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EXAMPLES OF CLASSICAL AND FUZZY RIESZ PROXIMITIES

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ABSTRACT. Examples of proximities which are Riesz (respectively fuzzy Riesz) but not Lodato (respectively fuzzy Lodato) have been constructed.

1. Introduction

In the classical theory of proximities, the notion of f-proximities and, in particular, of Riesz (or RI) proximity is due to Thron [6], and that of a symmetric generalized proximity (now known as Lodato or LO-proximity) is due to Lodato [2]. A relationship between these two, that "every LO-proximity on a nonempty set is an RI-proximity", is given by Thron [6]. In [5], we have continued the study of fuzzy f-proximities introduced in [3] and generalized the notion of classical RI-proximity to fuzzy Riesz (or RI) proximity. Fuzzy RI-proximity turns out to be a particular case of fuzzy f-proximities. In the fuzzy subset, setting also the result that "every fuzzy LO-proximity [4] on a set is a fuzzy RI-proximity" holds good [5].

In the present paper, we have constructed

(i) an example (Example 3.1) of an RI-proximity which is not an LO-proximity,

(ii) two examples of fuzzy RI-proximities both of which are not fuzzy LO-proximities.

Example 3.2 has been obtained with the help of Example 3.1, while Example 3.3 uses purely fuzzy behaviour in the sense that one cannot derive this example from a classical proximity using the technique of Example 3.2 (cf. Remark 3.4).

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2. Preliminaries

Let X be a nonempty set, $P(X)$ be the power set of X, and $I=[0,1]$ be the closed unit interval of the real line $\mathbb{R}$. A *fuzzy set* λ in X is an element of the family I^X of all functions from X to I. A *fuzzy point* x_p, $x \in X$, $0<p\le 1$, is a fuzzy set in X defined by

$$x_p(y)=\begin{cases} p & \text{if } y=x\,,\\ 0 & \text{otherwise.}\end{cases}$$

For $A \in P(X)$, $\chi_A \in I^X$ is defined by

$$\chi_A(x)=\begin{cases} 1 & \text{if } x\in A\,,\\ 0 & \text{otherwise;}\end{cases}$$

and $|A|$ denotes the cardinality of A. For $\lambda \in I^X$, we write $\operatorname{supp}\lambda = \{x\in X:\ \lambda(x)\neq 0\}$. A fuzzy set which assigns the value t, $t\in I$, to each x in X is denoted by $\boldsymbol{t}$. For $\lambda\in I^X$ and a binary relation Π on I^X define $c_\Pi(\lambda)=\bigvee\{x_p:\ (x_p,\lambda)\in\Pi\}$.

A binary relation Π on I^X is called a *fuzzy Lodato* (or *LO*) *proximity on* X if, for $\lambda,\mu,\nu\in I^X$, the following hold:

F1. $(\lambda,\mu)\in\Pi \implies (\mu,\lambda)\in\Pi$,

F2. $(\mathbf{0},\mathbf{1})\notin\Pi$,

F3. $(\lambda\vee\mu,\nu)\in\Pi \iff (\lambda,\nu)\in\Pi$ or $(\mu,\nu)\in\Pi$,

F4. $\lambda\wedge\mu\neq\mathbf{0} \implies (\lambda,\mu)\in\Pi$,

F5. $(\lambda,\mu)\in\Pi$ and $(x_p,\nu)\in\Pi$ for all $x_p\le\mu \implies (\lambda,\nu)\in\Pi$ ([4]).

A binary relation Π on I^X is called a *fuzzy Riesz* (or *RI*) *proximity on* X if it satisfies F1, F2, F3, F4, and

F5'. $c_\Pi(\lambda)\wedge c_\Pi(\mu)\neq\mathbf{0} \implies (\lambda,\mu)\in\Pi$ ([5]).

3. Examples

Example 3.1. Let $X=\mathbb{R}\times\mathbb{R}$, d be the Euclidean metric on X, and $d(A,B)=\inf\{d(\xi,\eta):\ \xi\in A,\ \eta\in B\}$ for subsets A, B of X. Denote by ω_0 the first infinite cardinal. Define

$$\begin{aligned}\delta=\{(A,B):\ d(A,B)=0\}\\ \cup\Big\{(A,B):\ \big|A\cap\{(0,y):\ -1\le y\le 1\}\big|\ge\omega_0\\ \text{and } \big|B\cap\{(x,0):\ x<-1\}\big|\ge\omega_0\Big\}\\ \cup\Big\{(A,B):\ \big|A\cap\{(x,0):\ x<-1\}\big|\ge\omega_0\\ \text{and } \big|B\cap\{(0,y):\ -1\le y\le 1\}\big|\ge\omega_0\Big\}.\end{aligned}$$

Then δ is a *Čech proximity* ([1]) on X and $c_\delta(A) \equiv \{x : (x,A) \in \delta\} = \{x : d(x,A) = 0\}$. If $c_\delta(A) \cap c_\delta(B) \neq \emptyset$, then there exists x in X such that $d(x,A) = 0 = d(x,B)$. Consequently, $d(A,B) = 0$, and hence $(A,B) \in \delta$. Thus δ is an RI-proximity on X.

Next, put $A = \{(x,0) : x < -1/2\}$, $B = \{(0,y) : -1 \le y \le 1\}$ and $C = \{(x, \sin 1/x) : x > 0\}$. Then $(A,B) \in \delta$, $(b,C) \in \delta$ for all $b \in B$. But $(A,C) \notin \delta$. This proves that δ is not an LO-proximity.

Example 3.2. Consider the metric space (X,d) of Example 3.1. Define

$$\Pi = \{(\lambda,\mu) : (\operatorname{supp}\lambda, \operatorname{supp}\mu) \in \delta\}.$$

Then Π satisfies F1 to F4, and, for A, B, C, as taken in Example 3.1, $(\chi_A, \chi_B) \in \Pi$, $(x_p, \chi_C) \in \Pi$ for all $x_p \le \chi_B$; but $(\chi_A, \chi_C) \notin \Pi$. Thus Π is not a fuzzy LO-proximity on X.

Since $c_\Pi(\lambda) = \chi_{c_\delta(\operatorname{supp}\lambda)}$, if $c_\Pi(\lambda) \wedge c_\Pi(\mu) \neq \mathbf{0}$, then $c_\delta(\operatorname{supp}\lambda) \cap c_\delta(\operatorname{supp}\mu) \neq \emptyset$. Hence $(\operatorname{supp}\lambda, \operatorname{supp}\mu) \in \delta$, i.e., $(\lambda,\mu) \in \Pi$. Thus Π is a fuzzy RI-proximity on X.

Example 3.3. Let X be an infinite set. For $0 < t < 1$, define

$$\begin{aligned}\Pi = {} & \{(\lambda,\mu) : \lambda \wedge \mu \neq \mathbf{0}\} \\ & \cup \{(\lambda,\mu) : \lambda \neq \mathbf{0},\ \mu \neq \mathbf{0} \text{ and} \\ & \qquad (\lambda \vee \mu)(x) > t \text{ for infinitely many elements } x \text{ of } X\}.\end{aligned}$$

The relation Π satisfies F1 to F4. Let $c_\Pi(\lambda) \wedge c_\Pi(\mu) \neq \mathbf{0}$. Then $\lambda \neq \mathbf{0}$ and $\mu \neq \mathbf{0}$. If at least one of λ and μ takes values greater than t for infinitely many elements of X, then $(\lambda,\mu) \in \Pi$. Otherwise, $\operatorname{supp}\lambda \cap \operatorname{supp}\mu \neq \emptyset$, which implies that $\lambda \wedge \mu \neq \mathbf{0}$, and again $(\lambda,\mu) \in \Pi$. It may be noted that, for $\lambda \in I^X$,

$$c_\Pi(\lambda) = \begin{cases} \mathbf{1} & \text{if } \lambda(x) > t \text{ for infinitely many elements of } X, \\ \chi_{\operatorname{supp}\lambda} & \text{otherwise.} \end{cases}$$

That Π is not a fuzzy LO-proximity, follows from the following arguments:

Let $\lambda\,(\neq \mathbf{0})$, $\nu\,(\neq \mathbf{0}) \in I^X$ be such that $\lambda \wedge \nu \neq \mathbf{0}$ and $\lambda(x) \le t$, $\nu(x) \le t$, for all x in X. Choose $\mu \in I^X$ such that $\operatorname{supp}\mu = \operatorname{supp}\nu$ and $\mu(x) \ge t$ for infinitely many points x of X. Then $(\lambda,\mu) \in \Pi$. Also, for $x_p \le \mu$, $\mu(x) \neq 0$, and, consequently, $\nu(x) \neq 0$. Hence $(x_p, \nu) \in \Pi$. But $(\lambda,\nu) \notin \Pi$. Thus Π is not a fuzzy Lodato proximity on X.

Remark 3.4. Let δ be a relation on $P(X)$. Define $\hat{\delta} = \{(\lambda,\mu) : (\operatorname{supp}\lambda, \operatorname{supp}\mu) \in \delta\}$. It may be verified that δ is an LO-proximity if and only if $\hat{\delta}$ is a fuzzy LO-proximity. Suppose that the fuzzy proximity Π of Example 3.3 can be derived from a classical proximity δ as in Example 3.2, i.e.,

$\Pi = \hat{\delta}$. Since Π is not a fuzzy LO-proximity, δ is not an LO-proximity. But

$$\begin{aligned}\widetilde{\Pi} &\equiv \{(A,B):\ (\chi_A, \chi_B) \in \Pi\}\\ &= \{(A,B):\ (\chi_A, \chi_B) \in \hat{\delta}\}\\ &= \delta .\end{aligned}$$

and $\widetilde{\Pi}$ is an LO-proximity, i.e., δ is an LO-proximity. This provides a contradiction.

Thus Example 3.3 cannot be derived from a classical proximity using the technique of Example 3.2.

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