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NOTE ON A GENERALIZATION OF THE GENERALIZED VECTOR FIELD PROBLEM

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(*Communicated by Peter Zvengrowski*)

ABSTRACT. The main purpose of this note is to show the existence of a certain number of linearly independent cross-sections of certain multiples of the canonical non-trivial line bundle over any real Grassmann manifold whose dimension is at least 9 and congruent to one modulo 4.

The largest number of everywhere linearly independent cross-sections of a real vector bundle α is called the *span of* α (briefly $\operatorname{span}\alpha$).

Let $G_{n,k}$ denote the *Grassmann manifold* of all k-dimensional vector subspaces in $\mathbb{R}^n$, let $\tilde{G}_{n,k}$ denote the *oriented Grassmann manifold* of oriented k-dimensional vector subspaces in $\mathbb{R}^n$, and let $\zeta_{n,k}$ denote the line bundle associated with the obvious double covering $\tilde{G}_{n,k} \longrightarrow G_{n,k}$. Since $\tilde{G}_{n,k}$ is simply connected for $n \geqq 3$, we have then $H^1(G_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2$ for the first $\mathbb{Z}_2$-cohomology group. We will suppose that $k \leqq \frac{n}{2}$; this is justified by the canonical diffeomorphism $G_{n,k} \approx G_{n,n-k}$.

Write $m\zeta_{n,k}$ for the m-fold Whitney sum $\zeta_{n,k} \oplus \cdots \oplus \zeta_{n,k}$. Then a problem studied in [9; § 4] for $k \geqq 2$ can be stated as follows:

PROBLEM. *Find* $\operatorname{span} m\zeta_{n,k}$ *for all admissible* m, n, k.

Note that for $k = 1$ this coincides with the generalized vector field problem over the $(n-1)$-dimensional real projective space $\mathbb{R}P^{n-1}$; see e.g. [1], [2], [5], [6], [7], [10].

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Now let us consider the problem for $k \geqq 2$, and in particular in the lowest stable case, hence for $m = d+1$, where $d := \dim G_{n,k} = k(n-k)$. Clearly $\operatorname{span}(d+1)\zeta_{n,k}$ is always positive.

By R. Stong [14], one has the following result for the height of the first *Stiefel-Whitney characteristic class* $w_1(\zeta_{n,k}) \in H^1(G_{n,k};\mathbb{Z}_2)$, defined as

$$\text{height}\,\big(w_1(\zeta_{n,k})\big) := \max\{t;\ w_1^t(\zeta_{n,k}) \neq 0\}\,.$$

PROPOSITION 1. (R. Stong [14]) *Let* $2 \leqq k \leqq \frac{n}{2}$ *and* $2^s < n \leqq 2^{s+1}$. *Then*

$$\text{height}\,\big(w_1(\zeta_{n,k})\big) = \begin{cases} 2^{s+1}-2 & \text{if } k=2, \text{ or if } k=3 \text{ and } n = 2^s+1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

Applying this, we obtain that the Stiefel-Whitney class $w_{2^{s+1}-2}\big((2^{s+1}-1)\zeta_{2^s+1,2}\big)$ does not vanish, and therefore $\operatorname{span}(d+1)\zeta_{n,k} = 1$ for $n = 2^s+1$, $k=2$. On the other hand, by [9] we have

$$\operatorname{span} m\zeta_{n,k} \geqq \operatorname{span} m\zeta_{d,1} \qquad (*)$$

for any positive integer m if $(n,k) \neq (2^s+1,2)$. In addition to this, of course always $\operatorname{span}(m+1)\zeta_{m,1} \geqq 2$, and therefore $\operatorname{span}(d+1)\zeta_{n,k}$ is always at least 2. If $d \equiv 1 \pmod 4$, we cannot obtain a better estimate using $(*)$, because the Stiefel-Whitney class $w_{m-1}\big((m+1)\zeta_{m,1}\big)$ in the algebra $H^*(G_{m,1};\mathbb{Z}_2) = \mathbb{Z}_2\big[w_1(\zeta_{m,1})\big]/\big(w_1^m(\zeta_{m,1})\big)$ does not vanish when $m \equiv 1 \pmod 4$, and consequently then $\operatorname{span}(m+1)\zeta_{m,1}$ is precisely 2. However the following proposition shows that an improvement – at least by one – is still possible.

PROPOSITION 2. *Let* $3 \leqq k \leqq \frac{n}{2}$. *If* $\dim G_{n,k} = d \equiv 1 \pmod 4$, *then*

$$\operatorname{span}(d+t)\zeta_{n,k} \geqq 2+t \qquad \textit{for all} \quad t \geqq 1\,.$$

Proof. We apply the classical Steenrod obstruction theory (see e.g. Milnor, Stasheff [11; § 12]). Since the *Stiefel manifold* $V_{d+t,2+t}$ of orthonormal $(2+t)$-frames in $\mathbb{R}^{d+t}$ is $(d-3)$-connected, the vector bundle $(d+t)\zeta_{n,k}$ has $2+t$ linearly independent cross-sections over the $(d-2)$-skeleton of the manifold $G_{n,k}$. Then the primary obstruction to the existence of $2+t$ linearly independent cross-sections of the same vector bundle over the $(d-1)$-skeleton of $G_{n,k}$ is nothing but the Stiefel-Whitney class $w_{d-1}\big((d+t)\zeta_{n,k}\big) \in H^{d-1}(G_{n,k};\mathbb{Z}_2)$.

Now Stong's Proposition 1 implies that this obstruction vanishes. Indeed, if we have $k \geqq 5$, then $d-1 > 2^{s+1}-1 = \text{height}\,(w_1(\zeta_{n,k}))$, and for $k=3$ and $2^s < n \leqq 2^{s+1}$ with $s \geqq 4$ one also readily sees that $d-1 > 2^{s+1}-1$. The latter inequality can be directly checked for the remaining cases $G_{6,3}$, $G_{10,3}$ and $G_{14,3}$.

Hence there is a finite "singularity" set $S \subset G_{n,k}$ such that the reduction of the vector bundle $(d+t)\zeta_{n,k}$ to $G_{n,k} \setminus S$ has $2+t$ linearly independent cross-sections.

But by Paechter [12] the *homotopy groups* $\pi_{d-1}(V_{d+t,2+t})$ vanish if $d \equiv 1 \pmod 4$ and $t \geqq 1$. Therefore one can remove the singularity set S, and the proposition is proved. □

Hurwitz [8] and Radon [13] determined the (largest) number of orthogonal transformations $A_i \colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ such that

$$A_i^2 = -\,\mathrm{Id}, \quad \text{and} \quad A_iA_j + A_jA_i = 0 \quad \text{for} \quad i \neq j .$$

In particular, if $m \equiv 0 \pmod 4$ then there are at least three such transformations. They induce the same number of linearly independent "linear" cross-sections of the tangent bundle of the $(m-1)$-dimensional sphere S^{m-1}. Consequently, if $d \equiv 3 \pmod 4$, then we have $\operatorname{span} TG_{d+1,1} \geqq 3$.

Let ε^1 denote the trivial line bundle. Since the Whitney sum $TG_{d+1,1} \oplus \varepsilon^1$ is isomorphic to $(d+1)\zeta_{d+1,1}$ (see [11; 4.5]), and since obviously $\operatorname{span}(d+1)\zeta_{d,1} \geqq \operatorname{span}(d+1)\zeta_{d+1,1}$, one has $\operatorname{span}(d+1)\zeta_{d,1} \geqq 4$ if $d \equiv 3 \pmod 4$. This together with Proposition 2 and considerations analogous to those of [9; § 3] implies the following:

COROLLARY 3. *Let n be even and k odd, $3 \leqq k \leqq \frac{n}{2}$. If $3 \leqq q \leqq r+2$, then there exists a map $f : G_{n,k} \longrightarrow G_{d+r,q}$ such that the pull-back bundle $f^*(\zeta_{d+r,q})$ is $\zeta_{n,k}$.*

Remarks.

(1) In the situation of the corollary, one has the inequality $\operatorname{span} m\zeta_{n,k} \geqq \operatorname{span} m\zeta_{d+r,q}$ for any m, hence another result directly relevant to our Problem.

(2) Proposition 2 in the case when $t=1$ can also be proved using [4; 4.8] combined with Stong's Proposition 1 and with formulae for the first two Stiefel-Whitney classes of Grassmannians from [3].

(3) Note that for $G_{6,3}$ and $t \equiv 6 \pmod 8$, Proposition 2 gives exactly $\operatorname{span}(9+t)\zeta_{6,3}$ (see [9; § 4]).

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