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# ON COMPLETION OF MEASURES ON A $q$-$\sigma$-RING

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In the classical measure theory the system of all measurable sets is assumed to be a $\sigma$-ring. Because of this strong postulate the classical theory cannot describe some situations in nature (for examples see [1], [2]). That is why modern theory studies more general families of sets.

A nonempty system $\mathcal{Q}$ of subsets of a set $X$ is called a $q$-$\sigma$-ring if it satisfies the following conditions: (i) $A_i \in \mathcal{Q}$, $A_i \cap A_j = \emptyset$ for $i \neq j$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{Q}$, (ii) $E$, $F \in \mathcal{Q}$, $F \subset E$ implies $E \setminus F \in \mathcal{Q}$. If $X \in \mathcal{Q}$, then a $q$-$\sigma$-ring $\mathcal{Q}$ is called a $\sigma$-class. A measure on $\mathcal{Q}$ is a nonnegative $\sigma$-additive function, which can attain also the value $+\infty$. A measure $\mu$ defined on the $q$-$\sigma$-ring $\mathcal{Q}$ is said to be complete if $F \subset E$, $\mu(E) = 0$ implies $F \in \mathcal{Q}$. A measure $\bar{\mu}$, defined on a $q$-$\sigma$-ring $\bar{\mathcal{Q}}$, is called a completion of the measure $\mu$, defined on the $q$-$\sigma$-ring $\mathcal{Q}$, if $\bar{\mu}$ is an extension of $\mu$ and $\bar{\mu}$ is complete. It is a well-known fact (cf. [3], Theorem 13 B) that a measure $\mu$ defined on the $\sigma$-ring $\mathcal{S}$ has always a completion $\tilde{\mu}$, which is defined on the $\sigma$-ring $\tilde{\mathcal{S}}$ of all sets of the form $E \cup F$, where $E \in \mathcal{S}$, $F \subset N$, $\mu(N) = 0$ and $\tilde{\mu}$ is of the form $\tilde{\mu}(E \cup F) = \mu(E)$. In this paper we show that the completion of a measure on a $q$-$\sigma$-ring, if it exists, may be obtained in a similar way. We present a method of the construction of a completion. In the case when the domain of the measure is a $\sigma$-ring this construction turns to the usual construction described in [3]. Thus, the classical result is the special case of a more general result, which holds in the extended measure theory. In the paper there is given a necessary and sufficient condition for the existence of a completion. Further, we deal with the notion of a minimal completion and prove some existence and uniqueness theorems.

## 1. Existence of a completion

Througout this paper, $\mathcal{N}_\mu$ will denote the family of all null sets of the measure $\mu$. The following theorem gives a necessary condition for the existence of a completion.

**Theorem 1.1.** *Let $\mu$ be a measure on a q-$\sigma$-ring $\mathcal{Q}$, assume that $\mu$ has a completion. Then $A \subset B \cup \bigcup_{i=1}^{\infty} N_i$, where $A$, $B \in \mathcal{Q}$, $N_i \in \mathcal{N}_\mu$ for all i, implies $\mu(A) \leqq \mu(B)$.*

Proof: Let $\bar{\mu}$ defined on $\bar{\mathcal{Q}}$ be a completion of $\mu$. Denote $M_i = N_i \setminus \bigcup_{j=1}^{i-1} N_j$, $i = 1, 2, \ldots$ . $M_i$ are mutually disjoint sets of $\bar{\mathcal{Q}}$, hence the $\sigma$-additivity of $\bar{\mu}$ implies $\bigcup_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} N_i \in \mathcal{N}_{\bar{\mu}}$. Hence $A \setminus \bigcup_{i=1}^{\infty} N_i = A \setminus \left(A \cap \bigcup_{i=1}^{\infty} N_i\right) \in \bar{\mathcal{Q}}$. Finally, $\mu(A) = \bar{\mu}\left(A \setminus \bigcup_{i=1}^{\infty} N_i\right) \leqq \bar{\mu}(B) = \mu(B)$. The theorem is proved.

The above theorem enables us sometimes to prove that a measure has no completion. As we shall see in the following example, such a measure can be found even on a three-element set.

Example 1.1. Let $X, \mathcal{Q}$ and $\mu$ be defined as follows: $X = \{1, 2, 3\}$, $\mathcal{Q} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $\mu(\{1, 2\}) = 2$, $\mu(\{1, 3\}) = 1$, $\mu(\emptyset) = \mu(\{2, 3\}) = 0$. Then $\mu$ has no completion, because the necessary condition from Theorem 1.1. is not satisfied.

The following, somewhat more complicated example will show that the condition from Theorem 1.1. is not sufficient.

Example 1.2. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{Q} = \{\emptyset, X, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 6, 7\}, \{1, 2, 7\}, \{5, 6, 7\}, \{4, 6, 7\}, \{1, 3, 4, 5\}, \{3, 4, 5, 6\}\}$, $\mu(X) = \mu(\{1, 3, 4, 5\}) = 4$, $\mu(\{3, 4, 5, 6\}) = 3$, $\mu(\{1, 2, 3, 4\}) = \mu(\{1, 2, 3, 5\}) = \mu(\{5, 6, 7\}) = \mu(\{4, 6, 7\}) = 2$, $\mu(\{1, 2, 7\}) = 1$, $\mu(\{2, 6, 7\}) = \mu(\emptyset) = 0$. We leave the verification of the validity of the condition from Theorem 1.1. to the reader. If $\bar{\mu}$ were a completion of $\mu$, then $\bar{\mu}(\{1\}) = \bar{\mu}(\{1, 2, 7\} \setminus \{2, 7\}) = 1$, $\bar{\mu}(\{4\}) = \bar{\mu}(\{4, 6, 7\} \setminus \{6, 7\}) = 2$, and hence $\bar{\mu}(\{1, 2, 3, 4\}) = 2 < \bar{\mu}(\{1, 4\}) = 3$, a contradiction.

If $\mathcal{X}$ is a family of subsets o $X$, then the smallest $q$-$\sigma$-ring over $\mathcal{X}$ will be denoted by $\sigma_q(\mathcal{X})$. If $\mathcal{N}$ is a subsystem of $\mathcal{X}$, then $\mathcal{N}$ is called an ideal in $\mathcal{X}$ if for every $A \in \mathcal{X}$, $B \in \mathcal{N}$ we have $A \cap B \in \mathcal{N}$. If $\mathcal{A}_1$, $\mathcal{A}_2$ are systems of subsets of $X$, then the system of all set-theoretical differences $A_1 \setminus A_2$, where $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, will be denoted by $\mathcal{A}_1 - \mathcal{A}_2$.

**Lemma 1.1.** If $\mathcal{N}$ is an ideal in a $q$-$\sigma$-ring $\mathcal{Q}$, then $\mathcal{N}$ is a $\sigma$-ring.
The proof is obvious.

The following theorem gives a necessary and sufficient condition for the existence of a completion of a measure defined on a $q$-$\sigma$-ring.

**Theorem 1.2.** *Let $\mu$ be a measure on a q-$\sigma$-ring $\mathcal{Q}$. Then the following assertions are equivalent:*

(*i*) *There exists a completion of $\mu$.*

(*ii*) *For every ordinal $\Gamma$ there exists a transfinite sequence of measures $\{\mu_\alpha\}_{\alpha<\Gamma}$*

*with domains* $\mathcal{Q}_\alpha$ *such that* $\alpha < \beta < \Gamma$ *implies that* $\mu_\beta$ *is an extension of* $\mu_\alpha$, $\mu_1 = \mu$ *and* $\mathcal{Q}_\alpha = \sigma_q\left(\bigcup_{\beta<\alpha}\left(\mathcal{Q}_\beta - \mathcal{N}_{\mu_\beta}\right)\right)$ *for all* $\alpha < \Gamma$.

(*iii*) *There exists a measure* $\bar{\mu}$ *defined on a q-σ-ring* $\bar{\mathcal{Q}}$ *which is an extension of* $\mu$ *and* $\mathcal{N}_{\bar{\mu}}$ *is an ideal in* $\bar{\mathcal{Q}}$.

Proof: (i) ⇒ (ii) Let $\hat{\mu}$ defined on $\hat{\mathcal{Q}}$ be a completion of $\mu$, let $\Gamma$ be an arbitrary ordinal. We construct, by transfinite induction, the sequence $\{\mu_\alpha\}_{\alpha<\Gamma}$ with the demanded properties. We put $\mathcal{Q}_1 = \mathcal{Q}$ and $\mu_1 = \mu$. Let $\gamma < \Gamma$ and let, for every $\alpha < \gamma$, $\mu_\alpha$ be a measure defined on $\mathcal{Q}_\alpha$, where $\mathcal{Q}_\alpha \subset \hat{\mathcal{Q}}$, $\mu_\alpha = \hat{\mu}|_{\mathcal{Q}_\alpha}$, $\mathcal{Q}_\alpha = \sigma_q\left(\bigcup_{\beta<\alpha}\left(\mathcal{Q}_\beta - \mathcal{N}_{\mu_\beta}\right)\right)$. Define $\mathcal{Q}_\gamma = \sigma_q\left(\bigcup_{\alpha<\gamma}\left(\mathcal{Q}_\alpha - \mathcal{N}_{\mu_\alpha}\right)\right)$. Obviously, $\mathcal{Q}_\gamma \subset \hat{\mathcal{Q}}$. Now put $\mu_\gamma = \hat{\mu}|_{\mathcal{Q}_\gamma}$. In such a way the whole sequence $\{\mu_\alpha\}_{\alpha<\Gamma}$ can be constructed.

(ii) ⇒ (iii) Let $\Gamma$ be the ordinal number of the set $2^{2^X}$ with some well ordering. By the assumption there exists a transfinite sequence $\{\mu_\alpha\}_{\alpha<\Gamma}$ with the claimed properties. Let us assume that $\mathcal{Q}_\alpha \neq \bigcup_{\beta<\alpha} \mathcal{Q}_\beta$ for every $\alpha < \Gamma$. Hence there exists a system $\{A_\alpha\}_{\alpha<\Gamma}$ of mutually different subsets of $X$ such that $A_\alpha \in \mathcal{Q}_\alpha$, $A_\alpha \notin \bigcup_{\beta<\alpha} \mathcal{Q}_\beta$. Hence card $\{A_\alpha\}_{\alpha<\Gamma} =$ card $2^{2^X} >$ card $2^X$, which is a contradition. Thus there exists an ordinal $\alpha < \Gamma$ such that $\bigcup_{\beta<\alpha} \mathcal{Q}_\beta = \mathcal{Q}_\alpha = \sigma_q\left(\bigcup_{\beta<\alpha} (\mathcal{Q}_\beta - \mathcal{N}_{\mu_\beta})\right)$. This implies that $\mathcal{N}_{\mu_\alpha}$ is an ideal in $\mathcal{Q}_\alpha$.

(iii) ⇒ (i) Let $\bar{\mu}$ defined on $\bar{\mathcal{Q}}$ be an extension of $\mu$, let $\mathcal{N}_{\bar{\mu}}$ be an ideal in $\bar{\mathcal{Q}}$. Define $\tilde{\mathcal{Q}} = \{E \cup F;\ E \in \bar{\mathcal{Q}}, F \subset N, N \in \mathcal{N}_{\bar{\mu}}\}$. Let us prove that $\tilde{\mathcal{Q}}$ is a $q$-σ-ring. Let $\{A_i\}_{i=1}^\infty$ be the sequence of mutually disjoint sets of $\tilde{\mathcal{Q}}$. Hence $A_i = E_i \cup F_i$, where $E_i \in \bar{\mathcal{Q}}$, $F_i \subset N_i$, $N_i \in \mathcal{N}_{\bar{\mu}}$, $i = 1, 2, \ldots$. We can write $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty E_i \cup \bigcup_{i=1}^\infty F_i$, where $\bigcup_{i=1}^\infty E_i \in \bar{\mathcal{Q}}$, $\bigcup_{i=1}^\infty F_i \subset \bigcup_{i=1}^\infty N_i$. By Lemma 1.1., $\bigcup_{i=1}^\infty N_i \in \mathcal{N}_{\bar{\mu}}$, hence $\bigcup_{i=1}^\infty A_i \in \tilde{\mathcal{Q}}$. Let $E_i \in \bar{\mathcal{Q}}$, $F_i \subset N_i$, $N_i \in \mathcal{N}_{\bar{\mu}}$ for $i = 1, 2$. Further we shall assume that $E_i \cap N_i = \emptyset$. This is always possible, because $E_i \cup F_i = E_i \cup ((N_i \setminus E_i) \cap F_i)$, where $E_i \in \bar{\mathcal{Q}}$, $(N_i \setminus E_i) \cap F_i \subset N_i \setminus E_i$ and $N_i \setminus E_i = N_i \setminus (N_i \cap E_i) \in \mathcal{N}_{\bar{\mu}}$. Suppose $E_1 \cup F_1 \supset E_2 \cup F_2$. Then $(E_1 \cup F_1) \setminus (E_2 \cup F_2) = [E_1 \setminus ((E_2 \setminus N_1) \cup (E_1 \cap N_2))] \cup [E_1 \cap (N_2 \setminus F_2)] \cup [F_1 \setminus (E_2 \cup F_2)]$. $E_1 \setminus ((E_2 \setminus N_1) \cup (E_1 \cap N_2)) \in \bar{\mathcal{Q}}$ and $[E_1 \cap (N_2 \setminus F_2)] \cup [F_1 \setminus (E_2 \cup F_2)] \subset N_1 \cup N_2$, hence $(E_1 \cup F_1) \setminus (E_2 \cup F_2) \in \tilde{\mathcal{Q}}$. Thus, $\tilde{\mathcal{Q}}$ is $q$-σ-ring. Let $E_i \in \bar{\mathcal{Q}}$, $F_i \subset N_i$, $n_i \in \mathcal{N}_{\bar{\mu}}$, $i = 1, 2$. We prove that $E_1 \cup F_1 = E_2 \cup F_2$ implies $\bar{\mu}(E_1) = \bar{\mu}(E_2)$. We suppose again $E_i \cap N_i = \emptyset$, $i = 1, 2$. Obviously, $E_1 \cup F_1 = E_2 \cup F_2$

implies $E_1 \setminus N_2 = E_2 \setminus N_1$. Then we can write $\bar{\mu}(E_1) = \bar{\mu}((E_1 \setminus N_2) \cup (E_1 \cap N_2)) = = \bar{\mu}(E_1 \setminus N_2) = \bar{\mu}(E_2 \setminus N_1) = \bar{\mu}((E_2 \setminus N_1) \cup (E_2 \cap N_1)) = \bar{\mu}(E_2)$. Now we can define uniquely on $\tilde{\mathcal{Q}}$ the function $\tilde{\mu}$ as follows: if $A \in \tilde{\mathcal{Q}}$, $A = E \cup F$, where $E \in \bar{\mathcal{Q}}$, $F \subset N$, $N \in \mathcal{N}_{\bar{\mu}}$, then $\tilde{\mu}(A) = \bar{\mu}(E)$. Obviously, $\tilde{\mu}$ is a measure. The completeness of $\tilde{\mu}$ follows immediately. The theorem is proved.

Theorem 1.2. gives us not only a necessary and sufficient condition for the existence of a completion of $\mu$, but gives, in a sense, also a method how to find that completion. It is necessary to extend $\mu$ in the above way to a suitable ordinal, yielding a measure, whose null sets form an ideal. Then it will suffice to perform the same process as when completing a measure on a $\sigma$-ring. However, this process of extension to $q$-$\sigma$-rings $\mathcal{Q}_\alpha$, $\alpha < \Gamma$, need not be unique. It may happen that there exists a completion of $\mu$, but after an inapropriate extension we obtain a measure which has no completion. This is proved by the following example.

Example 1.3. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{Q} = \{\emptyset, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1,2,4,5\}, \{2,6,7\}, \{1,2,7\}\}$, $\mu(\emptyset) = \mu(\{2,6,7\}) = 0$ and $\mu(A) = \infty$ for each other $A \in \mathcal{Q}$. Let $\mathcal{Q}_2 = \sigma_q(\mathcal{Q} - \mathcal{N}_\mu)$; then $\mathcal{Q}_2$ consists of all disjoint unions of sets $\{1\}$, $\{2\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, $\{6\}$, $\{7\}$. Define on $\mathcal{Q}_2$ the measure $\mu_2$ as follows: $\mu_2(\{1\}) = \mu_2(\{3, 4\}) = \infty$ and $\mu_2(\{2\}) = \mu_2(\{3, 5\}) = \mu_2(\{4, 5\}) = \mu_2(\{6\}) = = \mu_2(\{7\}) = 0$. $\mu_2$ is defined uniquely in this way. $\{3,4\} \subset \{3,5\} \cup \{4,5\}$, but $\mu_2(\{3, 4\}) > \mu_2(\{3, 5\})$. The necessary condition from Theorem 1.1. is not satisfied, hence $\mu_2$ has no completion. However, there exists a completion of $\mu$, for example the measure $\bar{\mu}$ defined on $2^X$ as follows: $\bar{\mu}(A) = \infty$ if $1 \in A$, $\bar{\mu}(A) = 0$ if $1 \notin A$, $A \in 2^X$.

**Lemma 1.2.** *Let $\mu$ be a finite measure on a $\sigma$-class $\mathcal{Q}$, let $\Gamma$ be an arbitrary ordinal. Then there exists at most one transfinite sequence $\{\mu_\alpha\}_{\alpha<\Gamma}$ of extensions of $\mu$, which are defined on $\mathcal{Q}_\alpha$, where $\mathcal{Q}_1 = \mathcal{Q}$, $\mathcal{Q}_\alpha = \sigma_q\Big(\bigcup_{\beta<\alpha} (\mathcal{Q}_\beta - \mathcal{N}_{\mu_\beta})\Big)$ and $\beta < \alpha < \Gamma$ implies that $\mu_\alpha$ is an extension of $\mu_\beta$.*

Proof: Let $\{\mu_\alpha\}_{\alpha<\Gamma}$ and $\{\nu_\alpha\}_{\alpha<\Gamma}$ be two different sequences having the above properties. Let $\gamma$ be the smallest ordinal such that $\mu_\gamma \neq \nu_\gamma$. Hence $\mu_\gamma(A) = = \nu_\gamma(A)$ for all $A \in \bigcup_{\beta<\gamma} (\mathcal{Q}_\beta - \mathcal{N}_{\mu_\beta})$. Obviously, the family of all sets $A \in \mathcal{Q}_\gamma$ such that $\mu_\gamma(A) = \nu_\gamma(A)$ is a $\sigma$-class, thus it is equal to $\mathcal{Q}_\gamma$. Then $\mu_\gamma = \nu_\gamma$, a contradiction. The lemma is proved.

We see now that Theorem 1.2. is significant first of all in the case when $\mu$ is finite and $\mathcal{Q}$ is a $\sigma$-class. Then if the completion exists, the process described in that theorem will lead us to it.

From Theorem 1.2. there immediately follow some sufficient conditions for the existence of a completion. They are contained in the following two theorems, whose proofs are omitted.

**Theorem 1.3.** *If $\mu$ is defined on a $q$-$\sigma$-ring $\mathcal{Q}$ and $\mathcal{N}_\mu$ is an ideal in $\mathcal{Q}$, then there exists a completion of $\mu$.*

**Theorem 1.4.** *If $\mu$ defined on a $q$-$\sigma$-ring $\mathcal{Q}$ extends to some $\sigma$-ring, then the completion o $\mu$ exists.*

Remark 1.1. It follows from Theorem 1.2. and from the properties of ordinals that, for every measure $\mu$ defined on a $q$-$\sigma$-ring $\mathcal{Q}$ which has a completion, there exists the smallest ordinal $\alpha$ such that there exists a sequence $\{\mu_\beta\}_{\beta \leqq \alpha}$ of measures with domains $\mathcal{Q}_\beta$, where $\mu_1 = \mu$, $\mathcal{Q}_\beta = \sigma_q\left(\bigcup_{\gamma<\beta} (\mathcal{Q}_\gamma - \mathcal{N}_{\mu_\gamma})\right)$ for every $\beta \leqq \alpha$, $\gamma < \beta$ implies that $\mu_\beta$ is an extension of $\mu_\gamma$ and $\mathcal{N}_{\mu_\alpha}$ is an ideal in $\mathcal{Q}_\alpha$. We shall denote that ordinal by $\alpha_\mu$. For the measure $\mu$ from Example 1.3. $\alpha_\mu = 2$. In the following example, $\alpha_\mu = 3$.

Example 1.4. Let $X$, $\mathcal{Q}$ be the same as in Example 1.3. Define the measure $\mu$ as follows: $\mu(A) = 1$ if $1 \in A$, $\mu(A) = 0$ if $1 \notin A$, $A \in \mathcal{Q}$. We leave the verification of the equality $\alpha_\mu = 3$ to the reader.

## 2. Existence and uniqueness of the minimal completion

The system $\boldsymbol{L}$ of all completions of the measure $\mu$ may be partially ordered with relation $\leqq_R$ as follows: $\mu_1 \leqq_R \mu_2$ if $\mu_2$ is an extension of $\mu_1$. Minimal elements of the partially ordered set $(\boldsymbol{L}, \leqq_R)$ are called minimal completions of $\mu$. It is a well-known fact that, in the case of $\mu$ being defined on a $\sigma$-ring, there exists a unique minimal completion, which is, moreover, the smallest element of $\boldsymbol{L}$. It is the measure $\tilde{\mu}$ mentioned in the introduction of this paper. Given a measure on a $q$-$\sigma$-ring, we cannot, in general, guarantee the uniqueness of the minimal completion. However, if $\mu$ has a completion, the existence of a minimal completion can be guaranteed. This is stated in the following theorem, which is a simple conclusion of the Zorn lemma.

**Theorem 2.1.** *If $\mu$ is a measure defined on a $q$-$\sigma$-ring $\mathcal{Q}$ which has a completion, then $\mu$ has a minimal completion.*

The following example shows that the minimal completion need not be unique.

Example 2.1. Let $X = \{1, 2, 3, 4\}$, $\mathcal{Q} = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}$, $\mu(\{1, 2\}) = \mu(\{1, 3, 4\}) = \infty$, $\mu(\emptyset) = \mu(\{2, 3\}) = 0$. Then each measure $\bar{\mu}$ defined on $2^X$ by $\bar{\mu}(\{1\}) = \infty$, $\bar{\mu}(\{2\}) = \bar{\mu}(\{3\}) = 0$, $\bar{\mu}(\{4\}) = a$, $a \geqq 0$, is a minimal completion of $\mu$.

Now we give a sufficient condition for the uniqueness of a minimal completion.

**Theorem 2.2.** *If $\mu$ is a finite measure on a $q$-$\sigma$-ring $\mathscr{Q}$ and there exists a finite completion of $\mu$, then there exists a unique minimal completion of $\mu$.*

Proof: Let $\{\mathscr{Q}_{\tilde{\mu}}\}_{\tilde{\mu}\in \mathbf{L}}$ be the system of domains of all measures of $\mathbf{L}$. Denote $\hat{\mathscr{Q}} = \bigcap_{\tilde{\mu}\in \mathbf{L}} \mathscr{Q}_{\tilde{\mu}}$, $\mathscr{A} = \{E \in \hat{\mathscr{Q}};\ \tilde{\mu}_1(E) = \tilde{\mu}_2(E)$ for every $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathbf{L}\}$. The existence of a finite completion implies that $\mathscr{A}$ is a $q$-$\sigma$-ring. Define on $\mathscr{A}$ a measure $\nu$ as follows: $\nu(E) = \tilde{\mu}(E)$, $\tilde{\mu} \in \mathbf{L}$. Let $E \in \mathscr{A}$, $\nu(E) = 0$, $F \subset E$. Then $F \in \mathscr{Q}_{\tilde{\mu}}$ for every $\tilde{\mu} \in \mathbf{L}$. Hence $F \in \hat{\mathscr{Q}}$. Since $\tilde{\mu}(F) = 0$ for every $\tilde{\mu} \in \mathbf{L}$, we have $F \in \mathscr{A}$. Obviously $\mathscr{A} = \hat{\mathscr{Q}}$ and thus $\nu$ is the unique minimal completion of $\mu$.

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## ОБ ПОПОЛНЕНИИ МЕР НА $q$—$\sigma$—КОЛЬЦЕ

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### Резюме


Семейство $\mathscr{Q}$ подмножеств множества $X$ называется $q$-$\sigma$-кольцом, если из $E, F \in \mathscr{Q}$, $F \subset E$ следует $E \setminus F \in \mathscr{Q}$ и $\mathscr{Q}$ замкнуто относительно счетных объединений непересекающихся множеств. В работе изучается пополнение меры, определенной на $q$-$\sigma$-кольце. Показано необходимое и достаточное условие для существования пополнения. Определяется понятие минимального пополнения и доказывается, что из существования какого-нибудь конечного пополнения следует единтвенность минимального пополнения.