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ON THE INTERCHANGE HEURISTIC FOR LOCATING CENTERS AND MEDIANS IN A GRAPH

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1. Introduction

Our graph terminology is based on [1] or [6]. Given a graph G, the vertex and edge sets of G and their cardinalities are denoted by $V(G)$, $E(G)$, n and m, respectively. It is assumed that every vertex v has assigned a weight $w(v) \geqslant 0$ and every edge uv has a length $a(uv) > 0$. These lengths determine the distance $d(u, v)$ between any two vertices u and v ($d(u, v)$ is the minimal sum of the edge lengths of a u-v path). For any subset $X \subset V(G)$, the distance between a vertex $v \in V(G)$ and the set X is given by $d(v, X) := \min\{d(v, x) | x \in X\}$. A p-set is a set of cardinality p.

Given a graph G with weights and lengths and an integer p, $0 < p < n$, we consider two location problems.

(1) *The p-center problem*: Find a p-set $X \subset V(G)$ such that the objective function (weighted eccentricity)

$$\eta(X) := \max_{v \in V(G)} \{w(v)d(v, X)\}$$

is minimized. Any optimal set X is called a *p-center*.

(2) *The p-median problem*: Find a p-set $X \subset V(G)$ such that the objective function (sum of weighted distances)

$$\sigma(X) := \sum_{v \in V(G)} w(v)d(v, X)$$

is minimized. Any optimal set X is called a *p-median*.

If the number p is not specified, then these problems are referred to as the *multicenter* and *multimedian problems*, respectively. Both problems can be interpreted in terms of communication networks [2, 13]. If $V(G)$ represents a collection of cities and an edge represents a communication link, then one may be interested in selecting a p-set X of cities as sites for hospitals (the p-center problem) or stores (the p-median problem). The multicenter and multimedian problems were introduced by Hakimi [3, 4] and are discussed in many papers (see e.g. [5, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21]). It is clear that for any fixed p there are polynomial algorithms for our two problems. However, in general, these problems are NP-hard [11, 12, 15]. Moreover, the problem of finding a near optimal solution for the p-center problem is also NP-hard [9, 15]. Therefore various heuristics are often used.

One of the heuristics which were suggested for these problems is the so-called *interchange heuristic*. More precisely, for a given $k \leq n - p$, the *k-change heuristic* works as follows: Given a p-set X, if there are a k- et $Z' \subseteq V(G) - X$ and a k-set $Z \subset X$ such that $\eta((X - Z) \cup Z') < \eta(X)$ for the p-center problem and $\sigma((X - Z) \cup Z') < \sigma(X)$ for the p-median problem, then instead of X the set $(X - Z) \cup Z'$ is considered and the step is repeated; otherwise the set X is said to be *k-optimal*, stop. Any k-optimal p-set is considered as an approximate solution of the p-center or the p-median problem. While we have found no reference to using the interchange heuristic for the multicenter problem, there are several papers dealing with multimedians. The first paper is by Teitz and Bart [19]. Revelle et al. [16] were convinced that 1-optimal p-sets are just p-medians. Järvinen et al. [10], however, showed an example with $n = 6$ (where every weight equals 1) and such that there is a 2-median M with $\sigma(M) = 7$ and a 1-optimal 2-set S with $\sigma(S) = 9$. Thus the error ratio $\sigma(S)/\sigma(M) = 9/7$. Another example with error ratio 8/7 can be found in [2, p. 116]. The existence of examples with error ratio 10/7 is reported in a table of [10] ($n = 20$, $p = 5$ and 10, $k = 1$).

The purpose of this paper is to present examples with a greater error ratio. We will show that the ratio can be arbitrarily large in the case of multicenters. As regards multimedians, we give only a partial solution proving that the ratio can be nearly 3.

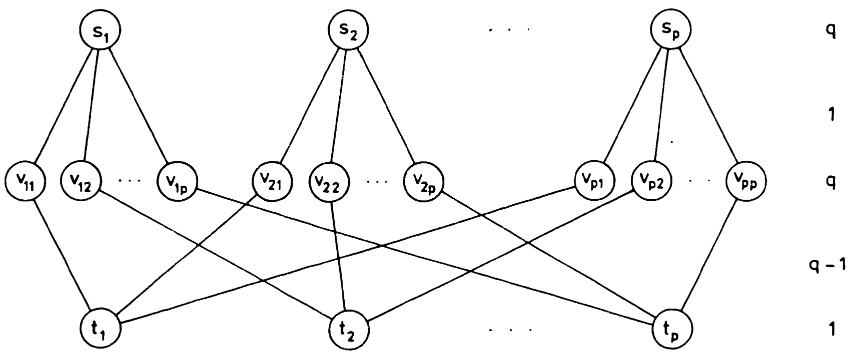

Fig. 1. The graph G from the proof of Theorem 1; the weights and lengths are on the right

2. The multicenter problem

Denote by η^* the optimal weighted eccentricity in a given example of the p-center problem under consideration, i.e. $\eta^*: = \eta(S)$ *where* S *is a* p-*center*.

Theorem 1. *Let p be a positive integer. Then for an arbitrarily large real number r there exists an example of the p-center problem such that*:

(i) *the coresponding graph is bipartite,*

(ii) *there is a p-set T which is k-optimal for* $k = 1, 2, \ldots, p - 1$,

(iii) $\eta(T)/\eta^* \geqslant r$.

Proof. Let q be an integer with $q \geqslant r$. Consider the bipartite graph G in Fig. 1 where

$$V(G): = \bigcup_{i=1}^{p} (\{s_i, t_i\} \cup \bigcup_{j=1}^{p} \{v_{ij}\}),$$

$$E(G): = \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \{s_i v_{ij}, t_j v_{ij}\}.$$

The weights and lengths are as follows:

$$w(s_i): = w(v_{ij}): = q, \quad w(t_j): = 1;$$

$$a(s_i, v_{ij}): = 1, \; a(v_{ij}, t_j): = q - 1,$$

for all i and j.

One can easily verify that the p-set $S: = \{s_1, s_2, \ldots, s_p\}$ is a p-center with $\eta(S) = q$. Further, it can be verified that the p-set $T: = \{t_1, t_2, \ldots, t_p\}$ has $\eta(T) = q^2$. Interchanging k vertices of T, $k < p$, we obtain a p-set M. Clearly, there is an index i such that

$$M \cap \{s_i, v_{i1}, v_{i2}, \ldots, v_{ip}\} = \emptyset.$$

Therefore we have

$$\eta(M) \geqslant w(s_i) d(s_i, M) = q^2 = \eta(T).$$

Hence the set T is k-optimal for every $k = 1, 2, \ldots, p - 1$. Since $\eta(T)/\eta(S) = q \geqslant r$, the theorem is proved. ∎

3. The multimedian problem

Considering an example of the p-median problem, we use σ^* to denote the optimal sum of weighted distances, that is, $\sigma^*: = \sigma(S)$ where S is any p-median.

Theorem 2. *Let K and q be positive integers with* $Kq \geqslant 2$. *Then for any real* *number $r < 3$ and integer $p \geqslant 2(Kq + r - 1)/[q(3 - r)]$ there exists an example of the p-median problem such that:*

(i) *the corresponding graph is bipartite,*

(ii) *there exists a p-set T which is k-optimal whenever $1 \leqslant k \leqslant K$ and $kq \geqslant 2$,*

(iii) *$\sigma(T)/\sigma^* \geqslant r$.*

Proof. Given integers $p \geqslant 2$ and $q \geqslant 1$, we first construct a graph G and then show the desired properties. We put

$$V(G) := S \cup T \cup \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} V_{ij},$$

where

$$S := \{s_1, s_2, \ldots, s_p\},$$
$$T := \{t_1, t_2, \ldots, t_p\},$$
$$V_{ij} := \{v_{ij}^1, \ldots, v_{ij}^q\} \qquad (i, j = 1, 2, \ldots, p);$$
$$E(G) := \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \bigcup_{h=1}^{q} \{s_i v_{ij}^h, v_{ij}^h t_j\}.$$

Thus G has $n := 2p + p^2 q$ vertices and $m := 2p^2 q$ edges. The graph G is illustrated in Fig. 2; clearly, it is bipartite. Every edge of G has assigned length 1; every vertex s_i has weight f (f is a positive integer which will be specified later) and every other vertex has weight 1.

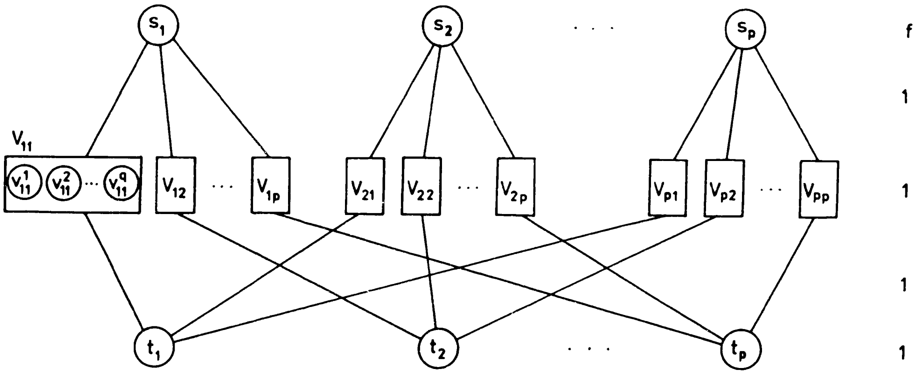

Fig. 2. Illustration of the proof of Theorem 2; the weights and lengths of G are on the right

One can easily verify that S is a p-median and the corresponding sum of weighted distances is equal to

$$\sigma^* := \sigma(S) = p^2 q + 2p.$$

Further the p-set T has

$$\sigma(T) = p^2q + 2pf.$$

We will show the k-optimality of T.

Let us consider a k-change of $T(1 \leq k \leq p-1)$. It gives a p-set M. Because of symmetry, we can suppose that $z := p - k$ vertices of T remain unchanged and these form a set $C := \{t_{p-k+1}, \ldots, t_p\}$. Further, we can assume that $A := M \cap S = \{s_1, \ldots, s_x\}$ $(x \leq k)$. The remaining $y := k - x$ vertices of M belong to $V(G) - S - T$ and form a set B. To prove the k-optimality of T it is sufficient to prove that $\sigma(M) \geqslant \sigma(T)$. Clearly, it is sufficient to deal with such a set B which minimizes $\sigma(M)$. Therefore we first prove that if

$$B = \{v^1_{i_1 j_1}, \ldots, v^1_{i_y j_y}\}$$

where

$$x + 1 \leqslant i_r \leqslant p, \quad 1 \leqslant j_r \leqslant p - z \qquad (r = 1, \ldots, y),$$

then $\sigma(M)$ is minimal. To calculate $\sigma(M)$ simply, a special set B is chosen. For this aim the following six cases are considered:

(1) If $|B \cap V_{ij}| > 1$ for some i and j, then there is an "unoccupied" set $V_{ij'}$ with $j' \leqslant p - z$, which enables us to replace v^h_{ij} by $v^h_{ij'}$ without increasing σ for any h.

(2) Instead of a vertex $v^h_{ij} \in B$, $h \neq 1$, we can take v^1_{ij} without increasing σ.

(3) If there are vertices $v^1_{i_g j_g}, v^1_{i_g j_h} \in B$, then there exists an index $i' \geqslant x + 1$ such that $v^1_{i'j} \notin B$ for every j and hence instead of $v^1_{i_g j_h}$, one can take $v^1_{i' j_h}$ without increasing σ.

(4) If some $v^1_{i_g j_g}, v^1_{i_h j_g} \in B$, then $i_g \neq i_h$ and therefore there exists an index $j' \leqslant p - z$ such that $v^1_{ij'} \notin B$ for every i. Thus instead of $v^1_{i_h j_h}$, one can take $v^1_{i_h j'}$ which produces a smaller value of σ.

(5) If $v^1_{i_h j_h} \in B$, where $j_h > p - z$, then there is an index j' such that $v^1_{i_h j'} \notin B$ and $t_{j'} \notin C$. Thus replacing $v^1_{i_h j_h}$ by $v^1_{i_h j'}$ does not increase σ.

(6) If there is $v^1_{i_h j_h} \in B$, where $i_h \leqslant x$, then there is a vertex $v^1_{i' j_h} \notin B$, which we can take instead of $v^1_{i_h j_h}$ without increasing σ.

Hence we can assume that M has the desired form. A symmetry of G enables us to assume that $B = \hat{B}$, where

$$\hat{B} := \{v^1_{x+1, x+1}, \ldots, v^1_{x+y, x+y}\}.$$

The corresponding set $\hat{M}$ has $\sigma(\hat{M}) \leq \sigma(M)$.

To calculate $\sigma(\hat{M})$, we explicitly write up the distance of every vertex to the set $\hat{M}$.

vertex		distance to $\hat{M}$
s_i,	$1 \leqslant i \leqslant x$	0
	$x + i \leqslant i \leqslant x + y$	1
	$x + y + 1 \leqslant i \leqslant p$	2
t_j,	$1 \leqslant j \leqslant x$	2
	$x + 1 \leqslant j \leqslant x + y$	1
	$x + y + 1 \leqslant j \leqslant p$	0
v_{ij}^h,	$1 \leqslant i \leqslant x,\ 1 \leqslant j \leqslant p,\ 1 \leqslant h \leqslant q$	1
	$x + 1 \leqslant i \leqslant x + y,\ 1 \leqslant j \leqslant x,\ 1 \leqslant h \leqslant q$	2
	$x + 1 \leqslant i \leqslant x + y,\ i = j,\ h = 1$	0
	$x + 1 \leqslant i \leqslant x + y,\ i = j,\ 2 \leqslant h \leqslant q$	2
	$x + 1 \leqslant i \leqslant x + y,\ x + 1 \leqslant j \leqslant x + y,\ i \neq j,\ 1 \leqslant h \leqslant q$	2
	$x + 1 \leqslant i \leqslant x + y,\ x + y + 1 \leqslant j \leqslant p,\ 1 \leqslant h \leqslant q$	1
	$x + y + 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant x,\ 1 \leqslant h \leqslant q$	3
	$x + y + 1 \leqslant i \leqslant p,\ x + 1 \leqslant j \leqslant x + y,\ 1 \leqslant h \leqslant q$	2
	$x + y + 1 \leqslant i \leqslant p,\ x + y + 1 \leqslant j \leqslant p,\ 1 \leqslant h \leqslant q$	1

Thus the sum of the weighted distances to the set $\hat{M}$ is

$$\begin{aligned}\sigma(\hat{M}) &= fy + 2f(p - x - y) + 2x + y + xpq + 2yxq + 2y(q - 1)\\ &\quad + 2(y^2 - y)q + y(p - x - y)q + 3(p - x - y)xq +\\ &\quad + 2(p - x - y)yq + (p - x - y)^2q\\ &= f(2p - 2x - y) + 2x - y + p^2q + 2xpq + ypq - 2x^2q - 2xyq.\end{aligned}$$

Substituting $k - x$ for y we obtain

$$\sigma(\hat{M}) = 2fp + p^2q + x(pq - 2kq + 3) + kpq - k - f(x + k).$$

Now we are going to specify f in order to ensure the k-optimality of T. We wish to prove the inequality $\sigma(\hat{M}) \geqslant \sigma(T)$, which gives:

$$x(pq - 2kq + 3) + kpq - k \geqslant f(x + k),$$

or

$$pq - 2kq + 3 + \frac{2k(kq - 2)}{x + k} \geqslant f.$$

Considering the left expression as a function of x we see that its minimum is attained if and only if x is as large as possible, i.e. $x = k$, because $kq \geqslant 2$ by the assumption in the theorem. Putting $x = k$, we obtain

$$pq - kq + 1 \geqslant f.$$

Thus we can let $f = pq - Kq + 1$ and the inequality $\sigma(\hat{M}) \geqslant \sigma(T)$ will hold. Hence the k-optimality of T is proved.

Now we can write:

$$\frac{\sigma(T)}{\sigma^*} = \frac{\sigma(T)}{\sigma(S)} = \frac{pq + 2f}{pq + 2} = \frac{3pq - 2Kq + 2}{pq + 2} \geqslant r.$$

(The last inequality holds by the assumption of the theorem on p.) This completes the proof. ■

Corollary 1. For any fixed $K \geqslant 1$ and $q \geqslant 2$, if $p \to \infty$, then $\sigma(T)/\sigma^* \to 3$.

Corollary 2. For $K = 1$ and $p = 2$, if $q \to \infty$, then $\sigma(T)/\sigma^* \to 2$.

4. Concluding remarks

As regards the multicenter problem, we have given a definite result. Nevertheless, there is stil an open question: How large can the worst-case error ratio be if all vertex weights are equal to one?

The same question for the multimedian problem seems to be even more interesting. However, the most important question which remains open is that concerning arbitrary weights. We conjecture that in this general case, the error ratio is always less than 3. Then Theorem 2 would be, in some sense, a best possible.

Another open question concerning both the multicenter and multimedian problems: How time consuming can finding k-optimal p-sets be? The main interest concerns small values of k and prilmarily $k = 1$. Is the time complexity polynomial or can it be of a higher order? If the former case holds and if the conjecture given above is true, then the interchange heuristic would be the first known polynomial heuristic for the multimedian problem with the error ratio bounded by a constant (equal to 3).

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О ПЕРЕСТАНОВОЧНОЙ ВРИСТИКЕ ДЛЯ РАЗМЕЩЕНИЯ ЦЕНТРОВ И МЕДИАН В ГРАФЕ

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Резюме

Показывается, что отношение величины результата перестановочной эвристики и минимума может быть произвольно большое число для проблемы p-центра и почти 3 для проблемы p-медианы.