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STICKELBERGER IDEAL OF A COMPOSITUM OF A REAL BICYCLIC FIELD AND A QUADRATIC IMAGINARY FIELD

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ABSTRACT. For a real abelian field with a non-cyclic Galois group of order l^2, l being an odd prime, a compositum with a suitable quadratic imaginary field is considered and its Stickelberger ideal in the sense of Sinnott is studied. Finally, the index of the Stickelberger ideal is computed.

1. Introduction

In abelian fields, two natural objects are linked to the structure of the ideal class group. In particular, they are linked to the class number $h = h^+ \cdot h^-$. The group C of circular units is linked to h^+, the class number of the maximum real subfield. In fact, Sinnott's formula from [3] gives us the index $[E : C]$ of the group of circular units as the product of h^+ and some other factors, one of which is the so-called Sinnott index $(e^+R : e^+U)$, that is explicit only in some special cases. In [1] we studied the simplest non-solved case of this non-explicit situation (by simplicity we mean the simplicity of the Galois group), namely the case of a bicyclic field. We obtained a fully explicit formula for both indices $[E : C]$ and $(e^+R : e^+U) = (R : U)$.

This paper is devoted to a similar problem concerning the Stickelberger ideal. We recall that the Stickelberger ideal, the elements of which annihilate the class group of K, is linked to $h^- = \frac{h}{h^+}$, the relative class number. In [3] a formula is derived which gives us the index of the Stickelberger ideal in terms of h^- and one other non-explicit factor $(e^-R : e^-U)$. Since for real fields the Stickelberger ideal is trivial, we consider the compositum of our bicyclic field K and an imaginary

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quadratic field F. We apply the methods used in [1] to get an explicit formula for the index of the Stickelberger ideal. This formula is stated in Theorem 7.2 at the end of this paper.

2. Notation

We shall introduce the following notation:

$\zeta_n = \mathrm{e}^{2\pi\,\mathrm{i}/n}$ is a primitive nth root of unity;
$\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$ is the nth cyclotomic field;
$\sigma_a\colon \zeta_n \mapsto \zeta_n^a$, $\sigma_a \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ are the automorphisms of $\mathbb{Q}_n$;
$\mathrm{Frob}(p, L) = \sigma_p\big|_L$ is the Frobenius automorphism of L for a prime p not ramified in L;
X_L is the group of Dirichlet characters corresponding to a field L;
$\langle x \rangle$ is the fractional part of x.

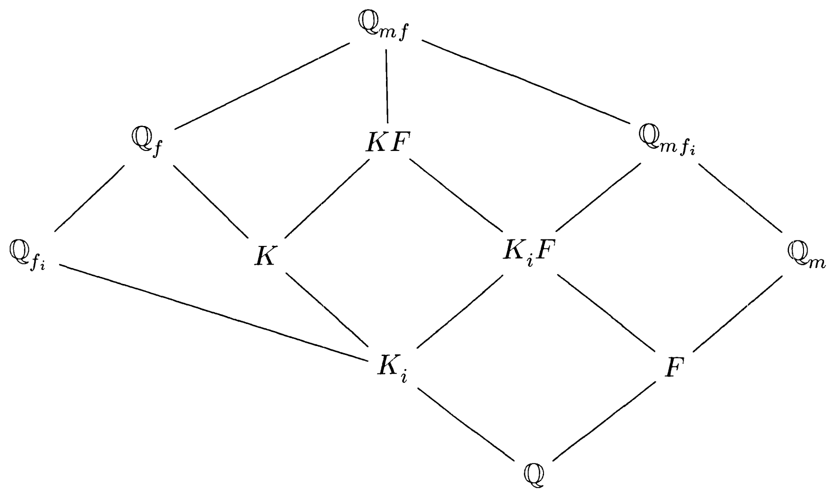
3. Defining a suitable quadratic imaginary extension of the bicyclic field

In this section we will — without explicit mention — use some facts about ramification properties of K proved in [1]. Let K be a real abelian field of degree l^2 with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{F}_l \times \mathbb{F}_l$, l being an odd prime not ramifying in K. Let K_i, $0 \le i \le l$, be all subfields of K of degree l over $\mathbb{Q}$. Let $f = p_1 p_2 \cdots p_s$ be the conductor of K and f_i the conductor of K_i.

Now we will define a quadratic imaginary extension of K. Let F be a quadratic imaginary field of conductor m such that all p_i split completely in F, in other words, for all $1 \le j \le s$ we have $\mathrm{Frob}(p_j, F) = 1$. Let $\mathrm{Gal}(F/\mathbb{Q}) = \{1, J\}$, where J denotes the complex conjugation. By abuse of notation, in what follows, J also denotes the complex conjugation in each considered Galois group.

The field FK of degree $2l^2$ has one real subfield K of degree l^2, $l+1$ real subfields K_i of degree l, $l+1$ imaginary subfields FK_i of degree $2l$ and one imaginary quadratic subfield F. Let Q be the Hasse unit index of FK and w the number of roots of unity in FK.

Let $\delta_i \in \mathrm{Gal}(KF/F)$ be fixed so that $\left\langle \delta_i|_{K_i F} \right\rangle = \mathrm{Gal}(K_i F/F)$. Let $G = \mathrm{Gal}(FK/F)$ and let g_i be a fixed generator of $G_i = \mathrm{Gal}(FK/FK_i)$. By abuse of notation, in the last part of the paper we will identify the elements of G with their restrictions in $\mathrm{Gal}(K/\mathbb{Q})$.



4. Stickelberger ideal for the field FK

From [2] it follows that the Stickelberger ideal S of the field FK in the sense of Sinnott can be defined as $S = S' \cap \mathbb{Z}\big[\mathrm{Gal}(FK/\mathbb{Q})\big]$, where

$$S' = \big\langle \theta'_n,\ n = 2 \text{ or } n|mf \big\rangle_{\mathrm{Gal}(FK/\mathbb{Q})}$$

and $\theta'_n = \mathrm{cor}_{FK/L_n} \mathrm{res}_{\mathbb{Q}_n/L_n} \theta_n$ with $\theta_n = \sum\limits_{(t,n)=1} \big\langle \frac{t}{n} \big\rangle \sigma_t^{-1}$ and $L_n = \mathbb{Q}_n \cap FK$.

From [3] it follows that the index of the Stickelberger ideal is

$$[A : S] = w \cdot (A : S') = w \cdot (R^- : e^- S') = w \cdot (R^- : S'^-)\,,$$

where

$$A = \Big\{ \alpha \in R :\ (1 + J)\alpha \in \Big(\sum_{\sigma \in \mathrm{Gal}(FK/\mathbb{Q})} \sigma \Big) \mathbb{Z} \Big\},$$

$R = \mathbb{Z}\big[\mathrm{Gal}(FK/Q)\big]$, $e^- = \frac{1-J}{2}$ and $R^- = R \cap e^- R = (1 - J)R$. Similarly $S'^- = S' \cap e^- S'$.

5. Relations between the Stickelberger elements

For $0 \le i \le l + 1$ let $\theta^-_{K_i} = \frac{1-J}{2}\theta'_{f_i}$, $\theta^-_K = \frac{1-J}{2}\theta'_f$, $\theta^-_F = \frac{1-J}{2}\theta'_m$, $\theta^-_{K_iF} = \frac{1-J}{2}\theta'_{mf_i}$, $\theta^-_{KF} = \frac{1-J}{2}\theta'_{mf}$ be the minus parts of the Stickelberger elements corresponding to different subfields of FK. Then

$$S'^{-} = \Big\langle \theta'^{-}_{2},\ \{\theta^{-}_{K_i} :\ 0 \le i \le l+1\},\ \theta^{-}_{K},\ \theta^{-}_{KF},\ \theta^{-}_{F},\\ \{\theta^{-}_{K_iF} :\ 0 \le i \le l+1\}\Big\rangle_{\mathrm{Gal}(FK/\mathbb{Q})}.$$

The following lemma shows that the Stickelberger elements corresponding to real subfields vanish.

LEMMA 5.1. *For all* $0 \le i \le l+1$ *we have*

$$\theta'^{-}_{2} = \theta^{-}_{K} = \theta^{-}_{K_i} = 0.$$

P r o o f. $\theta^{-}_{K} = \frac{1-J}{2}\theta'_f = \frac{1-J}{2}\operatorname{cor}_{FK/K}\operatorname{res}_{Q_f/K}\theta_f = \frac{1-J}{2}(1+J)\operatorname{res}_{Q_f/K}\theta_f = 0$, because $J^2 = 1$. The other identities follow similarly. □

COROLLARY 5.1. *We have*

$$S'^{-} = \Big\langle \theta^{-}_{KF},\ \theta^{-}_{F},\ \{\theta^{-}_{K_iF} :\ 0 \le i \le l+1\}\Big\rangle_{\mathrm{Gal}(FK/\mathbb{Q})}.$$

Using the same idea as in [1] we define the submodule

$$T'^{-} = \Big\langle \theta^{-}_{F},\ \{\theta^{-}_{K_iF} :\ 0 \le i \le l+1\}\Big\rangle_{\mathrm{Gal}(FK/\mathbb{Q})}.$$

The situation is analogous to the situation with the circular units dealt with in [1]. The Stickelberger elements $\theta^{-}_{K_iF}$ arising from the fields FK_i are the analogues of the circular numbers ε_i from the fields K_i, the element θ^{-}_{KF} arising from the field FK is the analogue of the circular unit η from the field K. The element θ^{-}_{F} arising from the field F is the only generator without an analogous circular unit.

In [1] we proved that

$$N_{K/K_i}(\eta) = \varepsilon_i^{\prod\limits_{j\in P_i}(1-\delta_i^{k_{ij}})} \qquad \text{and} \qquad N_{K_i/\mathbb{Q}}(\varepsilon_i) = 1 \quad \text{for } f_i \text{ composite.}$$

Similarly as in [1] we define $P_i = \{j :\ p_j \nmid f_i,\ 1 \le j \le s\}$ as the index set for the primes unramified in K_i. For $p_j \nmid f_i$ the numbers $k_{ij} \in \mathbb{F}_l$ are defined so that

$$\left(\delta_i\big|_{FK_i}\right)^{k_{ij}} = \mathrm{Frob}^{-1}(p_j, FK_i).$$

Now we will prove the analogous relations for the Stickelberger elements. The question whether f_i is composite or not will, however, not matter anymore. Before we state the theorem, we need a lemma:

LEMMA 5.2. *Let p be a prime not dividing n. Then*

$$\frac{1-J}{2}\operatorname{res}_{\mathbb{Q}_{pn}/\mathbb{Q}_n}\theta_{pn}=\frac{1-J}{2}\theta_n\cdot\left(1-\operatorname{Frob}^{-1}(p,\mathbb{Q}_n)\right).$$

P r o o f. Easily follows from [2; Lemma 12]. □

PROPOSITION 5.1. *For any $0\le i\le l+1$ we have*

$$N_{K_i/\mathbb{Q}}\cdot\theta^-_{K_iF}=0 \qquad \text{and} \qquad N_{K/K_i}\cdot\theta^-_{KF}=\theta^-_{K_iF}\cdot\prod_{j\in P_i}\left(1-\delta_i^{k_{ij}}\right),$$

where $N_{K_i/\mathbb{Q}}=\sum_{j=0}^{l-1}\delta_i^j$ *and* $N_{K/K_i}=\sum_{j=0}^{l-1}g_i^j$.

P r o o f. Using the last lemma, [2; Lemma 12] and the fact that all prime divisors of f split in F, we get

$$\begin{aligned}
N_{K_i/\mathbb{Q}}\cdot\theta^-_{K_iF}&=\frac{1-J}{2}\operatorname{cor}_{FK/FK_i}\left(\left(\sum_{j=0}^{l-1}\delta_i\big|_{FK_i}\right)\cdot\operatorname{res}_{\mathbb{Q}_{mf_i}/K_iF}\theta_{mf_i}\right)\\
&=\frac{1-J}{2}\operatorname{cor}_{FK/FK_i}\operatorname{cor}_{FK_i/F}\operatorname{res}_{K_iF/F}\operatorname{res}_{\mathbb{Q}_{mf_i}/K_iF}\theta_{mf_i}\\
&=\frac{1-J}{2}\operatorname{cor}_{FK/F}\operatorname{res}_{\mathbb{Q}_m/F}\operatorname{res}_{\mathbb{Q}_{mf_i}/\mathbb{Q}_m}\theta_{mf_i}\\
&=\frac{1-J}{2}\operatorname{cor}_{FK/F}\operatorname{res}_{\mathbb{Q}_m/F}\left(\theta_m\cdot\prod_{q|f_i}\left(1-\operatorname{Frob}^{-1}(q,\mathbb{Q}_m)\right)\right)\\
&=\frac{1-J}{2}\operatorname{cor}_{FK/F}\left(\prod_{q|f_i}\left(1-\operatorname{Frob}^{-1}(q,F)\right)\cdot\operatorname{res}_{\mathbb{Q}_m/F}\theta_m\right)\\
&=0\,,
\end{aligned}$$

because the corresponding Frobenius automorphisms are trivial. To prove the second part of the proposition let us at first suppose that $f_i=f$. Then

$$N_{K/K_i}\cdot\theta^-_{KF}=\frac{1-J}{2}\operatorname{cor}_{FK/FK_i}\operatorname{res}_{\mathbb{Q}_{mf}/FK_i}\theta_{fm}=\theta^-_{FK_i}\,,$$

which is the statement of the theorem for P_i empty.

Now let $f_i \neq f$. Then

$$
\begin{aligned}
&N_{K/K_i} \cdot \theta^-_{KF} \\
&= N_{K/K_i} \cdot \frac{1-J}{2} \operatorname{res}_{\mathbb{Q}_{mf}/FK} \theta_{fm} \\
&= \frac{1-J}{2} \operatorname{cor}_{FK/FK_i} \operatorname{res}_{\mathbb{Q}_{mf}/FK_i} \theta_{fm} \\
&= \frac{1-J}{2} \operatorname{cor}_{FK/FK_i} \operatorname{res}_{\mathbb{Q}_{mf_i}/FK_i} \operatorname{res}_{\mathbb{Q}_{mf}/\mathbb{Q}_{mf_i}} \theta_{mf} \\
&= \frac{1-J}{2} \operatorname{cor}_{FK/FK_i} \operatorname{res}_{\mathbb{Q}_{mf_i}/FK_i} \Big(\theta_{mf_i} \cdot \prod_{j \in P_i} (1 - \operatorname{Frob}^{-1}(p_j, \mathbb{Q}_{mf_i})) \Big) \\
&= \frac{1-J}{2} \operatorname{cor}_{FK/FK_i} \Big((\operatorname{res}_{\mathbb{Q}_{mf_i}/FK_i} \theta_{mf_i}) \cdot \prod_{j \in P_i} (1 - \operatorname{Frob}^{-1}(p_j, FK_i)) \Big) \\
&= \theta^-_{K_iF} \cdot \prod_{j \in P_i} (1 - \delta_i^{k_{ij}}) ,
\end{aligned}
$$

which proves the second part of the theorem. □

In exactly the same way as in [1; p. 306] we may derive the following formula:

COROLLARY 5.2. *We have*

$$
l \cdot \theta^-_{KF} = \sum_{i=0}^{l} \theta^-_{K_iF} \cdot \prod_{j \in P_i} (1 - \delta_i^{k_{ij}}) .
$$

Using the first part of the last proposition, we easily find a $\mathbb{Z}$-basis of T' :

PROPOSITION 5.2.

$$
T'^- = \big\langle \theta^-_F, \{ \delta_i^j \theta^-_{K_iF} : \ 0 \le i \le l+1, \ 0 \le j \le l-2 \} \big\rangle
$$

and the generating elements form a $\mathbb{Z}$-basis of T'^-.

Proof. The last proposition implies that $\delta_i^{l-1} \theta^-_{K_iF}$ can be expressed in terms of other elements and it is clear that $g_i^j \theta^-_{K_iF} = \theta^-_{K_iF}$ for any i, j. Thus the above system generates T'^-. This system has $1 + (l-1)(l+1) = l^2$ elements. But R^- has a basis of l^2 elements. From the last proposition it easily follows that T'^- has a finite index in R^-, thus the above system is a basis. □

Similarly as in [1], but with a slight simplification, we define:

DEFINITION 5.1. For $0 \le i \le l$ we define:

$$
a_i = \begin{cases} -\infty & \text{if there is a } j \in P_i \text{ such that } k_{ij} = 0 , \\ l - 1 - |P_i| & \text{if there is no such } j \in P_i . \end{cases}
$$

The groups G_i with the corresponding fields K_i were arranged arbitrarily, so we may now assume that G_i, K_i are chosen so that

$$a_0 \geq a_1 \geq \cdots \geq a_l .$$

Let σ, resp. τ, be the previously fixed generator of G_0, resp. G_1. For $2 \leq i \leq l$ we can modify the chosen generator g_i and suppose that $g_i = \tau\sigma^{l-n_i}$, which defines $n_i \in \mathbb{F}_l^\times$ in a unique way. We may now assume that $\delta_0 = \tau$, $\delta_i = \sigma$ for $1 \leq i \leq l$.

Definition 5.2. Let k, resp. q, be the smallest i such that

$$a_i \leq 0 , \qquad \text{resp.} \quad a_i = -\infty .$$

If no such i exists, then k, resp. q, equals $l+1$.

Similarly as in [1], we easily derive:

Proposition 5.3. *We have*

$$l \cdot \theta_{KF}^- = \sum_{0 \leq i < q} \theta_{K_iF}^- \cdot \mu_i(\delta_i) \cdot (1-\delta_i)^{l-1-a_i} ,$$

where $\mu_i(\delta_i) = \prod_{j \in P_i} (1 + \delta_i + \cdots + \delta_i^{k_{ij}-1})$ *and* $1-\delta_i$ *does not divide* μ_i *in* $\mathbb{F}_l[\delta_i]$. *By abuse of notation,* k_{ij} *here denotes a positive integer chosen from the corresponding residue class.*

6. Finding a basis of the module S'^-/T'^-

Reformulating the arguments from [1; p. 308–309], we get a criterion that gives us necessary and sufficient conditions for $\theta_{KF}^- \cdot g(\sigma,\tau) \in T'^-$:

Proposition 6.1. *For* $f \in \mathbb{F}_l[s,t]$ *the condition* $\theta_{KF}^- \cdot f(1-\sigma, 1-\tau) \in T'^-$ *is satisfied if and only if all of the following conditions hold:*

$$\begin{aligned} f(0,t) &\equiv 0 \pmod{t^{a_0}} && \text{if} \quad k > 0 , \\ f(s,0) &\equiv 0 \pmod{s^{a_1}} && \text{if} \quad k > 1 , \\ f\big(s, 1-(1-s)^{n_i}\big) &\equiv 0 \pmod{s^{a_i}} && \text{for all} \quad i , \ \ 2 \leq i < k \ \text{ if } \ k > 2 . \end{aligned}$$

The problem of finding a basis of S'^-/T'^- can be thus reduced to a problem formulated in terms of polynomial congruences; the congruences being the same as in [1], we may conclude that the basis of S'^-/T'^- has the same number of elements as the basis of C/B from [1] and thus:

Proposition 6.2. *We have:*

$$\big[S'^- : T'^-\big] = \begin{cases} 1 & \text{for } k = 0\,, \\ l^{a_0} & \text{for } k = 1\,, \\ l^{a_0+(a_1-1)+\cdots+(a_b-b)} & \text{for } k \geq 2\,, \end{cases}$$

where $b = 1$ for $k = 1$, for $k > 2$ it is the greatest index $i < k$ such that $a_i > i$.

7. Index of the Stickelberger ideal

Recall that the index of the Stickelberger ideal $[A : S] = w \cdot (R^- : S'^-)$. In order to compute the index $(R^- : S'^-)$, we will first compute the index $(R^- : T'^-)$ and then use the relation $\big(R^- : T'^-\big) = \big(R^- : S'^-\big) \cdot \big[S'^- : T'^-\big]$. We have already found a basis of T'^-, we may therefore write the transition matrix and compute its determinant, getting the index $\big(e^- R : T'^-\big)$ and thus the index $\big(R^- : T'^-\big)$:

Theorem 7.1.

$$\big(R^- : T'^-\big) = \frac{1}{Qw} \cdot h^-_{KF} \cdot l^{\frac{l^2-l}{2}}\,.$$

P r o o f. Let $\alpha = \frac{1-J}{2} \cdot \sum\limits_{g\in G} a_g g \in e^- R$. Then $\pi_g(\alpha)$ shall denote the integer coefficient a_g. The transition $l^2 \times l^2$ matrix from the canonical basis of $e^- R$ to the basis of T'^- found above is

$$M = \begin{pmatrix} \pi_g\big(\theta^-_F\big) \\ \vdots \\ \pi_g\big(\delta^j_i \theta^-_{K_i F}\big) \\ \vdots \end{pmatrix}$$

where all rows except the first are indexed by couples (i,j) with $0 \leq i \leq l$ and $0 \leq j \leq l-2$. The columns are indexed by the elements $g \in G$.

Now let χ_m be a fixed generator of X_{K_m} (the character group corresponding to the field K_m) and $Z = \big(1, \ldots, \chi^n_m(g), \ldots\big)$ be the character matrix, the first column corresponding to the trivial character. The other columns will be indexed with ordered pairs (m,n) with $0 \leq m \leq l$ and $1 \leq n \leq l-1$ and arranged in increasing lexicographical order. The first row corresponds to the identity automorphism, other rows correspond to $1 \neq g \in G$.

Now we will evaluate the product $M \cdot Z$:

1. In the first position of the first row we get

$$\begin{aligned}\sum_{g\in G}\pi_g(\theta_F^-) &= \sum_{g\in G}\pi_g\Big(\Big(\sum_{g\in G} g\Big)\tfrac{1-J}{2}\sum_{(t,m)=1}\langle\tfrac{t}{m}\rangle\sigma_t^{-1}|_F\Big)\\ &= \sum_{g\in G}\pi_g\Big(\Big(\sum_{g\in G} g\Big)\tfrac{1-J}{2}\sum_{(t,m)=1}\langle\tfrac{t}{m}\rangle\chi_F(t)\Big)\\ &= \sum_{g\in G}\pi_g\Big(\Big(\sum_{g\in G} g\Big)\tfrac{1-J}{2}B_{1,\chi_F}\Big)\\ &= B_{1,\chi_F}\cdot l^2,\end{aligned}$$

at other places we get zeroes.

2. At places $(i,j),(m,n)$, with $i\neq m$, we get

$$a_{(i,j),(m,n)} = \sum_{g\in G}\chi_m^n(g)\pi_g(\delta_i^j\theta_{K_iF}^-) = 0\,,$$

as can be easily shown by multiplying both sides by $\chi_m^n(g_i)\neq 1$.

3. At places $(i,j),(i,n)$ we get:

$$\begin{aligned}a_{(i,j),(i,n)} &= \sum_{g\in G}\chi_i^n(g)\pi_g(\delta_i^j\theta_{K_iF}^-) = \chi_i^n(\delta_i^j)\sum_{g\in G}\chi_i^n(g\delta_i^{-j})\pi_{g\delta_i^{-j}}(\theta_{K_iF}^-)\\ &= \chi_i^n(\delta_i^j)\sum_{g\in G}\chi_i^n(g)\pi_g(\theta_{K_iF}^-) = \chi_i^n(\delta_i^j)c_{i,n}\,,\end{aligned}$$

where

$$c_{i,n} = \sum_{g\in G}\chi_i^n(g)\pi_g(\theta_{K_iF}^-)\,.$$

Expanding the matrix $M\cdot Z$ along the first row, we get

$$|\det(M\cdot Z)| = |B_{1,\chi_F}|\cdot l^2\cdot|\det(Y)|\,,$$

where Y is the matrix consisting of $(l+1)\times(l+1)$ blocks

$$B_{im} = \big(a_{(i,j),(m,n)}\big)_{0\le j\le l-2,\,1\le n\le l-1}$$

of size $(l-1)\times(l-1)$. We have seen that only the blocks B_{ii} are non-zero, so Y is block-diagonal. Thus

$$\begin{aligned}|\det(Y)| &= \prod_{i=0}^{l}|\det(B_{ii})| = \prod_{i=0}^{l}\Big(\Big(\prod_{n=1}^{l-1}|c_{i,n}|\Big)\big|\det(\chi_i^n(\delta_i^j))\big|_{1\le j,\,n\le l-1}\Big)\\ &= l^{\frac{(l-2)(l+1)}{2}}\prod_{i=0}^{l}\prod_{n=1}^{l-1}|c_{i,n}|\,.\end{aligned}$$

Here we used the simple fact $\left|\det\left(\chi_i^n(\delta_i^j)\right)\right|_{1\le j,\,n\le l-1} = l^{\frac{l-2}{2}}$ derived in [1]. Using the similar fact $\det Z = l^{l^2}$ and substituting into the original formula we get

$$\begin{aligned}\left(e^- R : T'^-\right) &= |\det M| = |\det(MZ)|\cdot|\det Z|^{-1} \\ &= |B_{1,\chi_F}|\cdot l^2\cdot l^{\frac{(l-2)(l+1)}{2}}\cdot l^{-l^2}\cdot\prod_{i=0}^{l}\prod_{n=1}^{l-1}|c_{i,n}| \\ &= |B_{1,\chi_F}|\cdot l^{\frac{(l+2)(1-l)}{2}}\cdot\prod_{i=0}^{l}\prod_{n=1}^{l-1}|c_{i,n}|.\end{aligned}$$

We easily see that

$$\pi_g(\theta^-_{K_iF}) = 2\cdot\sum_{\substack{(t,mf_i)=1\\ \sigma_t|_{KF_i}=g|_{KF_i}}}\left(\left\langle\frac{t}{mf_i}\right\rangle-\frac12\right).$$

Thus

$$\begin{aligned}c_{i,n} &= 2\cdot\sum_{g\in G}\chi_i^n(g)\sum_{\substack{(t,mf_i)=1\\ \sigma_t|_{KF_i}=g|_{KF_i}}}\left(\left\langle\frac{t}{mf_i}\right\rangle-\frac12\right) \\ &= 2l\cdot\sum_{\substack{(t,mf_i)=1\\ \chi_F(t)=1}}\chi_i^n(t)\left(\left\langle\frac{t}{mf_i}\right\rangle-\frac12\right) \\ &= l\cdot\sum_{(t,mf_i)=1}\chi_i^n\chi_F(t)\left(\left\langle\frac{t}{mf_i}\right\rangle-\frac12\right) \\ &= l\cdot B_{1,\chi_i^n\chi_F}.\end{aligned}$$

Substituting we get

$$\left(e^-R : T'^-\right) = |B_{1,\chi_F}|\cdot l^{\frac{(l+2)(1-l)}{2}}\cdot l^{l^2-1}\prod_{i=0}^{l}\prod_{n=1}^{l-1}|B_{1,\chi_i^n\chi_F}| = l^{\frac{l^2-l}{2}}\cdot\prod_{\chi}|B_{1,\chi}|,$$

where χ runs through all odd characters on $\mathrm{Gal}(FK/\mathbb{Q})$. The well-known formula $h^- = Qw\prod_{\chi\,\mathrm{odd}}\left(-\frac12 B_{1,\chi}\right)$ implies in our case $\prod_{\chi\,\mathrm{odd}}|B_{1,\chi}| = \frac{1}{Qw}\cdot h^-_{FK}\cdot 2^{l^2}$. Finally we get $\left(e^-R : T'^-\right) = \frac{1}{Qw}\cdot h^-_{FK}\cdot 2^{l^2}\cdot l^{\frac{l^2-l}{2}}$. The theorem follows from $\left[e^-R : R^-\right] = 2^{l^2}$. □

Now we may substitute the results of Proposition 6.2 and Theorem 7.1 into the identity

$$[A : S] = w\cdot\left(R^- : S'^-\right) = w\cdot\left(R^- : T'^-\right)\cdot\left[S'^- : T'^-\right]^{-1},$$

getting a formula for the index of the Stickelberger ideal:

THEOREM 7.2. *Let K be an abelian field of degree l^2 with $G = \mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{F}_l \times \mathbb{F}_l$, l being an odd prime not ramifying in K. Let F be a quadratic imaginary field such that all primes ramifying in K split in F. Then the index of the Stickelberger ideal of the compositum KF is*

$$[A : S] = \frac{1}{Q} \cdot l^{\frac{1}{2}(l-1)l - \sum\limits_{i<a_i} (a_i - i)} \cdot h_{KF}^- .$$

Here, if $f = p_1 p_2 \cdots p_s$ is the conductor of K, K_i the non-trivial proper subfields of K and P_i the index set of all primes ramifying in K but not in K_i, the numbers a_i are defined as follows:

$$a_i = \begin{cases} -\infty & \text{if there is a } j \in P_i \text{ such that } p_j \text{ splits in } K_i , \\ l - 1 - |P_i| & \text{if there is no such } j \in P_i . \end{cases}$$

We order the fields K_i so that $a_0 \geq a_1 \geq \cdots \geq a_l$.

Comparing with the Sinnott formula for $[A : S]$ we can compute the Sinnott minus index:

COROLLARY 7.1. *For the Sinnott index $(e^- R : e^- U)$ we have:*

$$\left(e^- R : e^- U\right) = l^{\frac{1}{2}(l-1)l - \sum\limits_{i<a_i} (a_i - i)} .$$

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