Mathematica Slovaca

Pavel Klenovčan

Direct product decompositions of digraphs

Mathematica Slovaca, Vol. 38 (1988), No. 1, 3--10

Persistent URL: http://dml.cz/dmlcz/128707

DIRECT PRODUCT DECOMPOSITIONS OF DIGRAPHS

PAVEL KLENOVČAN

The direct, subdirect and weak direct product decompositions of partially ordered sets and the decompositions of their convering graphs were investigated, e. g., in [1], [3], [4], [5], [6]. Any almost discrete partially ordered set $(P, \leqq)$ may be represented as a directed graph. Some relations between the direct product decompositions of a covering graph $C(\bar{G})$ of a digraph $\bar{G}$ and the direct product decompositions of the digraph $\bar{G}$ will be studied in the present paper.

A *graph* $G = (V, E)$ consists of a nonempty set V of vertices together with a prescribed set E of unordered pairs of distinct vertices of V. Each pair $\{x, y\} \in E$ is an (*undirected*) *edge* of the graph G.

A *diagraph* $\bar{G} = (V, \bar{E})$ consists of a nonempty set V of vertices together with a prescribed set $\bar{E}$ of ordered pairs of distinct vertices. Each ordered pair $(x, y) \in \bar{E}$ is a (*directed*) *edge* of the digraph $\bar{G}$.

Let I be a nonempty set and $G_i = (V_i, E_i)$ $(i \in I)$ be graphs. Let elements of V will be denoted $a = (a_i)$, $i \in I$, where $a_i = a(i) \in V_i$. Let G be a graph whose set of vertices is V and whose set of edges consists of those pairs $\{x, y\}$, $x, y \in V$ which satisfy the following condition: there is $i \in I$ such that $\{x_i, y_i\} \in E_i$ and $x_j = y_j$ for each $j \in I \setminus \{i\}$. Then G is said to be the *direct product of the graphs* G_i $(i \in I)$ and we write $G = \prod_{i \in I} G_i$. We omit the symbol $i \in I$ very often if no misunderstanding is likely to arise.

The direct product of digraphs is defined similarly.

If a mapping f: $V_1 \to V_2$ is an isomorphism of a graph $G_1 = (V_1\, E_1)$ into a graph $G_2 = (V_2, E_2)$, then we shall write $G_1 \overset{f}{\simeq} G_2$ or shortly $G_1 \simeq G_2$.

If $G \overset{f}{\simeq} \prod G_i$, then we shall say that $\prod G_i$ is a *decomposition of the graph* G (with respect to the mapping f).

In the present paper every decomposition $\prod G_i$, where $G_i = (V_i, E_i)$, is supposed to be nontrivial (i. e. $|V_i| > 1$ for each $i \in I$).

Analogous terminology and notations are used for digraphs. For all further notions concerning digraphs and graphs we refer the reader to [2].

Let $\bar{G} = (V, \bar{E})$ be a digraph. By the *covering graph of* $\bar{G}$ we mean the graph $C(\bar{G}) = (V, E)$ whose edges are those pairs $\{a, b\}$, for which $(a, b) \in \bar{E}$ or $(b, a) \in \bar{E}$.

In Fig. 1 we have a digraph $\bar{G}$ and its covering graph $C(\bar{G})$.

Let $\bar{G} = (V, \bar{E})$ be a digraph and $C(\bar{G}) \overset{f}{\simeq} \prod G_i$, where $G_i = (V_i, E_i)$ $(i \in I)$. We shall say that the *decomposition* $\prod G_i$ *of the graph* $C(\bar{G})$ *induces a decomposition of the diagraph* $\bar{G}$ if there exist such digraphs $\bar{G}_i = (V_i, \bar{E}_i)$ that $C(\bar{G}_i) = G_i$ for each $i \in I$ and $\bar{G} \overset{f}{\simeq} \prod \bar{G}_i$.

The decomposition of $C(\bar{G})$ does not induce a decomposition of $\bar{G}$ in general. In Fig. 1 the digraph $\bar{G}$ is not isomorphic to the direct product of any two digraphs but its covering graph is isomorphic to the direct product of two complete graphs K_2.

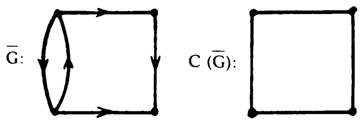

Fig. 1.

If a digraph $\bar{G}$ is isomorphic to $\prod \bar{G}_i$, then its covering graph $C(\bar{G})$ is isomorphic to $\prod C(\bar{G}_i)$ and the decomposition $\prod C(\bar{G}_i)$ induces the decomposition $\prod \bar{G}_i$ of $\bar{G}$. Thus the existence of a decomposition of $C(\bar{G})$ is a necessary condition for the existence of a decomposition of $\bar{G}$. A sufficient condition for the existence of a decomposition of $\bar{G}$ is the existence of such a decomposition of $C(\bar{G})$ which induces a decomposition of $\bar{G}$. hence the digraph $\bar{G}$ has a decomposition if and only if its covering graph $C(\bar{G})$ has a decomposition inducing a decomposition of the digraph $\bar{G}$. That is why we are going to investigate when the decomposition of $C(\bar{G})$ induces a decomposition of $\bar{G}$.

The subgraph of a graph $G = (V, E)$ induced by a set $W \subseteq V$ will be denoted by $G\langle W \rangle$.

Let $G = (V, E)$ be a graph. If there exists a four-element set $W = \{a, b, c, d\} \subseteq V$ such that $G\langle W \rangle = (W, F)$, where $F = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$, then we say that the graph $G\langle W \rangle$ is a *square* (*in* G) and we denote it by $S(a, b, c, d)$, If $\bar{G}$ is a digraph and $C(\bar{G}\langle W \rangle) = S(a, b, c, d)$, then the digraph $\bar{G}\langle W \rangle$ is called a *square* (*in* $\bar{G}$) and denoted by $\bar{S}(a, b, c, d)$.

The following lemma is easy to verify (see, e. g., appendix 2 in [2]).

Lemma 1. *Let* $\bar{S}_i$, $i \in \{1, 2, \ldots, 15\}$ *be digraphs in Fig.* 2. *If a digraph* $\bar{S}(a, b, c, d)$ *is a square, then there exists* $i \in \{1, 2, \ldots, 15\}$ *such that* $\bar{S}(a, b, c, d) \simeq \bar{S}_i$.

The edge $\{a, b\}$ of a graph $\prod G_i$ will be called a *k-edge* whenever $a_j = b_j$ for each $j \in I \setminus \{k\}$.

Remark. It is easy to see that every edge of a direct product of graphs is a k-edge for some $k \in I$.

We say that ordered pairs (a, b) and (c, d) of vertices of a direct product $\prod G_i$ are *r-equivalent* and write $(a, b) \overset{r}{\sim} (c, d)$ if $\{a, b\}$ and $\{c, d\}$ are r-edges and $a_r = c_r$, $b_r = d_r$.

Let $\prod G_i = (V, E)$. If $W \subseteq V$, then we denote $O_i(W) = \{a_i; a \in W\}$.

Lemma 2. *Let $\prod G_i = (V, E)$. Let $\{a, b, c, d\} = W \subseteq V$ and $S(a, b, c, d)$ be a square (in $\prod G_i$). If $\{a, b\}$, $\{b, c\}$ are r-edges, then $\{c, d\}$, $\{a, d\}$ are r-edges too, and $|O_r(W)| = 4$, $|O_j(W)| = 1$ for each $j \in I \setminus \{r\}$.*

Proof. Let $\{a, b\}$, $\{b, c\}$ be r-edges. Then $a_r \neq b_r$, $b_r \neq c_r$ and $a_j = b_j = c_j$ for each $j \in I \setminus \{r\}$. Since $a \neq c$, we have $a_r \neq c_r$. Suppose $\{a, d\}$ is not an r-edge. Then $d_r = a_r$ and there exists $s \in I \setminus \{r\}$ with $a_s \neq d_s$. Since $a_s = c_s$, we get $c_s \neq d_s$. Thus $d_r = c_r$, a contradiction. So $\{a, d\}$ is an r-edge. Similarly we obtain that $\{c, d\}$ is an r-edge. Further, $b \neq d$ and consequently $b_r \neq d_r$. Now we have $|O_r(W)| = 4$ and $|O_j(W)| = 1$ for each $j \in I \setminus \{r\}$ immediately.

Lemma 3. *Let $S(a, b, c, d)$ be a square in $\prod G_i$. If $\{a, b\}$ is an r-edge and $\{b, c\}$ is an s-edge, $r \neq s$, then $a_r = d_r$ and $c_s = d_s$.*

Proof. Suppose that $a_r \neq d_r$ or $c_s \neq d_s$. Let $a_r \neq d_r$. Then $\{a, d\}$ is an r-edge and (Lemma 2) $\{b, c\}$, $\{c, d\}$ are r-edges, a contradiction. In the case when $c_s \neq d_s$ we obtain a contradiction in a similar way.

Using Lemma 2 and Lemma 3 the following lemma is easy to verify.

Lemma 4. *Let $S(a, b, c, d)$ be a square in $\prod G_i$. If $\{a, b\}$ is an r-edge and $\{b, c\}$ an s-edge, then $\{c, d\}$ is an r-edge and $\{a, d\}$ an s-edge.*

A square $S(a, b, c, d)$ in $\prod G_i$ will be called an *r-square* whenever all its edges are r-edges for some $r \in I$. If such $r \in I$ does not exist, it will be called a *mixed square*.

Lemma 3 and Lemma 4 imply the following

Lemma 5. *Let $S(a, b, c, d)$ be a mixed square in $\prod G_i$, where $\{a, b\}$ is an r-edge and $\{b, c\}$ an s-edge. Then $(a, b) \overset{r}{\sim} (d, c)$, $(b, c) \overset{s}{\sim} (a, d)$.*

Lemma 6. *Let $\prod G_i = (V, E)$ be a connected graph and the ordered pairs (a, b), (c, d) of vertices of $\prod G_i$ be r-equivalent. Then there exist vertices $x^0 = a, x^1, \ldots, x^n = c, y^0 = b, y^1, \ldots, y^n = d \in V$ such that $S(x^j, x^{j+1}, y^{j+1}, y^j)$ is a mixed square for each $j \in \{0, 1, \ldots, n-1\}$.*

Proof. Since $a_r = c_r$ and the graph $\prod G_i$ is connected, there exist vertices $x^0 = a, x^1, \ldots, x^n = c \in V$ such that $a_r = x^1_r = \ldots = c_r$ and the sequence $x^0 = a, x^1, \ldots, x^n = c$ is a path in $\prod G_i$. For each $j \in \{1, 2, \ldots, n-1\}$ we define $y^j \in V$ as follows: $y^j_r = b_r$, $y^j_k = x^j_k$ for each $k \in I \setminus \{r\}$. It is clear that $S(x^j, x^{j+1}, y^{j+1}, y^j)$ is a square for each $j \in \{0, 1, \ldots, n-1\}$. Further, $\{x^j, y^j\}$ is an r-edge and since $x^j_r = x^{j+1}_r$, $\{x^j, x^{j+1}\}$ is not an r-edge. Hence all the squares $S(x^j, x^{j+1}, y^{j+1}, y^j)$ are mixed.

Let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$. We shall say that the edge (a, b) of the digraph $\bar{G}$ and the edge $\{a, b\}$ of the covering graph $C(\bar{G})$ are *k-edges* (with respect to the isomorphism f) if $\{f(a), f(b)\}$ is a k-edge of the graph $\prod G_i$. In an analogous way the other notions concerning the direct product $\prod G_i$ can be introduced for the digraph $\bar{G}$ and the covering graph $C(\bar{G})$.

Let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$, where $\bar{G} = (V, \bar{E})$. We shall say that the r-equivalent ordered pairs (a, b), (c, d) of vertices of $\bar{G}$ are *similarly oriented* if

$$(1) \qquad (a, b) \in \bar{E} \text{ if and only if } (c, d) \in \bar{E}.$$

Lemma 7. *Let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$, where $\bar{G} = (V, \bar{E})$ and $G_i = (V_i, E_i)$. The decomposition $\prod G_i$ of $C(\bar{G})$ induces a decomposition of $\bar{G}$ if and only if every two r-equivalent ordered pairs of vertices of $\bar{G}$ are similarly oriented for each $r \in I$ (i. e. if and only if the relation $(a, b) \overset{r}{\sim} (c, d)$ implies (1)).*

Proof. It suffices to define $\bar{G}_i$ for each $i \in I$ as follows: $\bar{G}_i = (V_i, \bar{E}_i)$, where $(f(a)_i, f(b)_i) \in \bar{E}_i$, if and only if there exists an i-edge $(a, b) \in \bar{E}$.

In the sequel we shall write "*connected*", instead of the more precise "*weakly connected*".

Theorem 1. *Let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$, where $\bar{G} = (V, \bar{E})$ is a connected digraph. The decomposition $\prod G_i$ of $C(\bar{G})$ induces a decomposition of $\bar{G}$ if and only if the following condition is fulfilled:*

(A) If $\bar{S}(a, b, c, d)$ is a mixed square in $\bar{G}$, then there exists $i \in \{1, 2, 3\}$ with $\bar{S}(a, b, c, d) \simeq \bar{S}_i$ (see Fig. 2).

Proof. Let the decomposition $\prod G_i$ of $C(\bar{G})$ induce a decomposition of $\bar{G}$ and $\bar{S}(a, b, c, d)$ be its mixed square. Then there exist (Lemma 5) $r, s \in I$, $r \neq s$, such that $(a, b) \overset{r}{\sim} (d, c)$ and $(b, c) \overset{s}{\sim} (a, d)$. From Lemma 7 it follows that (a, b), (d, c) are similarly oriented and (b, c), (a, d) are similarly oriented, too. Thus there exists $i \in \{1, 2, 3\}$ with the property $\bar{S}(a, b, c, d) \simeq \bar{S}_i$. Suppose, to prove the reverse implication, that (A) is fulfilled. With respect to Lemma 7, it suffices to prove that the edges (x, y) and (u, v) of $\bar{G}$ are similarly oriented whenever $(x, y) \overset{r}{\sim} (u, v)$. Then, by Lemma 6, there exist a nonnegative integer n and vertices $x^0 = x, x^1, \ldots, x^n = u, y^0 = y, y^1, \ldots, y^n = v \in V$ such that $\bar{S}(x^j, x^{j+1}, y^{j+1}, y^j)$ is a mixed square in $\bar{G}$ for each $j \in \{0, 1, \ldots, n-1\}$. In $n = 0$, then (x, y) and $u, v)$ are similarly oriented, since $(x, y) = (u, v)$. If $n = 1$, then $\bar{S}(x, u, v, y)$ is a mixed square and (x, y), (u, v) are similarly oriented according to (A). Now it is easy to complete the proof by induction on n.

From Theorem 1, Lemma 1 and Lemma 2 we immediately obtain:

Theorem 2. *Let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$, where $\bar{G} = (V, \bar{E})$ is a connected digraph and*

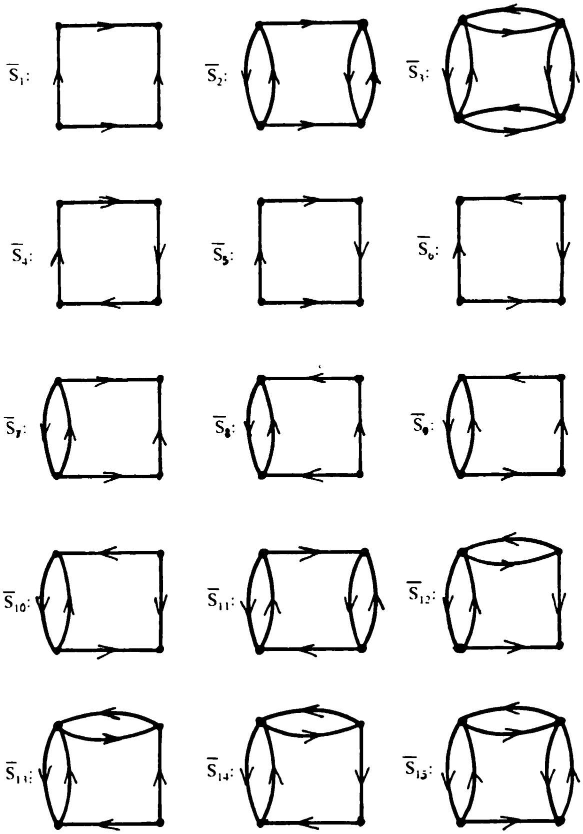

Fig. 2.

$G_i = (V_i, E_i)$. *The decomposition* $\prod G_i$ *of* $C(\bar{G})$ *induces a decomposition of* $\bar{G}$ *if and only if the following condition is fulfilled*:

(*B*) *If* $\bar{S}(a, b, c, d) \simeq \bar{S}_i$, $i \in \{4, 5, \ldots, 15\}$, *where* $W = \{a, b, c, d\} \subseteq V$, *then there exists* $k \in I$ *such that* $\bar{S}(a, b, c, d)$ *is a k-square* (*in other words, there exists* $k \in I$ *such that* $|f_j(W)| = |O_j(f(W))| = 1$ *for each* $j \in I \setminus \{k\}$, *where* f_j: $V \to V_j$ *is a projection corresponding to the mapping* f: $V \to \prod V_i$).

The following theorem follows from Theorem 1 or Theorem 2 immediately.

Theorem 3. *Let* $\bar{G}$ *be a connected digraph which contains no square isomorphic to* $\bar{S}_i$, $i \in \{4, 5, \ldots, 15\}$. *Then every decomposition* $\prod G_i$ ($i \in I$) *of the covering graph* $C(\bar{G})$ *of the digraph* $\bar{G}$ *induces a decomposition of the digraph* $\bar{G}$.

Let $\bar{G} = (V, \bar{E})$ be a digraph. An edge $(a, b) \in \bar{E}$ will be called *transitive* if there exists a vertex $c \in V$, $c \neq a$, $c \neq b$ such that there is a (directed) path from a to c and also a (directed) path from c to b.

It is easy to verify the following

Lemma 8. *Let $\bar{S}(a, b, c, d)$ be a square in $\bar{G}$. Then*
(*a*) *if $\bar{G}$ is an acyclic digraph, then $\bar{S}(a, b, c, d) \simeq \bar{S}_i$, $i \in \{1, 5, 6\}$,*
(*b*) *if $\bar{G}$ is a digraph with no transitive edge, then $\bar{S}(a, b, c, d) \simeq \bar{S}_i$, $i \in \{1, 4, 6\}$.*

Theorem 2 and Lemma 8 imply

Theorem 4. *Let $\bar{G} = (V, \bar{E})$ be an acyclic connected digraph with no transitive edge and let $C(\bar{G}) \overset{f}{\simeq} \prod G_i$. The decomposition $\prod G_i$ of $C(\bar{G})$ induces a decomposition of $\bar{G}$ if and only if the following condition is fulfilled:*
(*C*) *If $\bar{S}(a, b, c, d) \simeq \bar{S}_6$, where $W = \{a, b, c, d\} \subseteq V$, then there exists $k \in I$ such that $|f_j(W)| = 1$ for each $j \in I \setminus \{k\}$.*

Every almost discrete partially ordered set $(P, \leq)$ (shortly "poset P") may be represented as a digraph $\bar{G} = (P, \bar{E})$ such that $(a, b) \in \bar{E}$ if and only if b covers a. Clearly, this digraph is acyclic and has no transitive edge.

The *covering graph $C(P)$ of a poset P* is the graph whose vertices are the elements of P and whose edges are those pairs $\{a, b\}$, $a, b \in P$, for which a covers b or b covers a.

Obviously, $C(P) = C(\bar{G})$, where $\bar{G}$ is the digraph corresponding to P.

Let a, b, c, d be distinct elements of P such that $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{a, d\}$ are edges of $C(P)$. Then $Q = (a, b, c, d)$ is said to be an *elementary quadruple in P*. The elementary quadruple isomorphic to the poset in Fig. 3a or Fig. 3b is denoted by Q_1 or Q_2, respectively. If Q is an elementary quadruple in P, then the poset Q is isomorphic either to Q_1 or to Q_2 (see, e. g., [5]).

A subset K of P is said to be *saturated* if, whenever $a, b \in K$ and a covers b in the poset K, then a covers b in the poset P.

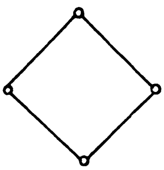

Fig. 3a.

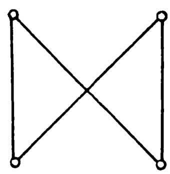

Fig. 3b.

If $\bar{G} = (P, \bar{E})$ is the digraph corresponding to P, then the saturated elementary quadruple Q_1 or Q_2 in P is represented by the square $\bar{S}_1$ or $\bar{S}_6$, respectively.

From the above mentioned facts and from Theorem 4 the following corollary proved by Jakubík in [6] follows easily.

Corollary 1. *Let $C(P) \overset{f}{\simeq} \prod G_i$, where P is a connected almost discrete poset (i. e. $C(P)$ is a connected graph) and $G_i = (P_i, E_i)$ are graphs. The decomposition*

$\prod G_i$ *of* $C(P)$ *induces a decomposition of* P *if and only if the following condition is fulfilled*:

(*D*) *If* Q *is a saturated elementary quadruple in* P *isomorphic to* Q_2, *then there exists* $k \in I$ *such that* $|f_j(Q)| = 1$ *for each* $j \in I \setminus \{k\}$.

Proof. Let $\bar{G}$ be a digraph corresponding to P. If (D) is fulfilled, then, by Theorem 4, $\bar{G} \overset{f}{\simeq} \prod \bar{G}_i$, where $\bar{G}_i = (P_i, \bar{E}_i)$. Clearly, digraphs $\bar{G}_i$ are acyclic and have no transitive edges. Let as define a partial ordering on each of the sets P_i as follows: b covers a if and only if $(a, b) \in \bar{E}_i$ (the ordering on P_i is determined by this covering relation). Then the digraphs $\bar{G}_i$ correspond to the posets P_i and $P \overset{f}{\simeq} \prod P_i$, $C(P_i) = C(\bar{G}_i) = G_i$. The necessity of the condition (D) is obvious.

If P is a semilattice, then every saturated elementary quadruple in P is isomorphic to Q_1. By Theorem 3, this implies the following corollary proved in [6], which is a generalization of a result from [1].

Corollary 2. *Let* P *be a semilattice. Then every decomposition* $\prod_{i \in I} G_i$, $I = \{1, 2, \ldots, n\}$ *of* $C(P)$ *induces a decomposition of* P.

REFERENCES

[1] GEDEONOVÁ, E.: The orientability of the direct product of graphs. Math. Slovaca 31, 1981, 71—78.

[2] HARARY, F.: Graph Theory, Addison-Wesley, Reading, 1969.

[3] JAKUBÍK, J.: Weak product decompositions of discrete lattices. Czechoslov. Math. J. 21, 1971, 399—412.

[4] JAKUBÍK, J.: Weak product decompositions of partially ordered sets. Colloquium math. 25, 1972, 177—190.

[5] JAKUBÍK, J.: On weak direct decompositions of lattices and graphs. Czechoslov. Math. J. 35, 1985, 269—277.

[6] JAKUBÍK, J.: Covering graphs and subdirect decompositions of partially ordered sets. Math. Slovaca 36, 1986, 151—162.

Received January 31, 1986

Katedra matematiky
Pedagogickej fakulty
Tajovského 40
975 49 Banská Bystrica

О РАЗЛОЖЕНИИ ОР-ГРАФОВ ПО ПРЯМЫМ ПРОИЗВЕДЕНИЯМ

Pavel Klenovčan

Резюме

Пусть $\bar{G} = (V, \bar{E})$ является орграфом. Граф $C(\bar{G}) = (V, E)$, у которого ребра суть те пары $\{a, b\}$, $a, b \in V$, что $(a, b) \in \bar{E}$ или $(b, a) \in \bar{E}$, называется покрывающим графом орграфа $\bar{G}$.

В статье автор рассматривает некоторые отношения между разложениями $\bar{G}$ и разложениями $C(\bar{G})$.