be denoted as u^* and v^*. Clearly, $u^*X = \emptyset$ and $v^*X = P$ for every subset $X \subseteq P$.

Let N denotes the set of all positive integers.

The reader can easily prove the following assertion:

Theorem 1. (1) $\mathscr{P}_O$, $\mathscr{P}_I$, $\mathscr{P}_M$ *are complete sublattices of* $\mathscr{P}$.
(2) $\mathscr{P}_A$, $\mathscr{P}_S$ *are complete join-subsemilattices of* $\mathscr{P}$.
(3) $\mathscr{P}_{MU}$ *is a complete meet-subsemilattice of* $\mathscr{P}$.

Remark 1. a) In the example 3.4a of [11] it is shown that for any set P with card $P \geqq 3$ the system $\mathscr{P}_{OIMAU}$ is not a meet-subsemilattice of $\mathscr{P}$. From this it follows that $\mathscr{P}_{OIMSU}$ is not a meet-subsemilattice of $\mathscr{P}$ whenever $3 \leqq \text{card}\, P < \aleph_0$. But from the same example it can also be easily seen that $\mathscr{P}_{OIMAU}$ is not a join-subsemilattice of $\mathscr{P}$ for any set P with card $P \geqq 3$. Thus, in consequence of Theorem 1, neither $\mathscr{P}_A$ nor $\mathscr{P}_S$ are meet-subsemilattice of $\mathscr{P}$, and $\mathscr{P}_{MU}$ is not a join-subsemilattice of $\mathscr{P}$, generally.

b) The system $\mathscr{P}_U$ is neither a join-semilattice nor a meet-semilattice in general — see the following example. Let $P = \{x, y, z, t\}$ and let u_1, u_2, u_3, u_4 be topologies on P defined as follows: $u_1\{x\} = \{x\}$, $u_2\{x\} = \{y\}$, $u_3\{x\} = \{x, y, z\}$, $u_4\{x\} = \{x, y, t\}$, $u_i\{x, y\} = \{z\}$ for $i = 1, 2, 3, 4$, and $X \subseteq P$, $\{x\} \neq X \neq \{x, y\} \Rightarrow$ $\Rightarrow u_iX = X$ for $i = 1, 2, 3, 4$. Evidently, $u_1, u_2, u_3, u_4 \in \mathscr{P}_U$. The topologies u_3 and u_4 are minimal upper bounds of $\{u_1, u_2\}$ in $\mathscr{P}_U$ and thus there exists no join of $\{u_1, u_2\}$ in $\mathscr{P}_U$. Similarly, u_1 and u_2 are maximal lower bounds of $\{u_3, u_4\}$ in $\mathscr{P}_U$ and thus there exists no meet of $\{u_3, u_4\}$ in $\mathscr{P}_U$

As the proofs of the following three Theorems are somewhat alike we present only the last.

Theorem 2. *$\mathscr{P}_A$ is a complete lattice. If $\mathscr{T} \subseteq \mathscr{P}_A$ is a non-empty system, then its meet $\bigwedge\mathscr{T}$ in $\mathscr{P}_A$ is defined by* $(\bigwedge\mathscr{T})X = \bigcap\left\{Y \subseteq P \mid Y = \bigcup_{i=1}^{m}\left(\bigcap_{u\in\mathscr{T}} uX_i\right), \bigcup_{i=1}^{m} X_i = X,\ m \in N\right\}$ *for any subset $X \subseteq P$.*

Theorem 3. *$\mathscr{P}_S$ is a complete lattice. If $\mathscr{T} \subseteq \mathscr{P}_S$ is a non-empty system, then its meet $\bigwedge\mathscr{T}$ in $\mathscr{P}_S$ is defined by* $(\bigwedge\mathscr{T})\emptyset = \bigcap_{u\in\mathscr{T}} u\emptyset$ *and* $(\bigwedge\mathscr{T})X =$

$$= \left[\bigcup_{x\in X}\left(\bigcap_{u\in\mathcal{T}} u\{x\}\right)\right] \cap \bigcap_{u\in\mathcal{T}} uX \textit{ for } \emptyset \neq X \subseteq P.$$

Theorem 4. *$\mathcal{P}_{MU}$ is a complete lattice. If $\mathcal{T} \subseteq \mathcal{P}_{MU}$ is a non-empty system, then its join $\bigvee\mathcal{T}$ in $\mathcal{P}_{MU}$ is defined by $(\bigvee\mathcal{T})X = \bigcap\left\{Y \subseteq P | \bigcup_{u\in\mathcal{T}} (uX \cup uY) \subseteq Y\right\}$ for any subset $X \subseteq P$.*

Proof. As $v^* \in \mathcal{P}_{MU}$, from Theorem 1 it follows that $\mathcal{P}_{MU}$ is a complete lattice. Let $\mathcal{T} \subseteq \mathcal{P}_{MU}$ be a non-empty system. For any subset $X \subseteq P$ put $vX = \bigcup_{u\in\mathcal{T}} uX$ and $wX = \bigcap\{Y \subseteq P|vX \cup vY \subseteq Y\}$. By Theorem 1, v is an M-topology on P. Let $X \subseteq Y \subseteq P$ be subsets and $x \in wX$ a point. Then $x \in Z$ for any subset $Z \subseteq P$ fulfilling $vX \cup vZ \subseteq Z$. Let $T \subseteq P$ be a subset such that $vY \cup vT \subseteq T$. As $vX \subseteq vY$, there holds $vX \cup vT \subseteq T$, and hence $x \in T$. Therefore $x \in wY$ and the inclusion $wX \subseteq vY$ is proved. Thus w is an M-topology on P. Let $X \subseteq P$ be a subset, $x \in wwX$ a point. Then $x \in Y$ holds for every subset $Y \subseteq P$ fulfilling $vwX \cup vY \subseteq Y$. There holds $vwX = v[\bigcap\{Y \subseteq P|vX \cup vY \subseteq Y\}] \subseteq \subseteq \bigcap\{vY \subseteq P|vX \cup vY \subseteq Y\} \subseteq \bigcap\{Y \subseteq P|vX \cup vY \subseteq Y\} = wX$. Now, putting $Y = wX$ we get $Y \subseteq P$, $vwX \cup vY \subseteq Y$. Consequently, $x \in Y = wX$ and the inclusion $wwX \subseteq wX$ is proved. Hence $w \in \mathcal{P}_U$, thus $w \in \mathcal{P}_{MU}$. It is easy to see that $v \leqq w$. Let $w_1 \in \mathcal{P}_{MU}$ be a topology on P such that $v \leqq w_1$. Let $X \leqq P$ be a subset and $x \in wX$ a point. Then $x \in Y$ for every subset $Y \subseteq P$ with$vX \cup vY \subseteq Y$. From $v \leqq wq_1$ the implication $vX \cup w_1Y \subseteq Y \Rightarrow vX \cup vY \subseteq Y$ follows. Therefore $x \in Y$ for every subset $Y \subseteq P$ with $vX \cup w_1Y \subseteq Y$. Put $Y = w_1X$. Then $Y \subseteq P$, $vX \cup w_1Y \subseteq Y$. Thus $x \in Y = w_1X$, and consequently $wX \subseteq w_1X$. This yields $w \leqq w_1$. We have proved that w is the weakest of all MU-topologies on P which are stronger than v. Consequently, since $v \leqq \bigvee\mathcal{T}$, we have $w \leqq \bigvee\mathcal{T}$. As $u \leqq v \leqq w$ for every $u \in \mathcal{T}$, there holds $\bigvee\mathcal{T} \leqq w$. Thus $\bigvee\mathcal{T} = w$ and the proof is complete.

Let us introduce the following denotation. By the symbol $\leftarrow$ ($\overset{\vee}{\leftarrow}$, $\overset{\wedge}{\leftarrow}$) we denote the relation "complete sublattice of" ("complete join-subsemilattice of", "complete meet-subsemilattice of"). Then we have:

Theorem 5. *There holds* Diagram 1

Proof. Throughout the proof, $\mathcal{T}$ will be a non-empty system of topologies on P, and by v_1, v_2, w_1, w_2, w_3 we shall denote the topologies on P defined as follows: $X \subseteq P \Rightarrow v_1X = \bigcap_{u\in\mathcal{T}} uX$, $v_2X = \bigcup_{u\in\mathcal{T}} uX$, $w_1X = \bigcap\left\{Y \subseteq P | Y = \bigcup_{i=1}^{m} v_1X_i, \bigcup_{i=1}^{m} X_i = X, m \in N\right\}$, $w_2X = \bigcap\{Y \subseteq P|v_2X \cup v_2Y \subseteq Y\}$, $w_3X = v_1X$ for

$X = \emptyset$ and $w_3X = \left(\bigcup_{x\in X} v_1\{x\}\right) \cap v_1X$ for $X \neq \emptyset$.

$\mathscr{P}_{\mathrm{MA}} \leftarrow \mathscr{P}_{\mathrm{A}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MA}}$. There holds $v_1 \in \mathscr{P}_{\mathrm{M}}$ by Theorem 1. Let $X \subseteq Y \subseteq P$ be subsets, $x \in w_1X$ a point. Let $\{Y_i | i = 1, \ldots, m\}$ be a system of sets such that $\bigcup_{i=1}^{m} Y_i = Y$. Put $X_i = Y_i \cap X$ for each $i \in \{1, \ldots, m\}$. Then $\bigcup_{i=1}^{m} X_i = X$, and hence $x \in \bigcup_{i=1}^{n} v_1X_i \subseteq \bigcup_{i=1}^{m} v_1 Y_i$. Consequently, $x \in w_1 Y$. Thus $w_1X \subseteq w_1 Y$, i.e. w_1 is an M-topology on P. Therefore $w_1 \in \mathscr{P}_{\mathrm{MA}}$. Let $\bigwedge$ and $\bigvee$ denote the meet and join in $\mathscr{P}_{\mathrm{A}}$. By Theorem 2, $w_1 = \bigwedge\mathscr{T}$, hence $\bigwedge\mathscr{T} \in \mathscr{P}_{\mathrm{MA}}$. From Theorem 1 it follows that $\bigvee\mathscr{T} \in \mathscr{P}_{\mathrm{MA}}$. The relation $\mathscr{P}_{\mathrm{MA}} \leftarrow \mathscr{P}_{\mathrm{A}}$ is proved.

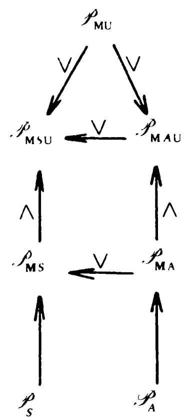


Diagram 1

$\mathscr{P}_{\mathrm{MAU}} \overset{\wedge}{\leftarrow} \mathscr{P}_{\mathrm{MA}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MAU}}$. Let $X \subseteq P$ be a subset. Then $v_1v_1X = \bigcap_{v\in\mathscr{T}}\left(v \bigcap_{u\in\mathscr{T}} uX\right) \subseteq \bigcap_{v\in\mathscr{T}}\left(\bigcap_{u\in\mathscr{T}} vuX\right) \subseteq \bigcap_{u\in\mathscr{T}} uuX \subseteq \bigcap_{u\in\mathscr{T}} uX = v_1X$. Hence v_1 is a U-topology on P. Let $X \subseteq P$ be a subset and $x \in w_1w_1X$ a point. Then $x \in \bigcup_{i=1}^{n} v_1 Y_i$ for any system of sets $\{Y_i | i = 1, \ldots, n\}$ fulfilling $\bigcup_{i=1}^{n} Y_i = w_1X$. Let $\{X_i | i = 1, \ldots, m\}$ be a system of sets such that $\bigcup_{i=1}^{m} X_i = X$. Put $Y_i = w_1X_i$ for each $i \in \{1, \ldots, m\}$. Let the meet in $\mathscr{P}_{\mathrm{MA}}$ be denoted by $\bigwedge$. Since $w_1 = \bigwedge\mathscr{T}$ is an MA-topology on P, we have $\bigcup_{i=1}^{m} Y_i = \bigcup_{i=1}^{m} w_1X_i = w_1\bigcup_{i=1}^{m} X_i = w_1X$. Therefore $x \in \bigcup_{i=1}^{m} v_1 Y_i = \bigcup_{i=1}^{m} v_1w_1X_i \subseteq \bigcup_{i=1}^{m} v_1v_1X_i \subseteq \bigcup_{i=1}^{m} v_1X_i$ because $w_1 = \bigwedge\mathscr{T} \leqq v_1$ and $v_1 \in \mathscr{P}_{\mathrm{U}}$. Consequently, $x \in w_1X$, which implies $w_1w_1X \subseteq w_1X$. Thus w_1 is a U-topology on P. Hence $w_1 = \bigwedge\mathscr{T} \in \mathscr{P}_{\mathrm{MAU}}$, i.e. $\mathscr{P}_{\mathrm{MAU}} \overset{\wedge}{\leftarrow} \mathscr{P}_{\mathrm{MA}}$.

$\mathscr{P}_{\mathrm{MAU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MA}}$.

$\mathscr{P}_{\mathrm{MAU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MU}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MAU}}$. Let $X, Y \subseteq P$ be subsets. Then $v_2(X \cup Y) = \bigcup_{u \in \mathscr{T}} u(X \cup Y) \subseteq \bigcup_{u \in \mathscr{T}} (uX \cup uY) = \bigcup_{u \in \mathscr{T}} uX \cup \bigcup_{u \in \mathscr{T}} uY = v_2 X \cup v_2 Y$. This implies that v_2 is an A-topology on P. Let $x \in w_2 (X \cup Y)$ be a point. Then $x \in Z$ for every subset $Z \subseteq P$ fulfilling $v_2(X \cup Y) \cup v_2 Z \subseteq Z$. Thus, $x \in Z$ for each subset $Z \subseteq P$ with $v_2 X \cup v_2 Y \cup v_2 Z \subseteq Z$. Let $T, U \subseteq P$ be subsets fulfilling $v_2 X \cup v_2 T \subseteq T$, $v_2 Y \cup v_2 U \subseteq U$. Then $v_2 X \cup v_2 Y \cup v_2 (T \cup U) \subseteq T \cup U$. Therefore $x \in T \cup U$, i.e. $x \in T$ or $x \in U$. Consequently, $x \in w_2 X$ or $x \in w_2 Y$. From here we get $x \in w_2 X \cup w_2 Y$, and the inclusion $w_2(X \cup Y) \subseteq w_2 X \cup w_2 Y$ is proved. Hence $w_2 \in \mathscr{P}_{\mathrm{A}}$. Now, denoting the join in $\mathscr{P}_{\mathrm{MU}}$ by $\bigvee$, according to Theorem 4 we have $w_2 = \bigvee \mathscr{T}$. This implies $\bigvee \mathscr{P} \in \mathscr{P}_{\mathrm{MAU}}$, so $\mathscr{P}_{\mathrm{MAU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MU}}$.

$\mathscr{P}_{\mathrm{MS}} \leftarrow \mathscr{P}_{\mathrm{S}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MS}}$. By Theorem 1, v_1 is an M-topology on P. Let $X \subseteq Y \subseteq P$ be subsets. For $X = \emptyset = Y$ there holds $w_3 X = v_1 X \subseteq v_1 Y = w_3 Y$. If $X = \emptyset$ and $Y \neq \emptyset$, then $w_3 X = v_1 X = v_1 \emptyset \subseteq \left(\bigcup_{x \in Y} v_1\{x\} \right) \cap v_1 Y = w_3 Y$. Finally, supposing $X \neq \emptyset \neq Y$ we have $w_3 X = \bigcup_{x \in X} v_1\{x\} \cap v_1 X \subseteq \bigcup_{x \in Y} v_1\{x\} \cap v_1 Y = w_3 Y$. Hence w_3 is an M-topology on P. Denote the meet and join in $\mathscr{P}_{\mathrm{S}}$ by $\bigwedge$ and $\bigvee$. According to Theorem 3, $w_3 = \bigwedge \mathscr{T}$. Thus $\bigwedge \mathscr{T} \in \mathscr{P}_{\mathrm{MS}}$. From Theorem 1 it follows that $\bigvee \mathscr{T} \in \mathscr{P}_{\mathrm{MS}}$. The relation $\mathscr{P}_{\mathrm{MS}} \leftarrow \mathscr{P}_{\mathrm{S}}$ is proved.

$\mathscr{P}_{\mathrm{MSU}} \overset{\wedge}{\leftarrow} \mathscr{P}_{\mathrm{MS}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MSU}}$. As $\mathscr{P}_{\mathrm{MSU}} \subseteq \mathscr{P}_{\mathrm{MAU}}$, from the proof of the relation $\mathscr{P}_{\mathrm{MAU}} \overset{\wedge}{\leftarrow} \mathscr{P}_{\mathrm{MA}}$ it follows that v_1 is a U-topology on P. By Theorem 1, v_1 is an M-topology on P. Thus $v_1 \in \mathscr{P}_{\mathrm{MU}}$. Denote the meet in $\mathscr{P}_{\mathrm{MS}}$ by $\bigwedge$. Then $w_3 = \bigwedge \mathscr{T}$ because $\mathscr{P}_{\mathrm{MS}} \leftarrow \mathscr{P}_{\mathrm{S}}$. Let $X \subseteq P$ be a subset. Suppose $X = \emptyset$. Then $w_3 w_3 X = w_3 v_1 X$. If $v_1 X = \emptyset$, then $w_3 v_1 X = v_1 v_1 X \subseteq v_1 X = w_3 X$. Otherwise, let $v_1 X \neq \emptyset$. Then $w_3 v_1 X = \bigcup_{x \in v_1 X} v_1\{x\} \cap v_1 v_1 X \subseteq v_1 X = w_3 X$. Thus, for the empty set X we have $w_3 w_3 X \subseteq w_3 X$. Now, suppose $X \neq \emptyset$. If $w_3 X = \emptyset$, then there is $w_3 w_3 X = w_3 \emptyset \subseteq w_3 X$ because $w_3 \in \mathscr{P}_{\mathrm{M}}$. Otherwise, let $w_3 X \neq \emptyset$. As $v_1 \in \mathscr{P}_{\mathrm{M}}$, the inclusion $\bigcup_{x \in Y} v_1\{x\} \subseteq v_1 Y$ holds whenever $\emptyset \neq Y \subseteq P$. This implies $w_3 Y = \bigcup_{x \in Y} v_1 Y$ for every non-empty subset $Y \subseteq P$. Hence $w_3 w_3 X = \bigcup_{x \in w_3 X} v_1\{x\} = \bigcup_{x \in \bigcup_{y \in X} v_1\{y\}} v_1\{x\} =$

$$= \bigcup_{\substack{y \in X \\ v_1\{y\} \neq \emptyset}} \bigcup_{x \in v_1\{y\}} v_1\{x\} = \bigcup_{\substack{y \in X \\ v_1\{y\} \neq \emptyset}} w_3 v_1\{y\} \subseteq \bigcup_{\substack{y \in X \\ v_1\{y\} \neq \emptyset}} v_1 v_1\{y\} \subseteq \bigcup_{\substack{y \in X \\ v_1\{y\} \neq \emptyset}} v_1\{y\} = \bigcup_{y \in X} v_1\{y\} =$$

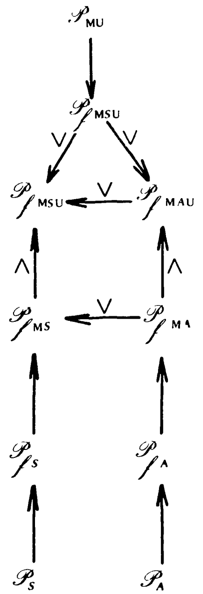
$= w_3 X$. Consequently, the inclusion $w_3 w_3 X \subseteq w_3 X$ is valid for any subset $X \subseteq P$.

Therefore w_3 is a U-topology on P. Thus $w_3 = \bigwedge \mathscr{T} \in \mathscr{P}_{\mathrm{SU}}$, which yields $\mathscr{P}_{\mathrm{MSU}} \overset{\wedge}{\leftarrow} \mathscr{P}_{\mathrm{MS}}$.

$\mathscr{P}_{\mathrm{MSU}} \leftarrow \mathscr{P}_{\mathrm{MU}}$: Let $\mathscr{T} \subseteq \mathscr{P}_{\mathrm{MSU}}$. Let $\emptyset \neq X \subseteq P$ be a subset, $y \in w_2 X$ a point. Suppose $y \notin \bigcup_{x \in X} w_2\{x\}$. Then we have $y \notin w_2\{x\}$ for every $x \in X$. Consequently, for every $x \in X$ there exists a subset $Y_x \subseteq P$ such that $v_2\{x\} \cup v_2 Y_x \subseteq Y_x$ and $y \notin Y_x$. Put $Y = \bigcup_{x \in X} Y_x$. Now, from $v_2 \in \mathscr{P}_{\mathrm{MS}}$ it follows that $v_2 X = \bigcup_{x \in X} v_2\{x\} \subseteq \bigcup_{x \in X} Y_x = Y$ and $v_2 Y = v_2 \bigcup_{x \in X} Y_x = \bigcup_{x \in X} v_2 Y_x \subseteq \bigcup_{x \in X} Y_x = Y$. This yields $v_2 X \cup v_2 Y \subseteq Y$. Hence, as $y \in w_2 X$, we obtain $y \in Y$, which is a contradiction. Therefore $y \in \bigcup_{x \in X} w_2\{x\}$ and the inclusion $w_2 X \subseteq \bigcup_{x \in X} w_2\{x\}$ is true. Thus, $w_2 \in \mathscr{P}_{\mathrm{S}}$. By Theorem 4, $w_2 = \bigvee \mathscr{T}$ where $\bigvee$ denotes the join in $\mathscr{P}_{\mathrm{MU}}$. This results in $\bigvee \mathscr{T} \in \mathscr{P}_{\mathrm{MSU}}$, i.e. $\mathscr{P}_{\mathrm{MSU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{M}}$.

Finally, the relation $\mathscr{P}_{\mathrm{MS}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MA}}$ follows from Theorem 1, and $\mathscr{P}_{\mathrm{MSU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MAU}}$ is a consequence of $\mathscr{P}_{\mathrm{MSU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MU}}$ and $\mathscr{P}_{\mathrm{MAU}} \overset{\vee}{\leftarrow} \mathscr{P}_{\mathrm{MU}}$. The proof is complete.

Corollary 1. *Let $f \in \{0, I\}$. Then there holds*



Proof. By the help of Theorems 2, 3 and 4, the reader can easily prove that for $f \in \{0, I\}$ the following three statements hold:

(1) If $\mathcal{T} \subseteq \mathcal{P}_{fA}$, then the meet $\bigwedge\mathcal{T}$ in $\mathcal{P}_A$ fulfils $\bigwedge\mathcal{T} \in \mathcal{P}_f$.
(2) If $\mathcal{T} \subseteq \mathcal{P}_{fS}$, then the meet $\bigwedge\mathcal{T}$ in $\mathcal{P}_S$ fulfils $\bigvee\mathcal{T} \in \mathcal{P}_f$.
(3) If $\mathcal{T} \subseteq \mathcal{P}_{fMU}$, then the join $\bigvee\mathcal{T}$ in $\mathcal{P}_{MU}$ fulfils $\bigvee\mathcal{T} \in \mathcal{P}_f$.
Then, using also Theorem 1, we get Corollary 1 as a consequence of Theorem 5.

Remark 2. a) From Corollary 1 it follows that $\mathcal{P}_{OIMAU} \overset{\wedge}{\leftarrow} \mathcal{P}_{OIMA}$. But this relation is well known — see [4], 31 B.4.

b) From Theorems 1,3 and Corollary 1 it follows that in the lattice $\mathcal{P}_{MS}$ for the meet $\bigwedge\mathcal{T}$ of an arbitrary non-empty system $\mathcal{T} \subseteq \mathcal{P}_{MS}$ there holds $(\bigwedge\mathcal{T})\emptyset = \bigcap_{u\in\mathcal{T}} u\emptyset$ and $(\bigwedge\mathcal{T})X = \bigcup_{x\in X}(\bigcap_{u\in\mathcal{T}} u\{x\})$ whenever $\emptyset \neq X \subseteq P$.

c) As a consequence of Theorems 1,4 and Corollary 1 it can be easily seen that in the lattice $\mathcal{P}_{IMU}$ for the join $\bigvee\mathcal{T}$ of an arbitrary non-empty system $\mathcal{T} \subseteq \mathcal{P}_{IMU}$ there holds $(\bigvee\mathcal{T})X = \bigcap\left\{Y \subseteq P | X \subseteq Y = \bigcup_{u\in\mathcal{T}} uY\right\}$ for every subset $X \subseteq P$.

d) Corollary 1 implies that for the meet and join in $\mathcal{P}_{OIMAU}$ the formulae contained in Theorem 2 and in the section c) of this remark are valid. But these formulae for the meet and join in $\mathcal{P}_{OIMAU}$ can be obtained as consequences of [8] (3.2. and 3.7) and [11] (3.6.), too.

According to Remark 1, $\mathcal{P}_A$ is not a meet-sublattice of $\mathcal{P}$. In the following theorem it will be shown that even every element of $\mathcal{P}$ is the meet (in $\mathcal{P}$) of a certain non-empty subset of $\mathcal{P}_A$. Similar assertions will be proved for $\mathcal{P}_S$ and $\mathcal{P}_U$.

Theorem 6. *Let $u \in \mathcal{P}$ be a topology and let $\bigwedge$ denote the meet in $\mathcal{P}$. Then $u = \bigwedge\{v \in \mathcal{P}_A | v \geqq u\}$.*

Proof. Put $\mathcal{T} = \{v \in \mathcal{P}_A | v \geqq u\}$. We have $\mathcal{T} \neq \emptyset$ since the topology v defined by $v\emptyset = u\emptyset$ and $\emptyset \neq X \subseteq P \Rightarrow vX = P$ fulfils $v \in \mathcal{T}$. Put $w = \bigwedge\mathcal{T}$. Clearly, $u \leqq w$. For every subset $Y \subseteq P$ let us define a topology v_Y on P in the following way: $v_Y X = uX$ for $X = \emptyset$ or $X = Y$, and $v_Y X = P$ for $\emptyset \neq X \neq Y$. Evidently, $v_Y \geqq u$ holds for every subset $Y \subseteq P$. It can be easily shown that $v_Y \in \mathcal{P}_A$ for every subset $Y \subseteq P$. Consequently, $v_Y \in \mathcal{T}$ for every subset $Y \subseteq P$. Now, let $X \subseteq P$ be an arbitrary subset. Then $wX \subseteq v_X = uX$. This yields $w \leqq u$. Therefore $u = w$ and the statement is proved.

Theorem 7. *Let $u \in \mathcal{P}_M$ be a topology and let $\bigvee$ and $\bigwedge$ denote the join and meet in $\mathcal{P}$. Then*

(1) $u = \bigwedge\{v \in \mathcal{P}_S | v \geqq u\}$,
(2) $u = \bigvee\{v \in \mathcal{P}_U | v \leqq u\}$.

Proof. (1) Put $\mathcal{T} = \{v \in \mathcal{P}_S | v \geqq u\}$. We have $\mathcal{T} \neq \emptyset$ because the topology v defined in the proof of Theorem 6 fulfils $v \in \mathcal{T}$. Put $w = \bigwedge\mathcal{T}$. Clearly, $u \leqq w$. For any subset $Y \subseteq P$ let us define a topology v_Y on P as follows:

$$X \subseteq P \Rightarrow v_Y X = \begin{cases} uX \text{ for } X = \emptyset, \\ uY \text{ for } \emptyset \neq X \subseteq Y, \\ P \text{ for } X \nsubseteq Y. \end{cases}$$

As u is an M-topology, there holds $v_Y \geqq u$ for every subset $Y \subseteq P$. It can be easily seen that $v_Y \in \mathscr{P}_S$ for every subset $Y \subseteq P$. Therefore $v_Y \in \mathscr{T}$ for every subset $Y \subseteq P$. Now, let $X \subseteq P$ be an arbitrary subset. Then $wX \subseteq v_X X = uX$. This yields $w \geqq u$. We have $u = w$, which gives the equality (1).

(2) Put $\mathscr{T} = \{v \in \mathscr{P}_U | v \leqq u\}$. Then $\mathscr{T} \neq \emptyset$ because the topology $v = u^*$ fulfils $v \in \mathscr{T}$. Put $w = \bigvee \mathscr{T}$. Clearly, $w \leqq u$. For any subset $Y \subseteq P$ let us define a topology v_Y on P in the following way: $v_Y X = \emptyset$ for $Y \nsubseteq X$, and $v_Y X = uY$ for $Y \subseteq X$. As u is an M-topology, there holds $v_Y \leqq u$ for every subset $Y \subseteq P$. The reader can easily show that $v_Y \in \mathscr{P}_{MU}$ for every subset $Y \subseteq P$. Consequently, $v_Y \in \mathscr{T}$ for every subset $Y \subseteq P$. Let $X \subseteq P$ be an arbitrary subset. Then $uX = v_X X \subseteq wX$. This yields $u \leqq w$. Therefore $u = w$ and the proof is complete.

Theorem 8. *Let* $f \in \{\text{O, I, M, OI, OM, IM, OIM}\}$. *Let* $u \in \mathscr{P}_f$ *be a topology and let* $\bigvee$ *and* $\bigwedge$ *denote the join and meet in* $\mathscr{P}$. *Then there holds*:

(1) $u = \bigwedge \{v \in \mathscr{P}_{fA} | v \geqq u\}$ *whenever* $f \in \{\text{O, I, OI, OIM}\}$,

(2) $u = \bigwedge \{v \in \mathscr{P}_{fS} | v \geqq u\}$ *whenever* $f \in \{\text{M, OM, IM, OIM}\}$,

(3) $u = \bigvee \{v \in \mathscr{P}_{fU} | v \leqq u\}$ *whenever* $f \in \{\text{M, OM, OIM}\}$.

Proof. For $f \in \{\text{O, I, OI}\}$ the proof of the equality (1) is the same as that of Theorem 6 because provided that $u \in \mathscr{P}_f$ it can be easily seen that the topologies v and v_Y defined there fulfil $v \in \mathscr{T} = \{v \in \mathscr{P}_{fA} | v \geqq u\}$ and $v_Y \in \mathscr{P}_f$ for every subset $Y \subseteq P$. Analogously, the proof of (2) and for $f \in \{\text{M,OM}\}$ the proof of (3) are the same as those of (1) and (2) of Theorem 7. For $f = \text{OIM}$ the equalities (1) and (3) follow from [8] (3.1.1. and 3.8.1.).

Now, let us introduce the following denotation. If $\mathscr{T} \subseteq \mathscr{P}$ is a subsystem, then by $\langle \mathscr{T} \rangle$ we denote the complete sublattice of $\mathscr{P}$ generated by $\mathscr{T}$ (i.e. the least complete sublattice of $\mathscr{P}$ containing $\mathscr{T}$). From Theorems 6 and 8 it immediately follows:

Corollary 2. *There holds*

(1) $\langle \mathscr{P}_A \rangle = \mathscr{P}$, *and* $\langle \mathscr{P}_{fA} \rangle = \mathscr{P}_f$ *for* each $f \in \{\text{O, I, OI, OIM}\}$,

(2) $\langle \mathscr{P}_{fS} \rangle = \mathscr{P}_f$ *for* each $f \in \{\text{M, OM, IM, OIM}\}$,

(3) $\langle \mathscr{P}_{fU} \rangle = \text{P}_f$ *for* each $f \in \{\text{M, OM, OIM}\}$.

Remark 3. a) The equalities $\langle \mathscr{P}_{\text{OIMA}} \rangle = \mathscr{P}_{\text{OIM}}$ and $\langle \mathscr{P}_{\text{OIMU}} \rangle = \mathscr{P}_{\text{OIM}}$ contained in Corollary 2 follow also from the equality $\langle \mathscr{P}_{\text{OIMAU}} \rangle = \mathscr{P}_{\text{OIM}}$ proved in [10].

b) In [4], 31 D.3 it is shown that every topology $u \in \mathscr{P}_{\text{OIMA}}$ is the meet in $\mathscr{P}_{\text{OIMA}}$ of a certain non-empty subsystem of $\mathscr{P}_{\text{OIMS}}$. Consequently, denoting by $\langle \mathscr{T} \rangle_1$ the complete sublattice of $\mathscr{P}_{\text{OIMA}}$ generated by a subsystem $\mathscr{T} \subseteq \mathscr{P}_{\text{OIMA}}$, we have $\langle \mathscr{P}_{\text{OIMS}} \rangle_1 = \mathscr{P}_{\text{OIMA}}$.

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О РЕШЕТКАХ ОБОБЩЕННЫХ ТОПОЛОГИЙ

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Резюме

Обобщенной топологией мы понимаем топологию, определенную оператором замыкания, выполняющим какие-нибудь аксиомы, которые слабее, чем аксиомы Куратовского. В работе изучаются некоторые системы обобщенных топологий на данном множестве, являющиеся полными решетками относительно обычного упорядочения этих систем.