Milan Demko

Lexicographic product decompositions of partially ordered quasigroups

Mathematica Slovaca, Vol. 51 (2001), No. 1, 13--24

Persistent URL: http://dml.cz/dmlcz/128708

Mathematica Slovaca

Math. Slovaca, **51** (2001), No. 1, 13–24

LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF PARTIALLY ORDERED QUASIGROUPS

MILAN DEMKO

(*Communicated by Tibor Katriňák*)

ABSTRACT. In this paper there are investigated some properties of partially ordered quasigroups (briefly: p.o. quasigroups) and lexicographic product decompositions of p.o. quasigroups are studied. It will be shown that for a p.o. quasigroup Q with an idempotent element h the assertion analogous with Theorem 15 in [JAKUBÍK, J.: *Lexicographic products of partially ordered groupoids*, Czechoslovak Math. J. **14(89)** (1964), 281–305 (Russian)] is valid, i.e. arbitrary two lexicographic product decompositions of a p.o. quasigroup Q with a finite number of directed lexicographic factors have isomorphic refinements.

1. Introduction

Lexicographic product decompositions of a certain type of partially ordered groupoids, so-called u-groupoids, were discussed by J. Jakubík in [6]. He proved that any two lexicographic product decompositions of an u-groupoid G with a finite number ([6; Theorem 15]) but also with an infinite number ([6; Theorem 35]) of lexicographic factors have isomorphic refinements. In this paper we will study lexicographic product decompositions of a partially ordered quasigroup Q with an idempotent element h. Here we will prove the following assertion analogous with [6; Theorem 15]: Arbitrary two lexicographic product decompositions of the partially ordered quasigroup Q with a finite number of directed lexicographic factors have isomorphic refinements. Let us remark that a partially ordered quasigroup Q with idempotent element h need not be an u-groupoid; conversely, an u-groupoid, in general, need not be a partially ordered quasigroup.

2000 Mathematics Subject Classification: Primary 06F15, 20N05.

Key words: partially ordered quasigroup, directed quasigroup, lexicographic product decomposition.

Fundamental results on lexicographic product of linearly ordered groups have been proved by M a l' c e v [9]. Further, lexicographic product decompositions of some types ordered algebraic structures were dealt with in the papers [5], [7], [8].

2. Preliminaries

We recall that a *quasigroup* $(Q,\cdot)$ is defined (cf., e.g. [3]) as an algebra having a binary operation $a \cdot b$ which satisfies the condition that for any a, b the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions x and y. A quasigroup having an identity element 1 (i.e., such that $1 \cdot x = x \cdot 1 = x$ for each $x \in Q$) is called a *loop*. If $(Q,\cdot)$ is a quasigroup, then we define $a/b = c$ if and only if $a = c \cdot b$; in this case we also put $c\backslash a = b$. For any $a, x \in Q$ we set $L_a x = a \cdot x$, $R_a x = x \cdot a$. Then L_a and R_a are called *left translations* or *right translations*, respectively. We have $L_a^{-1}x = a\backslash x$, $R_a^{-1}x = x/a$. The group generated by all left and right translations of $(Q,\cdot)$ is called the *multiplication group* of $(Q,\cdot)$ and is denoted by $G(Q,\cdot)$.

We will say that two quasigroups $(Q,\circ)$, $(Q,\cdot)$ are isotopic (cf., e.g. [3]) if there exist permutations α,β,γ of Q such that $\gamma(x \circ y) = \alpha x \cdot \beta y$ for all $x, y \in Q$. In such case we will write $(\circ) = (\cdot)^{(\alpha,\beta,\gamma)}$ and say that $(Q,\circ)$ is an *isotope* of $(Q,\cdot)$. It is well known (see, e.g. [3]) that if $(Q,\cdot)$ is a quasigroup and $(\circ) = (\cdot)^{(R_a^{-1}, L_b^{-1}, I)}$, where $a, b \in Q$, I is the identity permutation of Q, then $(Q,\circ)$ is a loop with the identity element ba.

The *direct product* $Q_1 \times Q_2$ of quasigroups Q_1, Q_2 is defined in a natural way, i.e. $Q_1 \times Q_2$ is the set of all ordered pairs (q_1, q_2), $q_1 \in Q_1$, $q_2 \in Q_2$, with the operation defined componentwise. The concepts of a normal subquasigroup, normal congruence on a quasigroup are used by definitions of [3]. Let $(Q_1,\cdot)$ and $(Q_2,\circ)$ be quasigroups. Notation $Q_1 \cong Q_2$ means that there exists isomorphism of $(Q_1,\cdot)$ into $(Q_2,\circ)$.

For the sake of convenience, we summarize here some results which will be frequently used and quoted. These results had been proved by B e l y a v s k a y a in [1] and later quoted in [2]. We will formulate them according to [2].

Let $(Q,\cdot)$ be a quasigroup with an idempotent element h. Then

A1) (Cf. [1; Theorem 4, Lemma 4]) $Q \cong Q_1 \times Q_2$ if and only if there exist normal subquasigroups A, B of Q such that $A \cdot B = Q$, $A \cap B = \{h\}$. Then $Q/A \cong Q_2 \cong B$, $Q/B \cong Q_1 \cong A$.

A2) (Cf. [1; Theorem 3]) Let A, B be normal subquasigroups of Q, $h \in A \cap B$. Then $A \cdot B = Q$ and $A \cap B = \{h\}$ if and only if each element $q \in Q$ can be uniquely written in the form $q = a \cdot b$, $a \in A$, $b \in B$.

A3) (Cf. [2; Lemma 1]) Let A, B be normal subquasigroups of Q and let $A \cdot B = Q$, $A \cap B = \{h\}$. If $a_1, a_2 \in A$, $b_1, b_2 \in B$, then

$$(a_1 b_1)(a_2 b_2) = R_h^{-1}(a_1 h \cdot a_2 h) \cdot L_h^{-1}(h b_1 \cdot h b_2). \tag{2.1}$$

A4) (Cf. [2; Chapt. 1, Corollaries 1, 2]) Let A, B be normal subquasigroups of Q such that $A \cdot B = Q$, $A \cap B = \{h\}$. Let $a \in A$, $b, b_1 \in B$. Then

$$L_h(ab) = R_h^{-1} L_h R_h a \cdot L_h b, \qquad R_h(ab) = R_h a \cdot L_h^{-1} R_h L_h b, \tag{2.2}$$

$$L_h^{-1}(ab) = R_h^{-1} L_h^{-1} R_h a \cdot L_h^{-1} b, \qquad R_h^{-1}(ab) = R_h^{-1} a \cdot L_h^{-1} R_h^{-1} L_h b, \tag{2.3}$$

$$\begin{aligned} ab \cdot b_1 &= ah \cdot L_h^{-1}(hb \cdot b_1), \\ b \cdot ab_1 &= R_h^{-1} L_h R_h a \cdot L_h^{-1}(b \cdot hb_1). \end{aligned} \tag{2.4}$$

3. Some properties of partially ordered quasigroups

DEFINITION 3.1. (Cf., e.g. [4; p. 297].) A nonempty set Q with an operation $\cdot$ and a relation $\leq$ is called a *partially ordered quasigroup* (briefly: *p.o. quasigroup*) if

(i) $(Q, \cdot)$ is a quasigroup.
(ii) $(Q, \leq)$ is a partially ordered set.
(iii) For all $x, y, a \in Q$, $x \leq y$ if and only if $ax \leq ay$ if and only if $xa \leq ya$.

A partially ordered quasigroup will be denoted by $(Q, \cdot, \leq)$ (or, if no misunderstanding can occur, by Q). If $(Q, \cdot)$ is a loop, then the p.o. quasigroup $(Q, \cdot, \leq)$ is called a *partially ordered loop* (*p.o. loop*). Let h be an arbitrary element of Q. The set $U_h = \{x \in Q : x \geq h\}$ is said to be h-cone of p.o. quasigroup Q (cf. [10; Definition 2]). The set $\{x \in Q : x \leq h\}$ will be denoted by U_h^*. If $(Q, \cdot, \leq)$ is a p.o. loop and h is an identity element of Q, then U will be used instead of U_h and U^* instead of U_h^*, respectively.

Let $(Q_1, \cdot, \leq)$ and $(Q_2, \circ, \leq')$ be p.o. quasigroups. Notation $Q_1 \cong_o Q_2$ means that there exists isomorphism of $(Q_1, \cdot)$ onto $(Q_2, \circ)$ which is also isomorphism of the partially ordered set $(Q_1, \leq)$ onto $(Q_2, \leq')$. In such case it will be said that p.o. quasigroups are o-isomorphic.

LEMMA 3.1. *Let $(Q, \cdot, \leq)$ be a p.o. quasigroup and let x, y be arbitrary elements in Q. Then $x \leq y$ if and only if $x/a \leq y/a$, $a\backslash x \leq a\backslash y$, $a/y \leq a/x$, $y\backslash a \leq x\backslash a$, where a is an arbitrary element in Q.*

Proof. Since $x = a \cdot (a\backslash x) = (x/a) \cdot a$ and $y = a \cdot (a\backslash y) = (y/a) \cdot a$, by Definition 3.1 we have $x \leq y$ if and only if $x/a \leq y/a$, $a\backslash x \leq a\backslash y$. Further,

$x \leq y$ if and only if $(a/x) \cdot x \leq (a/x) \cdot y$ if and only if $a \leq (a/x) \cdot y$ if and only if $(a/y) \cdot y \leq (a/x) \cdot y$ if and only if $a/y \leq a/x$. Analogously, $x \leq y$ if and only if $y \backslash a \leq x \backslash a$. □

LEMMA 3.2. *Let $(Q, \cdot, \leq)$ be a p.o. quasigroup. Let $(\circ) = (\cdot)^{(\alpha,\beta,\gamma)}$, where $\alpha, \beta, \gamma \in G(Q, \cdot)$. Then $(Q, \circ, \leq)$ is a p.o. quasigroup.*

P r o o f . This is an immediate consequence of Lemma 3.1. □

A p.o. quasigroup $(Q, \cdot, \leq)$ is said to be directed, if $(Q, \leq)$ is directed set (i.e. for arbitrary elements $a, b \in Q$ there exist $c, d \in Q$ such that $a, b \leq c$ and $d \leq a, b$). By the same method as in the case of p.o. groups (cf., e.g., [4; p. 290, Lemma 1]) we obtain:

LEMMA 3.3. *A p.o. loop $(Q, \cdot, \leq)$ is directed if and only if each element $q \in Q$ can be written in the form $q = u \cdot u^*$, where $u \in U$, $u^* \in U^*$.*

A generalization of Lemma 3.3 (and also of [4; p. 290, Lemma 1]) is the following lemma:

LEMMA 3.4. *Let $(Q, \cdot, \leq)$ be a p.o. quasigroup and let h be its arbitrary element. Then the p.o. quasigroup $(Q, \cdot, \leq)$ is directed if and only if each element $q \in Q$ can be written in the form $q = u \cdot u^*$, where $u \in U_h$, $u^* \in U_h^*$.*

P r o o f . Assume that a p.o. quasigroup $(Q, \cdot, \leq)$ is directed and q is an arbitrary element in Q. Then there is $c \in Q$ such that $q \leq c$, $hh \leq c$. There exists $x \in Q$ such that $c = xh$. From $q \leq xh$ and from $hh \leq xh$ we get $x \backslash q \leq h$ and $h \leq x$. Since $q = x \cdot (x \backslash q)$, we can conclude that q has the indicated form. Conversely, assume that each element $q \in Q$ can be written in the form $q = u \cdot u^*$, $u \in U_h$, $u^* \in U_h^*$. Let $(\circ) = (\cdot)^{(R_h^{-1}, L_h^{-1}, I)}$. Then $(Q, \circ)$ is a loop with identity element $1 = hh$ (see Section 2). By Lemma 3.2 $(Q, \circ, \leq)$ is a p.o. loop. Since each element $q \in Q$ can be represented in the form $q = R_h u \circ L_h u^*$, where $1 \leq R_h u$ and $L_h u^* \leq 1$, by Lemma 3.3 the p.o. loop $(Q, \circ, \leq)$ is directed. The p.o. quasigroup $(Q, \cdot, \leq)$ is obviously directed as well. □

Let $(Q, \cdot, \leq)$ be a p.o. quasigroup. Suppose that $(A, \cdot)$ is a subquasigroup of $(Q, \cdot)$. Then the p.o. quasigroup $(A, \cdot, \leq)$ will be called a p.o. subquasigroup of the p.o. quasigroup $(Q, \cdot, \leq)$. We write A instead of $(A, \cdot, \leq)$ if no misunderstanding can occur. Let $(A, \cdot, \leq)$ be a p.o. subquasigroup of Q and let h be any element in A. The sets $\{x \in A : \ h \leq x\}$ and $\{x \in A : \ x \leq h\}$ will be denoted by A_h^+ and A_h^-, respectively.

LEMMA 3.5. *Let A, B be p.o. subquasigroups of a p.o. quasigroup Q. Let $h \in A \cap B$. Then*

(i) $A_h^+ \subseteq B_h^+$ *if and only if* $A_h^- \subseteq B_h^-$,

(ii) *If A is directed, then $A_h^+ \subseteq B_h^+$ implies $A \subseteq B$.*

Proof.

(i) Let $a \in A_h^-$, i.e. $a \in A$, $a \leq h$. Then $h/h \leq h/a$, hence $(h/a) \cdot h \in A_h^+$. From $A_h^+ \subseteq B_h^+$ we have $(h/a) \cdot h \in B$, hence $a \in B$. Since $a \leq h$, we get $a \in B_h^-$. Analogously we can prove that $A_h^- \subseteq B_h^-$ implies $A_h^+ \subseteq B_h^+$.

(ii) Since A is directed, by Lemma 3.4 we have that each element $a \in A$ can be written in the form $a = u \cdot u^*$, where $u \in A_h^+$, $u^* \in A_h^-$. From $A_h^+ \subseteq B_h^+$ and from (i) it follows that $u \in B_h^+$, $u^* \in B_h^-$, hence $a = u \cdot u^*$ belongs to B.

□

4. Lexicographic product decomposition of p.o. quasigroups

In this section we will study lexicographic product decompositions (with a finite number lexicographic factors) of a p.o. quasigroup Q with an idempotent element h.

Let A_i, $i = 1, 2, \dots, n$, be p.o. quasigroups. Let C be the set of all ordered n-tuples $(a_1, \dots, a_n)$, $a_i \in A_i$. The binary operation (denoted by $\cdot$) defined componentwise. For distinct elements $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ in C we put $(a_1, \dots, a_n) < (b_1, \dots, b_n)$ whenever $a_i < b_i$ for the first element $i = 1, 2, \dots, n$ such that $a_i \neq b_i$. It is a routine to verify that $(C, \cdot, \leq)$ is a p.o. quasigroup. The p.o. quasigroup C that arises in this way will be called lexicographic product of the p.o. quasigroups A_i and it will be denoted by $\mathop{\Gamma}\limits_{i=1}^{n} A_i$. By $[A \circ B]$ we will denote the lexicographic product of two p.o. quasigroups A, B.

Let Q be a p.o. quasigroup with an idempotent element h. Let there exist p.o. subquasigroups A, B of Q which contain the element h and let the following conditions be fulfilled:

C1) For each $q \in Q$ there exists exactly one pair (a, b) such that $a \in A$, $b \in B$ and $q = a \cdot b$.

C2) If $q_1, q_2 \in Q$, $q_1 = a_1 b_1$, $q_2 = a_2 b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$, then

$$q_1 \cdot q_2 = R_h^{-1}(R_h a_1 \cdot R_h a_2) \cdot L_h^{-1}(L_h b_1 \cdot L_h b_2).$$

C3) Under the notation as in C2), the relation $q_1 \leq q_2$ is valid if and only if either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leq b_2$.

In such case we will write

$$Q = (A \circ B)_h. \tag{4.1}$$

The mapping

$$\varphi\colon Q \to [A \circ B], \qquad \varphi(ab) = (R_h a, L_h b), \tag{4.2}$$

where $a \in A$, $b \in B$ is an o-isomorphism. In fact. from C1) it follows that φ is a bijection. Further, $\varphi(a_1 b_1) \cdot \varphi(a_2 b_2) = (R_h a_1 . L_h b_1) \cdot (R_h a_2 . L_h b_2) = (R_h a_1 \cdot R_h a_2, L_h b_1 \cdot L_h b_2) = \varphi\big(R_h^{-1}(R_h a_1 \cdot R_h a_2) \cdot L_h^{-1}(L_h b_1 \cdot L_h b_2)\big) = \varphi(a_1 b_1 \cdot a_2 b_2)$. Finally, $a_1 b_1 \leq a_2 b_2$ if and only if (either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leq b_2$ if and only if (either $R_h a_1 < R_h a_2$ or $R_h a_1 = R_h a_2$ and $L_h b_1 \leq L_h b_2$ if and only if $(R_h a_1, L_h b_1) \leq (R_h a_2\ L_h b_2)$. Thus φ is [illegible] o-isomorphi[illegible] [illegible] 4.1) d[illegible]fines the le[illegible]icographic produc[illegible] d[illegible]compo[illegible]ition [illegible]f the p[illegible] qu[illegible] with an idempotent element h.

LEMMA 4.1. *Let $(Q, \leq)$ b[illegible] a p[illegible]o[illegible] quasigroup[illegible] The f[illegible]lowin[illegible] [illegible]o[illegible]d[illegible]tio[illegible] [illegible]1[illegible] (2) are equivalent*

(1) $Q = (A \circ B)_h$.

(2) A, B *are normal subquasigroups of* Q *such that*

 (i) $A \cap B = \{h\}$,

 (ii) $Q = A \cdot B$,

 (iii) $a_1 b_1 \leq a_2 b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$ *if and only if either* $a_1 < a_2$ *or* $a_1 = a_2$ *and* $b_1 \leq b_2$.

P r o o f. Let $Q = (A \circ B)_h$. Let Θ be a relation on Q such that $a_1 b_1 \,\Theta\, a_2 b_2$ if and only if $b_1 = b_2$, where $a_1, a_2 \in A$, $b_1, b_2 \in B$. In view of C1) and C2) it is easy to verify that Θ is a normal congruence on Q. If $x \,\Theta\, h$, then $x = ah$, $a \in A$, hence $x \in A$. Conversely, each element $x \in A$ can be written in the form $x = (x/h) \cdot h$, where $(x/h) \in A$, $h \in B$; thus $x \,\Theta\, h$. This proves that A is a class of the normal congruence Θ which contains the idempotent element h. Therefore A is a normal subquasigroup of Q. Analogously, B is a normal subquasigroup of Q. Now, for completing the proof, it suffices to use assertions A2), C1) and C3). The converse follows from A2), A3). □

LEMMA 4.2. *Let Q, Q_1, Q_2 be p.o. quasigroups and let h be an idempotent element of Q. Then the following are equivalent:*

(1) $Q \cong_o [Q_1 \circ Q_2]$.

(2) $Q = (A \circ B)_h$ *such that* $A \cong_o Q_1$, $B \cong_o Q_2$.

P r o o f. Let $\varphi\colon [Q_1 \circ Q_2] \to Q$ be an o-isomorphism. Let $h = \varphi(r, s)$, $r \in Q_1$, $s \in Q_2$ (it is obvious that r and s are idempotent elements) and let $Q_1' = \{(q, s) : q \in Q_1\}$, $Q_2' = \{(r, q) : q \in Q_2\}$. It is easy to verify that Q_1', Q_2' are normal subquasigroups of $[Q_1 \circ Q_2]$ such that $Q_1' \cdot Q_2' = [Q_1 \circ Q_2]$ and $Q_1' \cap Q_2' = \{(r, s)\}$. Put $A = \varphi(Q_1')$, $B = \varphi(Q_2')$. Since φ is an o-isomorphism, we can conclude that A, B are the normal subquasigroups of Q such that $A \cdot B = Q$ and $A \cap B = \{h\}$. Finally, we will show that the condition (iii) in Lemma 4.1 is valid. From A2) in the Section 2 it follows that each element $q \in Q$ can be uniquely written in the form $q = ab$, $a \in A$, $b \in B$. Let $q_1 = a_1 b_1$,

$q_2 = a_2b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$. Since φ is an o-isomorphism, there exist (u_1, u_2), (v_1, v_2) belonging to $[Q_1 \circ Q_2]$ such that $q_1 = \varphi(u_1, u_2)$, $q_2 = \varphi(v_1, v_2)$. We can write $q_1 = \varphi(u_1, u_2) = \varphi[(u_1/r, s) \cdot (r, s\backslash u_2)] = \varphi(u_1/r, s) \cdot \varphi(r, s\backslash u_2)$ and analogously $q_2 = \varphi(v_1/r, s) \cdot \varphi(r, s\backslash v_2)$. From $A = \varphi(Q_1')$ and $B = \varphi(Q_2')$ it follows that $\varphi(u_1/r, s)$, $\varphi(v_1/r, s) \in A$ and $\varphi(r, s\backslash u_2)$, $\varphi(r, s\backslash v_2) \in B$. Since q_1, q_2 can be uniquely written in the form $q_1 = a_1b_1$, $q_2 = a_2b_2$, we have $a_1 = \varphi(u_1/r, s)$, $a_2 = \varphi(v_1/r, s)$, $b_1 = \varphi(r, s\backslash u_2)$, $b_2 = \varphi(r, s\backslash v_2)$. Now, using that φ is an o-isomorphism we obtain $a_1 < a_2$ if and only if $u_1 < v_1$ and $b_1 \leq b_2$ if and only if $u_2 \leq v_2$. Thus $q_1 \leq q_2$ if and only if either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leq b_2$. By Lemma 4.1 we conclude that $Q = (A \circ B)_h$. Finally, from $Q_1' \cong_o Q_1$ and $Q_2' \cong_o Q_2$ it follows that $A \cong_o Q_1$ and $B \cong_o Q_2$.

Conversely, if $Q = (A \circ B)_h$, then $Q \cong_o [A \circ B]$. From $A \cong_o Q_1$ and $B \cong_o Q_2$ we get $[A \circ B] \cong_o [Q_1 \circ Q_2]$ and hence $Q \cong_o [Q_1 \circ Q_2]$. □

COROLLARY 4.3. *Let $Q = (A \circ B)_h$ and let $g \neq h$ be an idempotent element in Q. Then there exist p.o. quasigroups C, D such that $Q = (C \circ D)_g$ and $C \cong_o A$, $D \cong_o B$.*

P r o o f. From (4.2) it follows that $\varphi\colon (a, b) \to R_h^{-1}a \cdot L_h^{-1}b$ is an o-isomorphism of $[A \circ B]$ onto $Q = (A \circ B)_h$. For completing the proof it suffices to use Lemma 4.2. □

Let $Q = \big((A_1 \circ A_2)_h \circ A_3\big)_h$. From (4.2) and C2) it follows that

$$\varphi_1\colon (a_1a_2)a_3 \to \big(R_h(a_1a_2), L_ha_3\big) = \big(R_ha_1 \cdot L_h^{-1}R_hL_ha_2, L_ha_3\big),$$

where $a_i \in A_i$, for $i = 1, 2, 3$, is an o-isomorphism Q onto $\big[(A_1 \circ A_2)_h \circ A_3\big]$. Since $\varphi_2\colon a_1a_2 \to (R_ha_1, L_ha_2)$ is an o-isomorphism $(A_1 \circ A_2)_h$ onto $[A_1 \circ A_2]$, we get

$$\varphi_3\colon (a_1a_2)a_3 \to \big(\varphi_2(R_ha_1 \cdot L_h^{-1}R_hL_ha_2), L_ha_3\big) = \big((R_h^2a_1, R_hL_ha_2), L_ha_3\big)$$

is an o-isomorphism Q onto $\big[[A_1 \circ A_2] \circ A_3\big]$. Hence

$$\varphi\colon (a_1a_2)a_3 \to \big(R_h^2a_1, R_hL_ha_2, L_ha_3\big) \tag{4.3}$$

is an o-isomorphism Q onto $\mathop{\Gamma}\limits_{i=1}^{3} A_i$. Analogously,

$$\varphi\colon a_1(a_2a_3) \to \big(R_ha_1, L_hR_ha_2, L_h^2a_3\big) \tag{4.3'}$$

is an o-isomorphism of $Q = \big(A_1 \circ (A_2 \circ A_3)_h\big)_h$ onto $\mathop{\Gamma}\limits_{i=1}^{3} A_i$.

LEMMA 4.4. *Let $Q = \big((A_1 \circ A_2)_h \circ A_3\big)_h$. Then*

(i) $A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \{h\}$,

(ii) A_1, A_2, A_3 *are normal subquasigroups of* Q,

(iii) $(A_1 \cdot A_2) \cdot A_3 = A_1 \cdot (A_2 \cdot A_3)$.

Proof. Let $Q = ((A_1 \circ A_2)_h \circ A_3)_h$ and let $\varphi\colon Q \to \mathop{\Gamma}\limits_{i=1}^{3} A_i$ be the isomorphism defined by (4.3). Then $\varphi(A_1) = \{(a_1, h, h) :\ a_1 \in A_1\}$, $\varphi(A_2) = \{(h, a_2, h) :\ a_2 \in A_2\}$, $\varphi(A_3) = \{(h, h, a_3) :\ a_3 \in A_3\}$. Since $\varphi(A_1) \cap \varphi(A_2) = \varphi(A_2) \cap \varphi(A_3) = \varphi(A_1) \cap \varphi(A_3) = \{(h,h,h)\} = \{\varphi(h)\}$, we have $A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \{h\}$. Thus (i) holds. It is a routine to verify that the relation Θ defined by the rule $(a_1, a_2, a_3)\ \Theta\ (a_1', a_2', a_3')$ if and only if $a_2 = a_2'$ and $a_3 = a_3'$ is a normal congruence on $\mathop{\Gamma}\limits_{i=1}^{3} A_i$ and the subquasigroup $\varphi(A_1)$ is a class of the normal congruence Θ. Therefore $\varphi(A_1)$ is a normal subquasigroup of $\mathop{\Gamma}\limits_{i=1}^{3} A_i$. Analogously $\varphi(A_2)$, $\varphi(A_3)$ are normal subquasigroups of $\mathop{\Gamma}\limits_{i=1}^{3} A_i$. Hence A_1, A_2, A_3 are normal subquasigroups of Q, i.e. (ii) is valid. Finally, from $(\varphi(A_1) \cdot \varphi(A_2)) \cdot \varphi(A_3) = \varphi(A_1) \cdot (\varphi(A_2) \cdot \varphi(A_3)) = \mathop{\Gamma}\limits_{i=1}^{3} A_i$ we have (iii). □

Lemma 4.5. *$Q = ((A_1 \circ A_2)_h \circ A_3)_h$ if and only if $Q = (A_1 \circ (A_2 \circ A_3)_h)_h$.*

Proof. Let $Q = ((A_1 \circ A_2)_h \circ A_3)_h$. Let us denote $E = A_2 \cdot A_3$ and let $\varphi\colon Q \to \mathop{\Gamma}\limits_{i=1}^{3} A_i$ be an o-isomorphism defined by (4.3). Then $\varphi(E) = \varphi(A_2 \cdot A_3) = \varphi(A_2) \cdot \varphi(A_3) = \{(h, a_2, a_3) :\ a_2 \in A_2,\ a_3 \in A_3\}$. Since $\varphi(E)$ is a normal subquasigroup of $\mathop{\Gamma}\limits_{i=1}^{3} A_i$ and $\varphi(A_2)$, $\varphi(A_3)$ are normal subquasigroups of $\varphi(E)$. E is a normal subquasigroup of Q and A_2, A_3 are normal subquasigroups of E. From Lemma 4.4(i) it follows that $A_2 \cap A_3 = \{h\}$. Further, let $a_2a_3, a_2'a_3' \in E$ $(a_i, a_i' \in A_i)$. Since $a_2, a_2' \in (A_1 \circ A_2)_h$ and $a_3, a_3' \in A_3$, from the assumption we get $a_2a_3 \le a_2'a_3'$ if and only if either $a_2 < a_2'$ or $a_2 = a_2'$ and $a_3 \le a_3'$. Thus by Lemma 4.1 we conclude that $E = (A_2 \circ A_3)_h$.

From Lemma 4.4(iii) it follows that $Q = A_1 \cdot E$. Since $\varphi(A_1) \cap \varphi(E) = \{(h,h,h)\}$, $A_1 \cap E = \{h\}$. For arbitrary elements $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$ we can write $a_1(a_2a_3) = [(a_1/h) \cdot h] \cdot (a_2a_3)$. Since a_1/h, $a_2 \in (A_1 \circ A_2)_h$ and $h, a_3 \in A_3$, then from the assumption of the lemma and by C2) we have $a_1(a_2a_3) = R_h^{-1}(a_1 \cdot R_h a_2) \cdot L_h^{-1}(h \cdot L_h a_3)$. Consequently in view of (2.3) we obtain

$$a_1(a_2a_3) = (R_h^{-1}a_1 \cdot L_h^{-1}R_h^{-1}L_hR_ha_2) \cdot L_ha_3 . \tag{4.4}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$. From (4.4) it follows (we take R_ha_1 instead of a_1, $R_h^{-1}L_h^{-1}R_hL_ha_2$ instead of a_2 and $L_h^{-1}a_3$ instead of a_3)

$$R_ha_1 \cdot (R_h^{-1}L_h^{-1}R_hL_ha_2 \cdot L_h^{-1}a_3) = (a_1a_2)a_3 \tag{4.4'}$$

According to (4.4) and from the assumption we obtain $a_1(a_2a_3) \le a_1'(a_2'a_3')$ if and only if $(R_h^{-1}a_1 \cdot L_h^{-1}R_h^{-1}L_hR_h\ a_2) \cdot L_ha_3 \le (R_h^{-1}a_1' \cdot L_h^{-1}R_h^{-1}L_hR_ha_2') \cdot L_ha_3'$

if and only if either $a_1 < a_1'$ or $a_1 = a_1'$ and $a_2a_3 \le a_2'a_3'$. Thus, by Lemma 4.1 we conclude that $Q = (A_1 \circ E)_h = \big(A_1 \circ (A_2 \circ A_3)_h\big)_h$. Analogously, using an o-isomorphism defined by (4.3′) and by (4.4′), we can prove that $Q = (A_1 \circ (A_2 \circ A_3)_h)_h$ implies $Q = \big((A_1 \circ A_2)_h \circ A_3\big)_h$. □

In view of Lemma 4.5 we can write $Q = (A_1 \circ A_2 \circ A_3)_h$ instead of $Q = \big((A_1 \circ A_2)_h \circ A_3\big)_h$. Analogously, by induction we can write

$$(A_1 \circ A_2 \circ A_3 \circ \cdots \circ A_{n-1} \circ A_n)_h = \big(\big(\big(\dots(A_1 \circ A_2)_h \circ A_3\big)_h \circ \cdots \circ A_{n-1}\big)_h \circ A_n\big)_h . \tag{4.5}$$

A p.o. quasigroup A is said to be the lexicographic factor of Q with an idempotent element h, if there are p.o. subquasigroups H, D of Q such that $Q = (H \circ A \circ D)_h$ (for an analogous notation in the theory of partially ordered u-groupoids cf. [6; Sect. 6]). Let us remark that Q and $\{h\}$ are lexicographic factors, because $Q = \big(\{h\} \circ Q \circ \{h\}\big)_h$ and also $Q = \big(Q \circ \{h\} \circ \{h\}\big)_h$.

Let $Q = (A_1 \circ A_2 \circ A_3 \circ \cdots \circ A_{n-1} \circ A_n)_h$. Then, using (4.5) and (4.2), we get by induction that

$$\varphi\colon \big(\dots\big((a_1a_2)a_3\big)\dots a_{n-1}\big)a_n \longrightarrow \big(R_h^{n-1}a_1, R_h^{n-2}L_ha_2, R_h^{n-3}L_ha_3, \dots, R_hL_ha_{n-1}, L_ha_n\big)$$

is an o-isomorphism $(A_1 \circ A_2 \circ A_3 \circ \cdots \circ A_{n-1} \circ A_n)_h$ onto $\Gamma_{i=1}^{n} A_i$. In such case we say that $Q = (A_1 \circ A_2 \circ \cdots \circ A_n)_h$ defines the lexicographic product decomposition of Q with the finite number of lexicographic factors.

From Lemma 3.5 it follows that if each lexicographic factor of a p.o. loop Q is directed, then Q is an u-groupoid. Therefore all results which hold for u-groupoids (see [6]) also hold for these p.o. loops. Now, we will show that some assertions analogous to those in [6] valid for u-groupoids can be proved for p.o. quasigroups.

LEMMA 4.6. *If $Q = (A \circ B)_h$, then B is a convex p.o. subquasigroup of Q.*

P r o o f . This proof is analogous to the proof in [6; Sect. 7]. □

LEMMA 4.7. *Let $Q = (A \circ B)_h$, $Q = (C \circ D)_h$ be two lexicographic product decompositions of a p.o. quasigroup Q. Let A, B, C, D are directed subquasigroups of Q. Then*

(i) *$B \subseteq D$ or $D \subseteq B$.*
(ii) *If $D \subseteq B$, then $B = \big((B \cap C) \circ D\big)_h$.*
(iii) *If $A = C$, then $B = D$.*

P r o o f .

(i) Let $D \not\subseteq B$. Then, by Lemma 3.5, $D_h^+ \not\subseteq B_h^+$ and $D_h^- \not\subseteq B_h^-$. Now, in the same way as when proving 9) in [6] we get $B \subset D$.

(ii) Let $D \subseteq B$. First, we will prove that $B = (B \cap C) \cdot D$. Each element $b \in B$ can be uniquely represented in the form $b = cd$, $c \in C$, $d \in D$. Since $D \subseteq B$, we have $d \in B$ and hence $c \in B$. Thus $c \in B \cap C$, therefore $b \in (B \cap C) \cdot D$. We have $B \subseteq (B \cap C) \cdot D$. The converse inclusion is trivial. From the assumption of the lemma it follows that $B \cap C$ and D are normal subquasigroups of Q; thus they are normal subquasigroups of B. It is clearly that $(B \cap C) \cap D = \{h\}$. For completing the proof we need show that the condition (iii) from Lemma 4.1 is valid. Let $b_1 = c_1 d_1$, $b_2 = c_2 d_2$, $c_1, c_2 \in B \cap C$, $d_1, d_2 \in D$. Since $Q = (C \circ D)_h$, $b_1 \le b_2$ if and only if either $c_1 < c_2$ or $c_1 = c_2$ and $d_1 \le d_2$; thus (iii) is valid. Therefore by Lemma 4.1 we can conclude that $B = \big((B \cap C) \circ D\big)_h$.

(iii) From (i) and (ii) we get either $B = \big((B \cap A) \circ D\big)_h = \big(\{h\} \circ D\big)_h$ or $D = \big((D \cap C) \circ B\big)_h = \big(\{h\} \circ B\big)_h$. Hence $B = D$. □

Let $Q = (A \circ B)_h$. From Lemma 4.1 it follows that A, B are normal subquasigroups of Q such that $A \cap B = \{h\}$. Let Q/B be a set of all classes xB, $x \in Q$, with the operation $xB \cdot yB = R_h^{-1}(R_h x \cdot R_h y) \cdot B$. Then Q/B is a quasigroup (see e.g. [3]). Every class xB contains exactly one element of A. In fact, let $a, a' \in A \cap xB$ and let $x = a_1 b_1$, $a_1 \in A$, $b_1 \in B$. Then from (2.4) we have $a = a_1 b_1 \cdot b = a_1 h \cdot L_h^{-1}(L_h b_1 \cdot b)$ and $a' = a_1 b_1 \cdot b' = a_1 h \cdot L_h^{-1}(L_h b_1 \cdot b')$, where $b, b' \in B$. Since, at the same time $a = (a/h) \cdot h$ and $a' = (a'/h) \cdot h$, we get $a_1 h = a/h$ and $a_1 h = a'/h$. Hence $a = a'$. Finally, if $x = a_1 b_1$, then by (2.4), $x \cdot (L_h b_1 \backslash h) = a_1 h \cdot h$, hence $x \cdot (L_h b_1 \backslash h) \in A \cap xB$, therefore $A \cap xB \neq \emptyset$.

In view of the assertion above we can write $Q/B = \{R_h^{-1}(a) \cdot B : \ a \in A\}$. Let $\le$ be a relation on the set Q/B which is defined as follows: $R_h^{-1}(a_1) \cdot B \le R_h^{-1}(a_2) \cdot B$ if and only if $a_1 \le a_2$. It is a routine to verify that $(Q/B, \cdot, \le)$ is a p.o. quasigroup. The mapping $\varphi(a) = R_h^{-1}(a) \cdot B$ is an o-isomorphism of A onto Q/B.

LEMMA 4.8. *Let $Q = (A \circ B)_h$ and $Q = (C \circ B)_h$. Then there exists an o-isomorphism φ of A onto C such that $\varphi(h) = h$.*

Proof. In view of the assumption, each element $a \in A$ can be uniquely written in the form $a = R_h^{-1}(c) \cdot b$, where $c \in C$, $b \in B$. Let φ be a mapping of A into C such that $\varphi(a) = c$ whenever $a = R_h^{-1}(c) \cdot b$. The map φ is a composition of two o-isomorphisms:

$\varphi_1 \colon A \to Q/B$; $\varphi_1(a) = R_h^{-1}(a) \cdot B$,

$\varphi_2 \colon Q/B \to C$; $\varphi_2(R_h^{-1}(a) \cdot B) = c$, where $c \in R_h^{-1}(a) \cdot B$.

Therefore $\varphi = \varphi_2 \varphi_1$ is an o-isomorphism of the quasigroup A onto quasigroup C. Clearly, $\varphi(h) = h$. □

Let

$$Q = (A_1 \circ A_2 \circ \cdots \circ A_n)_h \tag{4.6}$$

and suppose that there are given lexicographic product decompositions

$$A_i = (A_{i1} \circ A_{i2} \circ \cdots \circ A_{im(i)})_h$$

for each $i = 1, 2, \dots, n$. Then according to Lemma 4.5 and by (4.5) we can write

$$Q = \big(A_{11} \circ A_{12} \circ \cdots \circ A_{ij} \circ \cdots \circ A_{nm(n)}\big)_h . \tag{4.7}$$

We will say that the lexicographic product decomposition (4.7) is a refinement of (4.6). Further, let

$$Q = \big(B_1 \circ B_2 \circ \cdots \circ B_m\big)_h . \tag{4.8}$$

The lexicographic product decompositions (4.6) and (4.8) are said to be isomorphic, if $m = n$ and A_i and B_i are o-isomorphic for all $i = 1, 2, \dots, n$.

THEOREM 4.1. *Two lexicographic product decompositions $Q = (A_1 \circ \cdots \cdots \circ A_n)_h$ and $Q = (B_1 \circ \cdots \circ B_m)_h$, where $A_1, \dots, A_n$, $B_1, \dots, B_m$ are directed subquasigroups of p.o. quasigroup Q, have isomorphic refinements.*

Proof. We prove the theorem by induction on $n + m$, $n + m \geq 2$ (for an analogous proof cf. [6; Theorem 15]). It is clear for $n + m = 2$. Let $n + m > 2$. According to Lemma 4.7(i) we can suppose without loss of generality that $A_n \subseteq B_m$. Then, by Lemma 4.7(ii) we have $B_m = (E \circ A_n)_h$, where $E = B_m \cap (A_1 \circ A_2 \circ \cdots \circ A_{n-1})_h$. Since E is the first lexicographic factor and $B_m = (E \circ A_n)_h$ is directed, then E is also directed. From $Q = (B_1 \circ B_2 \circ \cdots \circ B_{m-1} \circ E \circ A_n)_h = (A_1 \circ A_2 \circ \cdots \circ A_n)_h$ and by Lemma 4.8 we have $(B_1 \circ B_2 \circ \cdots \circ B_{m-1} \circ E)_h \cong_{\circ} (A_1 \circ A_2 \circ \cdots \circ A_{n-1})_h$. By assumption of induction we can conclude that the theorem is proved. □

REFERENCES

[1] BELYAVSKAYA, G. B.: *Direct decomposition of quasigroups*, Mat. Issled. **5** (1987), 23–38. (Russian)

[2] BELYAVSKAYA, G. B.: *Complete direct decompositions of quasigroups with an idempotent*, Mat. Issled. **113** (1990), 21–36. (Russian)

[3] BELOUSOV, V. D.: *Foundations of the Theory of Quasigroups and Loops*, Nauka, Moscow, 1967. (Russian)

[4] BIRKHOFF, G.: *Lattice Theory*, Amer. Math. Soc., Providence, RI, 1967.

[5] ČERNÁK, Š.: *Lexicographic products of cyclically ordered groups*, Math. Slovaca **45** (1995), 29 38.

[6] JAKUBÍK, J.: *Lexicographic products of partially ordered groupoids*, Czechoslovak Math. J. **14(89)** (1964), 281–305. (Russian)

[7] JAKUBÍK, J.: *Lexicographic product decompositions of cyclically ordered groups*, Czechoslovak Math. J. **48(123)** (1998), 229–241.

[8] JAKUBÍK, J.: *Lexicographic products of half linearly ordered groups*, Czechoslovak Math. J. (To appear).

[9] MAL'CEV, A. I.: *On ordered groups*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 473–482. (Russian)

[10] TARARIN, V. M.: *Ordered quasigroups*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (1979), 82–86. (Russian)

Received October 30, 1998
Revised July 15, 1999

Department of Mathematics
FHPV PU
17. novembra 1
SK-081 16 Prešov
SLOVAKIA
E-mail: demko@unipo.sk