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REPRESENTATION OF LINEAR OPERATORS ON SPACES OF VECTOR VALUED FUNCTIONS

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This paper is concerned with an integral representation of some linear operators defined on an ordered space of vector valued functions.

The terminology used is that of [1].

1. Preliminaries

Let X be a locally convex vector lattice with the topology τ and $\mathcal{P}$ the set of all solid and (τ)-continuous semi-norms defined on X. Let Y be a complete vector lattice. For any $p \in \mathcal{P}$ we denote by $\mathcal{L}_p$ the set of all linear operators U: $X \to Y$ for which the set $\{U(x);\ p(x) \leqslant 1\}$ is order bounded. If $U \in \mathcal{L}_p$, we put

$$\|U\|_p = \sup \{|U(x)|;\ p(x) \leqslant 1\}.$$

We set also

$$\mathcal{L} = \bigcup_{p \in \mathcal{P}} \mathcal{L}_p$$

If $U \in \mathcal{L}$, then U is called a *(po)-bounded operator.* If $U \in \mathcal{L}_p$, we say that $U$is (po)-bounded with respect to p.

The set $\mathcal{L}$ is a normal subspace of the space $\mathcal{R}(X, Y)$ of all regular operators which map X into Y.

2. The space M(T, X)

Let T be a locally compact space and $\mathcal{K}$ the set of all compact subsets of T. For any $E \in \mathcal{K}$ we denote by $\mathcal{B}_E$ the set of all borelian subsets of E and we put

$$\mathcal{B} = \bigcup_{E \in \mathcal{K}} \mathcal{B}_E$$

A $\mathscr{B}$-*simple function* $f\colon T \to X$ is, by definition, of the form

$$f(t)=\sum_{i=1}^{k} \gamma_{A_i}(t)x_i, \qquad (t\in T) \tag{1}$$

where $A_i \in \mathscr{B}$, $x_i \in X$, and γ_A being the characteristic function of A.

We denote by $M(T, X)$ the set of the functions $f\colon T\to X$ having the following properties: there exists $E\in\mathscr{K}$ and a generalized sequence $\{f_\delta\}_{\delta\in\Delta}$ of $\mathscr{B}$-simple functions (mapping T into X) such that spt $f_\delta \subset E$ (where spt means "the support") and $\{f_\delta\}_{\delta\in\Delta}$ is uniformly convergent to f. We shall say that $\{f_\delta\}_{\delta\in\Delta}$ is an *approximating sequence* for f.

For any $p\in\mathscr{P}$ we define a semi-norm $\tilde{p}$ on the vector space $M(T, X)$ putting

$$\tilde{p}(f)=\sup\{p(f(t));\ t\in T\}$$

if $f\in M(T, X)$.

The set $M(T, X)$ is a locally convex vector lattice with respect to the pointwise order and the topology defined by the set $\{\tilde{p}\colon p\in\mathscr{P}\}$ of semi-norms.

The set $C_0(T, X)$ of continuous functions with compact support (mapping T into X) is a vector sublattice of the space $M(T, X)$.

For any $E\in\mathscr{K}$ we denote

$$M_E(T, X)=\{f\in M(T, X);\ \text{spt}\, f\subset E\}.$$

The set $M_E(T, X)$ is a vector sublattice of the vector lattice $M(T, X)$.

We shall consider on the vector subspaces $C_0(T, X)$ and $M_E(T, X)$ of $M(T, X)$ the induced topology.

3. The integral

Let $m\colon \mathscr{B}\to\mathscr{L}$ be an additive function which satisfies the condition: for any $E\in\mathscr{K}$ there exists $p\in\mathscr{P}$ such that $m(\mathscr{B}_E)\subset\mathscr{L}_p$ and the set

$$G(E;p)=\left\{\sum_{i=1}^{k}\|m(A_i)\|_p;\ (A_1,\dots,A_k)\ \mathscr{B}\text{-partition of } E\right\}$$

is (o)-bounded. We shall say that m is of the (bv)-type and we shall denote

$$v_m(E, p)=\sup G(E, p).$$

If $f\in M(T, X)$ is a $\mathscr{B}$-simple function (see formula (1)), we define

$$\int_T f\,\mathrm{d}m=\sum_{i=1}^{k} m(A_i)(x_i).$$

It is easily verified that the operator $f \to \int_T f \, dm$ (defined on the set of $\mathscr{B}$-simple functions) is linear and for any $E \in \mathscr{K}$ there exists $p \in \mathscr{P}$ such that

$$\left| \int_T f \, dm \right| \leq \tilde{p}(f) v_m(E, p) \tag{2}$$

if spt $f \subset E$.

Let now f be arbitrary in $M(T, X)$ and let $\{f_\delta\}_{\delta \in \Delta}$ be an approximating sequence for f, with spt $f_\delta \subset$ spt $f = E$.

There exists (see also (2)) $p \in \mathscr{P}$ such that

$$\left| \int_T f_{\delta'} \, dm - \int_T f_{\delta''} \, dm \right| \leq \tilde{p}(f_{\delta'} - f_{\delta''}) v_m(E, p).$$

Since Y is a complete vector lattice, the generalized sequence $\left\{ \int_T f_\delta \, dm \right\}_{\delta \in \Delta}$ is (ϱ)-convergent (convergent with regulator [1]). We shall define

$$\int_T f \, dm = (\varrho) - \lim_{\delta \in \Delta} \int_T f_\delta \, dm$$

the limit being independent of the approximating sequence.

The integral is a linear operator (mapping $M(T, X)$ into Y) and the inequality (2) holds for any $f \in M_E(T, X)$.

4. Representation of some operators

If U: $M(T, X) \to Y$ is a linear operator and $E \in \mathscr{K}$, we shall denote by U_E the restriction of U to the subspace $M_E(T, X)$. If U_E is (po)-bounded with respect to $\tilde{p}$, we shall denote

$$|||U|||_{E, p} = \|U_E\| \tilde{p}.$$

Theorem. A linear operator U: $M(T, X) \to Y$ satisfies the condition

(i) $$U_E \text{ is } (po)\text{-bounded}, \forall E \in \mathscr{K},$$

if and only if

(ii) $$U(f) = \int_T f \, dm, \quad (f \in M(T, X))$$

where m: $\mathscr{B} \to \mathfrak{X}$ is an additive function of the (bv)-type. If (i) holds, then m can be chosen in (ii) such that the equality

(iii) $$|||U|||_{E, p} = v_m(E, p)$$

holds, as soon as the left-hand member exists.

Proof. As we saw in §3, the operator defined by (ii) satisfies the condition (i). From (2), which holds for any $f \in M_E(T, X)$, it follows that

$$\|U_E\|\tilde{p} \leqslant v_m(E, p). \tag{3}$$

Conversely, let U: $M(T, X) \to Y$ be a linear operator satisfying (i). Hence, for any $E \in \mathcal{K}$ there exist $p \in \mathcal{P}$ and $y_0 \in Y$ such that

$$|U(f)| \leqslant \tilde{p}(f)\, y_0, \quad (\forall f \in M_E(T, X)) \tag{4}$$

Define m: $\mathcal{B} \to \mathcal{L}$ by setting

$$(m(A))(x) = U(\gamma_A . x), \qquad (\forall x \in X)$$
$$(\text{where } (\gamma_A . x)(t) = \gamma_A(t) . x;\ \forall t \in T).$$

The operator $m(A)$: $X \to Y$ is obviously linear. With (4), there exists $p \in \mathcal{P}$, such that $m(A) \in \mathcal{L}_p$ (and the function m: $\mathcal{B} \to \mathcal{L}$ is obviously additive). By considering a $\mathcal{B}$-partition $(A_1, \ldots, A_k)$ of a set $E \in \mathcal{K}$, one has

$$\sum_{i=1}^{k} \|m(A_i)\|_p = \sum_{i=1}^{k} \sup\{|m(A_i)(x_i)|;\ p(x) \leqslant 1\} =$$

$$= \sup\left\{\sum_{i=1}^{k} |m(A_i)(x_i)|;\ p(x_i) \leqslant 1;\ i = 1, \ldots, k\right\} \leqslant$$

$$\leqslant \sup\left\{\sum_{i=1}^{k} |U|(\gamma_{A_i}|x_i|);\ p(x_i) \leqslant 1;\ i = 1, \ldots, k\right\} \leqslant$$

$$\leqslant \sup\{|U|(|f|);\ f \in M_E(T, X);\ \tilde{p}(f) \leqslant 1\}$$

by taking into account that (4) implies

$$|U|(|f|) \leqslant \tilde{p}(f)\, y_0, \quad (\forall f \in M_E(T, X)).$$

It follows that

$$\sum_{i=1}^{k} \|m(A_i)\|_p \leqslant y_0 \tag{5}$$

The equality in (ii) obviously holds if f is a $\mathcal{B}$-simple function. Let now f be arbitrary in $M(T, X)$ and $\{f_\delta\}_{\delta \in \Delta}$ an approximating sequence for f such that $\operatorname{spt} f_\delta \subset \operatorname{spt} f = E$. With (4) it follows that

$$|U(f_\delta) - U(f)| \leqslant \tilde{p}(f_\delta - f)\, y_0$$

(where p and y_0 are suitably taken for E). Hence $U(f) = (\varrho) - \lim_{\delta \in \Delta} U(f_\delta)$ and consequently (ii) hold.

If U_E is (po)-bounded with respect to $\tilde{p}$, then we can take $y_0 = \|U_E\|_{\tilde{p}}$ in (4) and from (5) we get

$$w_m(E, p) \leq \|U_E\|_{\tilde{p}};$$

with (3) it follows that (iii) holds.

Corollary. Any (po)-bounded linear operator U: $C_0(T, X) \to Y$ can be expressed in the form

$$U(f) = \int_T f \, \mathrm{d}m$$

where m: $\mathcal{B} \to \mathcal{X}$ is an additive function of the (bv)-type.

Indeed, U can be extended as a (po)-bounded linear operator on the space $M(T, X)$.

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ПРЕДСТАВЛЕНИЕ ЛИНЕЙНЫХ ОПЕРАТОРОВ НА ПРОСТРАНСТВАХ ВЕКТОРНЫХ ФУНКЦИЙ

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Резюме

В данной работе устанавливается интегральное представление некоторых линейных операторов, заданных на упорядочных пространствах, состоящих из векторных функций.