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VARIATIONAL METRICS ON $\mathbb{R} \times TM$ AND THE GEOMETRY OF NONCONSERVATIVE MECHANICS

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ABSTRACT. We introduce a variational metric on $\mathbb{R} \times TM$ which is a generalization of Riemannian and Finslerian metrics and is suitable for a geometric description of time-dependent mechanical systems. We show that a manifold endowed with a variational metric carries a canonical metric semispray connection. Connections associated with a variational metric are shown to be a global counterpart of the nonconservative Euler-Lagrange equations, and they can be viewed as a generalization of the Levi-Civita connection for a Riemannian structure, of the Cartan connection for a Finslerian structure, and of the Grifone connection for a generalized Finslerian structure. We also investigate metrizability and variationality of general semispray connections on $\mathbb{R} \times TM$, and obtain a generalization of Krupka-Sattarov's theorem on variationality of a Finslerian structure.

1. Introduction

The aim of this paper is to propose a generalization of the concept of Finslerian manifold, suitable for a geometric description of time-dependent nonconservative mechanical systems.

The dynamics of a regular time-dependent mechanical system on a manifold M is described by a semispray (a "second order vector field") on the fibered manifold $\mathbb{R} \times M \to \mathbb{R}$, or equivalently, by a *semispray connection* which is an Ehresmann connection on $\mathbb{R} \times TM$ (i.e. a section $\mathbb{R} \times TM \to \mathbb{R} \times T^2M$, where T^2M denotes the tangent bundle of order 2 of M). Locally it is represented by a regular system of second order differential equations for sections of the fibered manifold $\mathbb{R} \times M \to \mathbb{R}$. In case that the manifold M is endowed

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with a Riemannian metric g, it carries a canonical semispray connection Γ such that the geodesics of Γ coincide with the graphs of geodesics of the Levi-Civita connection ∇ of g; a similar situation occurs in the case of a Finslerian manifold. Moreover, we know that both in Riemannian and Finslerian geometries the equations for geodesics are variational (i.e. they are the Euler-Lagrange equations of a lagrangian called the "kinetic energy" of Riemannian and Finslerian structures, respectively). Hence, the geodesics in Riemannian and Finslerian geometries can be viewed as geodesics (paths) of semispray connections describing the dynamics of a Riemannian and Finslerian free particle, respectively. These two important particular cases of mechanical systems suggest us an idea to investigate the structure of semispray connections on $\mathbb{R} \times TM$, and to search for all (semispray) connections describing the dynamics of "free particles". Naturally, we will require these connections be *variational.*

In classical Finslerian geometry, a Finslerian manifold is a manifold M endowed with a Finslerian metric g on TM which is a regular symmetric fibered morphism $g\colon TM \to T_2^0 M$ over id_M (where $T_2^0 M$ denotes the bundle of all tensors of type $(0,2)$ over M), satisfying the following two conditions:

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j} \quad (\text{"integrability"}), \qquad \frac{\partial g_{ij}}{\partial \dot{x}^k}\, \dot{x}^k = 0 \quad (\text{"homogeneity"}). \tag{1.1}$$

Omitting the "integrability" condition one obtains a class of metrics which is studied in a generalized Finslerian geometry (cf. e.g. [13] and the references therein).

In this paper, we consider regular symmetric fibered morphisms $g\colon \mathbb{R}\times TM \to T_2^0 M$ over id_M (time-dependent metrics on TM) which satisfy the "integrability" condition, but not necessarily the "homogeneity" condition; we call these metrics *variational metrics* on $\mathbb{R} \times TM$. A manifold M endowed with a variational metric is then called a *semi-finslerian manifold.* Using the results of [7] we show in Sec. 4 that every semi-finslerian manifold (M, g) carries a *canonical* (semispray) *connection* Γ_g. The property of variationality of the canonical connection enables us to introduce naturally the concept of a *kinetic energy* λ_g associated with the variational metric g. Since any semispray connection on a semi-finslerian manifold (M, g) is uniquely determined by the fundamental connection and a soldering form, the equations for geodesics of a connection on a semi-finslerian manifold take the form of the Euler-Lagrange equations for a general nonconservative mechanical system,

$$\frac{\partial T}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}^i} = F_i\,, \qquad 1 \le i \le \dim M\,. \tag{1.2}$$

In comparison to [2], where certain connections determined by a lagrangian defined on $\mathbb{R} \times TM$ are constructed, we are interested in semispray connections, since they naturally arise as a geometric counterpart of the "equations of motion".

In Sec. 5, we study the structure of semispray connections on $\mathbb{R}\times TM$. We propose a concept of *metrizability* of such connections, and we get a classification of metrizable semispray connections. We study conditions of *variationality* [6] of a semispray connection (the so called "inverse variational problem" for connections), and investigate the relation between variationality and metrizability. We obtain a generalization (to metrizable semispray connections) of a theorem by Krupka and Sattarov [5].

In Sec. 6, we show on a few easy examples from geometry and physics that our concepts of semi-finslerian manifold and mechanical system are a general background for mechanical systems connected with Riemannian and Finslerian geometries and/or described by Grifone's connections [1]. Finally, we show that applying our theorem on variationality of semispray connections to *linear* connections on M and on TM one gets the results known in Riemannian and Finslerian geometries (cf. [5]).

We use some results on Ehresmann's connections, semispray connections (see e.g. [6], [10], [11], [12], [14], [15]), and the calculus of variations on fibered manifolds ([3], [4], [9] and references therein); notations and the main concepts are briefly explained in Sec. 2 and Sec. 3.

The present paper is an enlarged version of the Preprint [8].

2. Semispray connections and regular second order equations

Throughout the paper, all manifolds and mappings are supposed to be smooth, and the summation convention is used. We denote by * the pull-back, T the tangent functor, and ∂ the Lie derivative. $\mathcal{F}$ denotes the ring of smooth functions on $\mathbb{R}\times TM$.

We shall consider a fibered manifold $\pi\colon \mathbb{R}\times M \to \mathbb{R}$, where M is an m-dimensional manifold, and π is the first canonical projection. The first (resp. second) jet prolongation of π will be denoted by $\pi_1\colon J^1(\mathbb{R}\times M)\to\mathbb{R}$ (resp. $\pi_2\colon J^2(\mathbb{R}\times M)\to\mathbb{R}$). Note that $J^1(\mathbb{R}\times M)$ (resp. $J^2(\mathbb{R}\times M)$) is canonically identified with $\mathbb{R}\times TM$ (resp. $\mathbb{R}\times T^2M$, where $T^2M\subset T(TM)$ is the tangent bundle of M of order 2).

The global coordinate on $\mathbb{R}$ will be denoted by t. If (x^i), $1\le i\le m$, are coordinates on an open subset of M, we obtain a fiber chart (V,ψ), $\psi=(t,x^i)$ on $\mathbb{R}\times M$. The associated fiber chart on $\mathbb{R}\times TM$ (resp. on $\mathbb{R}\times T^2M$) will be denoted by (V_1,ψ_1), $\psi_1=(t,x^i,\dot{x}^i)$ (resp. (V_2,ψ_2), $\psi_2=(t,x^i,\dot{x}^i,\ddot{x}^i)$). Obviously, for any two fiber charts (V,ψ), $\psi=(t,x^i)$ and $(\bar{V},\bar{\psi})$, $\bar{\psi}=(t,\bar{x}^i)$

such that $V \cap \bar{V} \neq 0$, the overlap mapping is defined by

$$\bar{x}^i = \bar{x}^i(x^k), \quad \dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^k}\dot{x}^k, \quad \ddot{\bar{x}}^i = \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}\dot{x}^j \dot{x}^k + \frac{\partial \bar{x}^i}{\partial x^k}\ddot{x}^k, \qquad 1 \leq i \leq m. \tag{2.1}$$

If $\gamma\colon \mathbb{R} \to \mathbb{R} \times M$ is a section, then the first (resp. second) jet prolongation of γ is denoted by $J^1\gamma$ (resp. $J^2\gamma$); $J^1\gamma$ (resp. $J^2\gamma$) is a section of the fibered manifold $\pi_1\colon J^1(\mathbb{R} \times M) \to \mathbb{R}$ (resp. $\pi_2\colon J^2(\mathbb{R} \times M) \to \mathbb{R}$). Since $\gamma(t) = (t, c(t))$, where $c(t)$ is a curve in M defined on an open subset of $\mathbb{R}$, we have $J^1\gamma(t) = (t, c(t), \mathrm{d}c/\mathrm{d}t)$ and $J^2\gamma = (t, c(t), \mathrm{d}c/\mathrm{d}t, \mathrm{d}^2c/\mathrm{d}t^2)$.

Recall that a (*Ehresmann*) *connection* on $\mathbb{R} \times M$ is a section of the fibered manifold $\mathbb{R} \times T^2M \to \mathbb{R} \times M$. In fibered coordinates, a connection Γ is represented by means of its *components* Γ^i, $1 \leq i \leq \dim M$, defined by $(t, x^i) \circ \Gamma = (t, \Gamma^i)$. A connection can be identified with the so called *horizontal form* h_Γ, or with the *vertical form* v_Γ, or with a horizontal distribution H_Γ on $\mathbb{R} \times M$, which, in fibered coordinates, are expressed as follows:

$$h_\Gamma = \Big(\frac{\partial}{\partial t} + \Gamma^i \frac{\partial}{\partial x^i}\Big) \otimes \mathrm{d}t, \qquad v_\Gamma = \frac{\partial}{\partial x^i} \otimes (\mathrm{d}x^i - \Gamma^i\,\mathrm{d}t),$$
$$H_\Gamma = \operatorname{span}\Big\{\frac{\partial}{\partial t} + \Gamma^i \frac{\partial}{\partial x^i}\Big\}.$$

A (local) section γ of $\mathbb{R} \times M \to \mathbb{R}$ is called a *geodesic* (a *path*, or an *integral section*) of a connection Γ if $\Gamma \circ \gamma = J^1\gamma$; this equation, when expressed in fibered coordinates, gives a system of m first order ordinary differential equations for γ.

The *dynamics* of a time-dependent mechanical system on M is described by a *semispray connection* on $\mathbb{R} \times TM$, which is a section $\Gamma\colon \mathbb{R} \times TM \to \mathbb{R} \times T^2M$ (hence, it is a kind of Ehresmann's connection on $\mathbb{R} \times TM$). In a fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$, Γ is expressed by

$$(t, x^i, \dot{x}^i, \ddot{x}^i) \circ \Gamma = (t, x^i, \dot{x}^i, \Gamma^i), \tag{2.2}$$

where Γ^i are functions on V_1, called the *components* of Γ. Obviously, Γ^i, $1 \leq i \leq m$, transform like the coordinates $\ddot{x}^i$ under transformations of fibered coordinates (cf. (2.1)). A semispray connection Γ on $\mathbb{R} \times TM$ is identified with the *horizontal form* h_Γ of Γ, or the *vertical form* v_Γ of Γ,

$$h_\Gamma = \Big(\frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial \dot{x}^i}\Big) \otimes \mathrm{d}t,$$
$$v_\Gamma = \frac{\partial}{\partial x^i} \otimes (\mathrm{d}x^i - \dot{x}^i\,\mathrm{d}t) + \frac{\partial}{\partial \dot{x}^i} \otimes (\mathrm{d}\dot{x}^i - \Gamma^i\,\mathrm{d}t),$$

or the π_1-horizontal distribution $H_\Gamma = \operatorname{Im} h_\Gamma \subset T(\mathbb{R}\times TM)$ spanned by the vector field

$$\zeta = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial \dot{x}^i},$$

called a *semispray.*

A (local) section γ of π is called a *path* or an *integral section* or a *geodesic* of a semispray connection Γ if

$$\Gamma \circ J^1\gamma = J^2\gamma . \tag{2.3}$$

If $\gamma(t) = \big(t, c(t)\big)$, where c is a curve defined on an open subset of $\mathbb{R}$, we obtain that γ is a geodesic of Γ if and only if

$$\frac{\mathrm{d}^2 c^i}{\mathrm{d}t^2} = \Gamma^i\Big(t, c(t), \frac{\mathrm{d}c}{\mathrm{d}t}\Big), \qquad 1 \le i \le m, \tag{2.4}$$

in each fiber chart (V,ψ), $\psi = (t, x^i)$ on $\mathbb{R}\times M$. It is clear that integral sections of a connection Γ and of its horizontal distribution H_Γ coincide.

Every $\pi_{1,0}$-vertical valued π_1-horizontal one-form on $\mathbb{R}\times TM$ is called a *soldering form.* Soldering forms on $\mathbb{R}\times TM$ can be roughly characterized as "differences of semispray connections". More precisely, if Γ, Γ' are two semispray connections on $\mathbb{R}\times TM$, then the vector valued one-form s defined by

$$s = h_\Gamma - h_{\Gamma'} \tag{2.5}$$

is a soldering form; conversely, if s is a soldering form on $\mathbb{R}\times TM$, then there exist semispray connections Γ, Γ' such that $s = h_\Gamma - h'_\Gamma$. We shall denote by $\mathcal{S}(\mathbb{R}\times TM)$ the $\mathcal{F}$-module of all soldering forms on $\mathbb{R}\times TM$.

For more details on connections and semispray connections on fibered manifolds we refer e.g. to [10], [11], [12], [14] and [15].

A semispray connection describes the motion of a mechanical system but does not represent the mechanical system itself. It is easy to find different mechanical systems represented by the same semispray connection (i.e. possessing the same "trajectories"): this situation occurs if the corresponding equations of motion differ from the equations for geodesics by a so called "regular integrating factor", i.e. if they are of the form

$$\big[g^1_{ji}(\ddot{x}^i - \Gamma^i)\big] \circ J^2\gamma = 0 \quad \text{ and } \quad \big[g^2_{ji}(\ddot{x}^i - \Gamma^i)\big] \circ J^2\gamma = 0,$$

where (g^1_{ij}) and (g^2_{ij}) are regular matrices (at each point of $\mathbb{R}\times TM$). As an example, let us consider a damped harmonic oscillator of mass m, frequency

ω and damping constant k, and a harmonic oscillator of frequency ω whose mass-accretion is ruled by me^{kt}. The corresponding equations of motion are $m\ddot{x} + mk\dot{x} + m\omega^2 x = 0$ and $me^{kt}(\ddot{x} + k\dot{x} + \omega^2 x) = 0$, which means that the motion of both these physical systems is described by the same semispray connection $\ddot{x} \circ \Gamma = -k\dot{x} - \omega^2 x$. Examples from classical mechanics show that the "integrating factor" carries an important physical information, since it is related to the "kinetic energy" of the system. Therefore, to avoid confusion, it is better to work with the equations of motion in their "covariant form". Within the range of the theory of second (resp. first) order ordinary differential equations on a fibered manifold $\mathbb{R} \times M \to \mathbb{R}$ this means that we have to consider the *$\mathcal{F}$-module of one-contact 2-forms on $\mathbb{R} \times T^2M$ (resp. on $\mathbb{R} \times TM$), which are horizontal with respect to the projection* $\pi_{2,0} \colon \mathbb{R} \times T^2M \to \mathbb{R} \times M$ (see e.g [3], [4], [6]); this module is denoted by $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times T^2M)$ (resp. $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$). For our purpose it is sufficient to recall that this module consists of 2-forms, which in each fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$ are expressed in the form

$$E = E_i \,\mathrm{d}x^i \wedge \mathrm{d}t\,, \tag{2.6}$$

where E_i are functions on V_2 (resp. on V_1), i.e. $E_i = E_i(t, x^k, \dot{x}^k, \ddot{x}^k)$ (resp. $E_i = E_i(t, x^k, \dot{x}^k)$), $1 \le i \le \dim M$. A (local) *section* γ of $\pi \colon \mathbb{R} \times M \to \mathbb{R}$ is called a *solution* of such a form E on $\mathbb{R} \times T^2M$ (resp. on $\mathbb{R} \times TM$) if $E \circ J^2\gamma = 0$ (resp. $E \circ J^1\gamma = 0$). Clearly, a section $\gamma\big(t, c(t)\big)$ of π is a solution of $E \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times T^2M)$ (resp. of $E \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$) if and only if it satisfies the system of $m = \dim M$ second (resp. first) order ordinary differential equations

$$E_i\big(t, c(t), \mathrm{d}c/\mathrm{d}t, \mathrm{d}^2c/\mathrm{d}t^2\big) = 0\,, \qquad \text{resp.} \quad E_i\big(t, c(t), \mathrm{d}c/\mathrm{d}t\big) = 0\,. \tag{2.7}$$

In this paper, we shall consider a submodule $\Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ (resp. $\Omega^{\mathrm{lin}}(\mathbb{R} \times TM)$) of the module $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times T^2M)$ (resp. $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$), which is defined as a module of 2-forms on $\mathbb{R}\times T^2M$ (resp. $\mathbb{R}\times TM$) satisfying in each fiber chart the condition

$$E = E_i \,\mathrm{d}x^i \wedge \mathrm{d}t\,, \qquad E_i = A_i + B_{ik}\ddot{x}^k\,, \tag{2.8}$$

resp.

$$E = E_i \,\mathrm{d}x^i \wedge \mathrm{d}t\,, \qquad E_i = A_i + B_{ik}\dot{x}^k\,. \tag{2.9}$$

where A_i, B_{ik}, $1 \le i, k \le m$ are functions of t, x^j, $\dot{x}^j$ (resp. of t, x^j), $1 \le j \le m$.

A form $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ (resp. $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times TM)$) is called *regular* if

$$\det(B_{ik}) \ne 0\,. \tag{2.10}$$

It is easy to see (cf. [6], [14], [15]) the following:

PROPOSITION 1. *Let $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ (resp. $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times TM)$) be a regular form. Then there exists a unique semispray connection Γ on $\mathbb{R} \times TM$ (resp. an Ehresmann connection Γ on $\mathbb{R} \times M$) such that the geodesics of Γ coincide with the solutions of E. The connection Γ is obtained as the solution of the equation $\Gamma^* E = 0$.*

The connection Γ satisfying the equation $\Gamma^* E = 0$ is called *associated* to E.

Regular forms E_1, $E_2 \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ (resp. in $\Omega^{\mathrm{lin}}(\mathbb{R} \times TM)$) are called *equivalent* if the semispray connections (resp. Ehresmann's connections) associated to E_1 and E_2 coincide. This means that equivalent forms have the same solutions. Hence, we can say that a semispray connection on $\mathbb{R} \times TM$ (resp. a connection on $\mathbb{R} \times M$) represents an equivalence class of regular forms in $\Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ (resp. in $\Omega^{\mathrm{lin}}(\mathbb{R} \times TM)$).

3. Locally variational forms and variational connections

We shall need a few concepts from the calculus of variations on fibered manifolds. Our exposition is adapted to the case of second (and first) order ordinary differential equations on a fibered manifold $\mathbb{R} \times M \to \mathbb{R}$; for more complete information we refer e.g. to [3], [4], [6], [9] and references therein.

Recall that a first order *lagrangian* on a fibered manifold $\pi\colon \mathbb{R} \times M \to \mathbb{R}$ is defined as a π_1-horizontal one-form λ on $\mathbb{R} \times TM$; in fibered coordinates it is expressed by $\lambda = L\,\mathrm{d}t$, where L is a function of t, x^i and $\dot{x}^i$. If λ is a first order lagrangian, we denote by E_λ the *Euler-Lagrange* form of λ; we have $E_\lambda = E_i\,\mathrm{d}x^i \wedge \mathrm{d}t$, where

$$E_i = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i}, \qquad 1 \le i \le m,$$

are called the *Euler-Lagrange expressions* of the lagrangian λ. It is easy to see that $E_\lambda \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$.

Let $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2M)$ be a form. E is called *globally variational* if there exists a lagrangian λ defined on $\mathbb{R} \times TM$ such that $E = E_\lambda$. E is called *locally variational* if $\mathbb{R} \times TM$ can be covered by open sets such that the restriction of E to each of these sets is variational. Recall that E is locally variational if and only if in each fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$ the functions E_i, $1 \le i \le m$, satisfy the *Helmholtz conditions*

$$\begin{gathered}\frac{\partial E_i}{\partial \ddot{x}^k} - \frac{\partial E_k}{\partial \ddot{x}^i} = 0, \qquad \frac{\partial E_i}{\partial \dot{x}^k} + \frac{\partial E_k}{\partial \dot{x}^i} - 2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_k}{\partial \ddot{x}^i} = 0, \\ \frac{\partial E_i}{\partial x^k} - \frac{\partial E_k}{\partial x^i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_k}{\partial \dot{x}^i} - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial E_k}{\partial \ddot{x}^i} = 0.\end{gathered} \tag{3.1}$$

If E is projectable onto $\mathbb{R} \times M$, then the Helmholtz conditions are obviously reduced to

$$\frac{\partial E_i}{\partial \dot{x}^k} + \frac{\partial E_k}{\partial \dot{x}^i} = 0\,, \qquad \frac{\partial E_i}{\partial x^k} - \frac{\partial E_k}{\partial x^i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_k}{\partial \dot{x}^i} = 0\,. \tag{3.2}$$

We note that the existence of local lagrangians in general does not imply the existence of a global lagrangian.

Solutions of a locally variational form are called *extremals*, and the correspoding equations for extremals are called the *Euler-Lagrange equations*.

A semispray connection Γ on $\mathbb{R} \times TM$ (resp. an Ehresmann connection on $\mathbb{R} \times M$) is called *variational* [6] if there exists a locally variational form E such that

$$\Gamma^* E = 0\,. \tag{3.3}$$

4. Variational metrics, semi-finslerian manifolds

Denote by $T_2^0 M$ the bundle of all tensors of type $(0,2)$ over M. Let $g\colon \mathbb{R} \times TM \to T_2^0 M$ be a fibered morphism over id_M; g will be called a *metric on* $\mathbb{R} \times TM$ if it is regular and symmetric (i.e. if in every fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$ the matrix (g_{ij}), built from the components of g, is regular and symmetric).

We shall say that a metric g on $\mathbb{R} \times TM$ is *variational* if there exists a regular locally variational form E on $\mathbb{R} \times T^2 M$ such that

$$g_{ij} = -\frac{\partial E_i}{\partial \ddot{x}^j}\,, \qquad 1 \le i, j \le m \tag{4.1}$$

in each fiber chart on $\mathbb{R} \times M$. Every (local) lagrangian λ such that the form E is the Euler-Lagrange form of λ will be called a *dynamical lagrangian* for the metric g. Every 2-form $E \in \Omega^{\mathrm{lin}}(\mathbb{R} \times T^2 M)$ satisfying (4.1) will be called a *dynamical 2-form* associated with the variational metric g.

PROPOSITION 2. *A metric g on $\mathbb{R} \times TM$ is variational if and only if the components of g satisfy in each fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$ the conditions*

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}\,, \qquad 1 \le i, j, k \le m\,. \tag{4.2}$$

Proof. Let g be variational. Then the relations (4.2) follow from the Helmholtz conditions (3.1).

We shall prove the converse. Consider an open ball $W \subset \mathbb{R}^m$ with the center at the origin, and denote by (x^i) the canonical coordinates on W. Let g be a metric on $\mathbb{R} \times TW$ satisfying (4.2). Define a mapping $\bar{\chi}\colon [0,1] \times (\mathbb{R} \times TW) \to \mathbb{R} \times TW$ setting

$$\bar{\chi}(v,(t,x^i,\dot{x}^i)) = (t,x^i,v\dot{x}^i)\,, \tag{4.3}$$

and put

$$T = \dot{x}^i\dot{x}^j \int_0^1 \left(\int_0^1 (g_{ij} \circ \bar{\chi})\,\mathrm{d}v \right) \circ \bar{\chi} v\,\mathrm{d}v\,. \tag{4.4}$$

Then $T\,\mathrm{d}t$ is a lagrangian on the fibered manifold $\pi\colon \mathbb{R} \times W \to \mathbb{R}$ satisfying

$$g_{ij} = \frac{\partial^2 T}{\partial \dot{x}^i \partial \dot{x}^j} = -\frac{\partial E_i(T)}{\partial \ddot{x}^j}\,,$$

where $E_i(T)$ are the Euler-Lagrange expressions of $T\,\mathrm{d}t$.

Now, let $\pi\colon \mathbb{R} \times M \to \mathbb{R}$ be a fibered manifold, g a metric on $\mathbb{R} \times TM$, satisfying the conditions (4.2). Then there exists an open covering $\mathcal{O}$ of $\mathbb{R}\times TM$ such that on every open set of $\mathcal{O}$ the lagrangian $T\,\mathrm{d}t$ (4.4) is defined. From the transformation properties of the components g_{ij}, $1 \le i,j \le m$, of g and of the coordinates $\dot{x}^i$, $1 \le i \le m$, it is easy to see that the local lagrangians $T\,\mathrm{d}t$ define a (global) lagrangian λ_g on $\mathbb{R} \times TM$ such that for each $U \in \mathcal{O}$, $\lambda_{g|_U} = T\,\mathrm{d}t$. For the Euler-Lagrange form E_g of the lagrangian λ_g we have (4.1), i.e. the metric g is variational.

This completes the proof.

If g is a variational metric on $\mathbb{R}\times TM$, then the (global) dynamical lagrangian λ_g of g defined in the proof of Proposition 2 will be called *kinetic energy* of the metric g. The Euler-Lagrange form E_g of the kinetic energy λ_g will be called a *canonical dynamical 2-form* of the metric g.

By Proposition 1, there exists a *unique* semispray connection $\Gamma_g\colon \mathbb{R}\times TM \to \mathbb{R}\times T^2M$ such that the geodesics of Γ_g coincide with the extremals of the kinetic energy λ_g, i.e. they coincide with the solutions of the Euler-Lagrange equations

$$\frac{\partial T}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}^i} = 0\,.$$

This connection is defined by the relation

$$\Gamma_g^* E_g = 0\,, \tag{4.5}$$

and will be called a *canonical connection* associated with the metric g. Expressing the relation (4.5) in a fiber chart (t,x^i) on $\mathbb{R} \times M$ one obtains for the components $\Gamma^i(g)$, $1 \le i \le m$ of Γ_g the following formulas:

$$\ddot{x}^i \circ \Gamma_g = \Gamma^i(g) = g^{ip}\Gamma_p(g)\,, \tag{4.6}$$

where (g^{ip}) is the inverse matrix to (g_{ip}), and the functions $\Gamma_p(g)$, $1 \le p \le m$, are given by

$$-\Gamma_p(g) = \Gamma_{pqr}(g)\dot{x}^q\dot{x}^r + \dot{x}^q \int_0^1 \left(\frac{\partial g_{pq}}{\partial t} \circ \bar{\chi} \right) \mathrm{d}v \,, \tag{4.7}$$

where

$$\Gamma_{pqr}(g) = \frac{1}{2} \int_0^1 \left(\frac{\partial g_{pq}}{\partial x^r} + \frac{\partial g_{pr}}{\partial x^q} - 2\frac{\partial g_{qr}}{\partial x^p} \right) \circ \bar{\chi} \,\mathrm{d}v + \int_0^1 \left(\frac{\partial g_{qr}}{\partial x^p} \circ \bar{\chi} \right) v \,\mathrm{d}v \,. \tag{4.8}$$

A manifold M endowed with a variational metric g will be called a *semi-finslerian manifold.* According to Propositions 2 and 1, on every semi-finslerian manifold (M, g) there exists a unique canonical dynamical 2-form E_g and a unique canonical semispray connection Γ_g.

Let (M, g) be a semi-finslerian manifold, E_1, E_2 dynamical 2-forms on $\mathbb{R} \times T^2M$ associated with g. Then obviously $E_1 - E_2 \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$. Conversely, if E_1 is a dynamical 2-form associated to g, and F is an element of $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$, then $E_2 = E_1 + F$ is another dynamical 2-form of g. This leads us to the following definition: A triple (M, g, F) will be called a *mechanical system in the force field* F if (M, g) is a semi-finslerian manifold and $F \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM)$; we shall also say that F is a *force* on a semi-finslerian manifold (M, g). A mechanical system (M, g, F) is characterized by the dynamical 2-form $E = E_g + F$. Hence, its motion is described by sections γ of $\mathbb{R} \times M \to \mathbb{R}$ which are solutions to the "Euler-Lagrange equations for a nonconservative mechanical system"

$$\left[\frac{\partial T}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}^i} - F_i \right] \circ J^2\gamma = 0 \,,$$

where T and F_i are the components of the kinetic energy λ_g and the force F, respectively. A mechanical system $(M, g, 0)$ will be also called a *free particle*, and will be identified with the semi-finslerian manifold (M, g).

Let (M, g) be a semi-finslerian manifold. Then there arises a canonical isomorphism

$$\tilde{g}\colon \mathcal{S}(\mathbb{R} \times TM) \ni s \to \tilde{g}(s) = E \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times TM) \tag{4.9}$$

of $\mathcal{F}$-modules. It is defined in each fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$, where

$$s = s^i \frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}t \,, \qquad E = E_i \,\mathrm{d}x^i \wedge \mathrm{d}t \,,$$

by the formula

$$E_i = g_{ij} s^j .$$

By this isomorphism, on a semi-finslerian manifold, forces can be identified with soldering forms. This means, however, that a mechanical system (M, g, F) can be equivalently represented by the semispray connection Γ such that $h_\Gamma = h_{\Gamma_g} + s$, where $s = \tilde{g}^{-1}(F)$.

Let Γ be a semispray connection on a fibered manifold $\mathbb{R} \times M \to \mathbb{R}$. Note that if it is chosen a variational metric g on $\mathbb{R} \times TM$, then Γ represents a unique mechanical system (M, g, F): it holds $F = \tilde{g}(s)$, where $s = h_\Gamma - h_{\Gamma_g}$.

5. Metrizable semispray connections

In this section, we shall define the concept of a *metrizable semispray connection*, and we shall study the conditions of metrizability. We shall be interested in the relation between variationality and metrizability of a semispray connection, and we shall obtain a classification of metrizable and variational connections.

Let us denote by $\mathcal{M}_2^0(\mathbb{R} \times TM)$ the set of all fibered morphisms $\mathbb{R} \times TM \to T_2^0 M$ over id_M.

PROPOSITION 3. *Let $\Gamma\colon \mathbb{R} \times TM \to \mathbb{R} \times T^2 M$ be a semispray connection. The formula*

$$(\mathcal{D}_\Gamma g)_{ij} = \frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{ij}}{\partial x^k} \dot{x}^k + \frac{\partial g_{ij}}{\partial \dot{x}^k} \Gamma^k + \frac{1}{2}\left(g_{ik} \frac{\partial \Gamma^k}{\partial \dot{x}^j} + g_{jk} \frac{\partial \Gamma^k}{\partial \dot{x}^i} \right), \qquad 1 \le i, j \le m, \tag{5.1}$$

defines a mapping $\mathcal{D}_\Gamma\colon \mathcal{M}_2^0(\mathbb{R} \times TM) \ni g \to \mathcal{D}_\Gamma g \in \mathcal{M}_2^0(\mathbb{R} \times TM)$.

Proof. Let $g \in \mathcal{M}_2^0(\mathbb{R} \times TM)$. We have to check the transformation properties of $\mathcal{D}_\Gamma g$ under transformations of fibered coordinates. Let (V, ψ), $\psi = (t, x^i)$ and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (t, \bar{x}^i)$ be two fiber charts on $\mathbb{R} \times M$, and denote by g_{ij} (resp. $\bar{g}_{ij}$) and Γ^i (resp. $\bar{\Gamma}^i$) the components of g and Γ in the chart (V, ψ) (resp. $(\bar{V}, \bar{\psi})$). Then

$$\bar{g}_{ij} = \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} g_{rs}, \quad \bar{\Gamma}^k = \frac{\partial^2 \bar{x}^k}{\partial x^r \partial x^s} \dot{x}^r \dot{x}^s + \frac{\partial \bar{x}^k}{\partial x^r} \Gamma^r, \qquad 1 \le i, j, k \le m.$$

Computing the components of $\mathcal{D}_\Gamma g$ in the chart $(\bar{V}, \bar{\psi})$ and using the relation

$$\frac{\partial^2 x^i}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^p}{\partial x^j} \frac{\partial \bar{x}^q}{\partial x^k} + \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k} = 0,$$

we obtain the transformation formula

$$\left(\overline{\mathcal{D}_\Gamma g}\right)_{ij} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}(\mathcal{D}_\Gamma g)_{rs}\,,$$

proving our assertion.

The mapping $\mathcal{D}_\Gamma$ will be called a *derivative along* Γ or a Γ-*derivative*. A semispray connection Γ on $\mathbb{R}\times TM$ is called *metrizable* if there exists a *variational* metric g on $\mathbb{R}\times TM$ such that the derivative of g along Γ vanishes, i.e.

$$\mathcal{D}_\Gamma g = 0\,. \tag{5.2}$$

The following proposition is a classification of metric connections on $\mathbb{R}\times TM$.

PROPOSITION 4. *Let Γ be a semispray connection on $\mathbb{R}\times TM$. The following two conditions are equivalent:*

(1) *Γ is a metrizable connection.*
(2) *There exists a variational metric g on $\mathbb{R}\times TM$ such that*

$$h_\Gamma = h_{\Gamma_g} + s\,,$$

where Γ_g is the canonical connection of g, and $\tilde{g}(s)$ is an element of $\Omega^{\mathrm{lin}}(\mathbb{R}\times TM)$ such that (in the notation of (2.9)) $B_{ij} = -B_{ji}$.

Proof.

Suppose (1). Let $g\in\mathcal{M}_2^0(\mathbb{R}\times TM)$ be a variational metric on $\mathbb{R}\times TM$ such that $\mathcal{D}_\Gamma g = 0$. Put in every fiber chart on $\mathbb{R}\times M$

$$\Gamma_i = g_{ij}\Gamma^j\,.$$

Then the relation $\mathcal{D}_\Gamma g = 0$ reads

$$\frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k + \frac{1}{2}\left(\frac{\partial \Gamma_i}{\partial \dot{x}^j} + \frac{\partial \Gamma_j}{\partial \dot{x}^i}\right) = 0\,, \qquad 1\le i,j\le m\,. \tag{5.3}$$

Solving this system of partial differential equations for the functions Γ_i, $1\le i\le m$, we obtain (cf. [7] for technical details)

$$\Gamma_i = -\frac{1}{2}\dot{x}^j\dot{x}^k\int_0^1\left(\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j}\right) + \int_0^1\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - 2\frac{\partial g_{jk}}{\partial x^i}\right)\circ\bar{\chi}\,\mathrm{d}v\right)\circ\bar{\chi}v\,\mathrm{d}v$$
$$- \dot{x}^k\int_0^1\left(\frac{\partial g_{ik}}{\partial t}\circ\bar{\chi}\right)\mathrm{d}v + b_{ik}\dot{x}^k + a_i\,,$$

where b_{ik} and a_i, $1 \leq i, k \leq m$, are functions depending only on t and x^p, $1 \leq p \leq m$, and satisfying the condition

$$b_{ik} = -b_{ki} .$$

Hence, $\Gamma_i = \Gamma_i(g) + E_i$, where the form $E = E_i \, dx^i \wedge dt$ belongs to $\Omega^{\text{lin}}(\mathbb{R} \times TM)$. Using (2.5) and the definition of the mapping $\tilde{g}$ we obtain $h_\Gamma = h_{\Gamma_g} + \tilde{g}^{-1}(E)$, as required.

Suppose (2). Denote $\tilde{g}(s) = E_i \, dx^i \wedge dt$. Since $\mathcal{D}_{\Gamma_g} g = 0$, we obtain

$$(\mathcal{D}_\Gamma g)_{ij} = \frac{1}{2} \left(\frac{\partial E_i}{\partial \dot{x}^j} + \frac{\partial E_j}{\partial \dot{x}^i} \right) = 0 .$$

Let g be a variational metric on $\mathbb{R} \times TM$. A soldering form $s \in S(\mathbb{R} \times TM)$ is called *potential with respect to* g if the form $\tilde{g}(s)$ is locally variational. The Helmholtz conditions (3.2) immediately lead to the following

PROPOSITION 5. *A soldering form s on $\mathbb{R} \times TM$ is potential with respect to a variational metric g on $\mathbb{R} \times TM$ if and only if in each fiber chart (V, ψ), $\psi = (t, x^i)$ on $\mathbb{R} \times M$*

$$\tilde{g}(s) = (a_i + b_{ik} \dot{x}^k) \, dx^i \wedge dt ,$$

where a_i, b_{ik}, $1 \leq i, k \leq m$, are functions on V satisfying the conditions

$$b_{ij} = -b_{ji} , \quad \frac{\partial b_{ij}}{\partial t} = \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} , \quad \frac{\partial b_{ij}}{\partial x^k} + \frac{\partial b_{ki}}{\partial x^j} + \frac{\partial b_{jk}}{\partial x^i} = 0 , \qquad 1 \leq i, j, k \leq m .$$

Obviously, if a soldering form s on $\mathbb{R} \times TM$ is potential with respect to g, then there exists an open covering $\mathcal{O}$ of $\mathbb{R} \times TM$ and a lagrangian ω for $\tilde{g}(s)$ on each $U \in \mathcal{O}$, called a (local) *potential energy* associated to g. It holds $\omega = V \, dt$,

$$V = -f_i \dot{x}^i + \varphi + \frac{d\phi}{dt} , \tag{5.4}$$

where ϕ, φ, f_i, $1 \leq i \leq m$, are functions depending only on t and x^p, $1 \leq p \leq m$, and such that

$$\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} = b_{ij} , \qquad \frac{\partial \varphi}{\partial x^i} + \frac{\partial f_i}{\partial t} = a_i .$$

The following proposition solves the so called *inverse variational problem* for semispray connections on a semi-finslerian manifold.

PROPOSITION 6. *A semispray connection Γ on $\mathbb{R}\times TM$ is variational if and only if there exists a variational metric g on $\mathbb{R}\times TM$ such that the following two conditions are satisfied:*

(1) $\mathcal{D}_\Gamma g = 0$,

(2) *the soldering form $s = h_\Gamma - h_{\Gamma_g}$ is potential.*

Proof. Let $E \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R}\times T^2M)$ be a locally variational form such that $\Gamma^*E = 0$. By the Helmholtz conditions (3.1), in each fiber chart on $\mathbb{R}\times M$, it holds $E = E_i\,\mathrm{d}x^i \wedge \mathrm{d}t$, where $E_i = \Gamma_i - g_{ij}\ddot{x}^j$, (g_{ij}) is a variational metric on $\mathbb{R}\times TM$ and the conditions (1), (2) are satisfied.

The converse follows from Propositions 4 and 5.

From Propositions 4 and 6 we immediately get the following assertion:

COROLLARY.

(1) *Every variational connection on $\mathbb{R}\times TM$ is metrizable.*

(2) *A metrizable connection Γ on $\mathbb{R}\times TM$ is variational if and only if the soldering form $s = h_\Gamma - h_{\Gamma_g}$ is potential.*

We shall call the assertion (2) of the above Corollary the *generalized Krupka-Sattarov theorem* (since it can be viewed as a generalization of the Theorem on variationality of a Finslerian structure by Krupka and Sattarov [5] to semispray connections on a semi-finslerian manifold).

6. Examples

(1) *Riemannian metric.* Let (M,g) be a Riemannian manifold, ∇ the Levi-Civita connection of g. Putting

$$\Gamma^i = -\Gamma^i_{jk}\dot{x}^j\dot{x}^k, \qquad 1 \le i \le \dim M,$$

where

$$\Gamma^i_{jk} = \frac{1}{2}g^{ip}\left(\frac{\partial g_{pj}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p}\right)$$

are the Christoffel symbols of ∇, we get a semispray connection Γ on $\mathbb{R}\times TM$ such that the geodesics of Γ coincide with the graphs of geodesics of ∇. Since the metric g satisfies trivially the variationality condition (4.2), it is a variational metric, hence the manifold (M,g) is a particular case of a semi-finslerian manifold. We shall show that this semispray connection Γ is the canonical connection for the semi-finslerian manifold (M,g). According to Sec. 4, the canonical connection Γ_g is defined by (4.6)–(4.8). Substituting the (time and velocity independent) metric g into the formulas (4.6)–(4.8) and performing integration, we get

$$\Gamma^i(g) = \frac{1}{2} g^{ip}\left(\frac{\partial g_{pj}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p}\right)\dot{x}^j\dot{x}^k = \Gamma^i .$$

For the kinetic energy we get from (4.4) the familiar formula $T = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$. Now, every choice of a force $F\in\Omega^{\mathrm{lin}}(\mathbb{R}\times TM)$ gives us a mechanical system (M,g,F) on the Riemannian manifold (M,g). The geodesics of the corresponding semispray connection (i.e. the "equations of motion") then are of the form

$$g_{ij}\ddot{x}^j + \Gamma_{ijk}\dot{x}^j\dot{x}^k = F_i ,$$

where F_i are the components of F.

(2) *Finslerian metric.* Let g be a Finslerian metric on a manifold M, i.e. a regular symmetric fibered morphism $g\colon TM\to T^0_2M$ over id_M, satisfying the conditions (1.1). A Finslerian metric on M is obviously a particular variational metric on $\mathbb{R}\times TM$. We shall compute the kinetic energy λ_g and the canonical connection Γ_g of g according to (4.6)–(4.8) and (4.4), respectively. Using the formulas

$$f = \int_0^1 (f\circ\bar{\chi})\;\mathrm{d}v + \dot{x}^i\int_0^1\left(\frac{\partial f}{\partial\dot{x}^i}\circ\bar{\chi}\right)v\;\mathrm{d}v = 2\int_0^1 (f\circ\bar{\chi})v\;\mathrm{d}v + \dot{x}^i\int_0^1\left(\frac{\partial f}{\partial\dot{x}^i}\circ\bar{\chi}\right)v^2\;\mathrm{d}v$$

for the functions $f=g_{ij}$ and $f=\partial g_{ij}/\partial x^k$, respectively, and using the "homogeneity condition" (1.1) we obtain

$$T = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j , \qquad \Gamma^i(g) = -\Gamma^i_{jk}(g)\dot{x}^j\dot{x}^k ,$$

where

$$\Gamma^i_{jk}(g) = \frac{1}{2}g^{ip}\left(\frac{\partial g_{pj}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p}\right). \tag{6.1}$$

Since the metric g satisfies the "homogeneity condition" (1.1), we get $\Gamma^i_{jk}(g)\dot{x}^j\dot{x}^k = \gamma^i_{jk}\dot{x}^j\dot{x}^k$, where γ^i_{jk}, $1\le i,j,k\le m$, are the components of the Cartan connection (which is a unique linear connection on TM such that the covariant derivative of the Finslerian metric g vanishes).

(3) *Time-independent variational metrics.* Recall that a *Grifone connection* is a vector-valued 1-form $\tilde{\Gamma}$ on TM satisfying the conditions $J\tilde{\Gamma} = J$, $\tilde{\Gamma}J = -J$, where J is the canonical almost tangent structure on TM. The equations for geodesics of a Grifone connection $\tilde{\Gamma}$ are of the form

$$\ddot{x}^i + \Gamma^i_k \dot{x}^k = 0\,,$$

where $\Gamma^i_k(x^j, \dot{x}^j)$ are the components of $\tilde{\Gamma}$. Grifone has shown in [1] that each manifold M endowed with a kinetic energy T carries a canonical Grifone connection such that the equations of geodesics of this connection coincide with the Euler-Lagrange equations of T. If the manifold M is endowed with a kinetic energy T and a *Grifone force* ϕ, which is defined as a 2-form on TM horizontal with respect to the projection $TM \to M$, he has shown in [1] that there is a canonical Grifone connection $\tilde{\Gamma}$ on TM satisfying the following two conditions:

(1) the equations for geodesics of $\tilde{\Gamma}$ coincide with the (nonconservative) Euler-Lagrange equations

$$\frac{\partial T}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}^i} = \phi_{ij} x^j\,,$$

where $\phi_{ij} = -\phi_{ji}$ are the components of ϕ, and

(2) the function

$$E = T - \frac{\partial T}{\partial \dot{x}^k}\dot{x}^k\,,$$

called the principal energy, is constant along the solutions of the equations for geodesics (hence it is a first integral of these equations).

We shall show that these results are in correspondence with the results of Sec. 4.

Denote by ψ the mapping assigning to each Grifone's connection $\tilde{\Gamma}$ a semispray connection Γ by

$$\Gamma^i = -\tilde{\Gamma}^k_i \dot{x}^k\,,$$

where $\tilde{\Gamma}^i_k$ are the components of $\tilde{\Gamma}$. Obviously, the geodesics of $\psi(\tilde{\Gamma})$ coincide with the graphs of geodesics of $\tilde{\Gamma}$. Similarly, by the same letter ψ, we denote the mapping assigning to each Grifone force ϕ a force $F = \psi(\phi) \in \Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times M)$ by $F_i = \phi_{ik}\dot{x}^k$, where ϕ_{ik} are the components of ϕ; note that this mapping is *not* surjective (even in case we restrict $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R} \times M)$ to time-independent forces). Let g be a metric on TM satisfying the variationality condition (4.2), and denote by $\lambda_g = T\,\mathrm{d}t$ the kinetic energy of g. (M, g) is a semi-finslerian manifold, and the canonical connection takes the form

$$-\Gamma^i_g = \Gamma^i_{qr}(g)\dot{x}^q\dot{x}^r$$
$$= g^{ip}\left(\frac{1}{2}\int_0^1\left(\frac{\partial g_{pq}}{\partial x^r} + \frac{\partial g_{pr}}{\partial x^q} - 2\frac{\partial g_{qr}}{\partial x^p}\right)\circ\bar{\chi}\,\mathrm{d}v + \int_0^1\left(\frac{\partial g_{qr}}{\partial x^p}\circ\bar{\chi}\right)v\,\mathrm{d}v\right)\dot{x}^q\dot{x}^r.$$

If $\tilde{\Gamma}$ is the canonical Grifone connection for the energy T, then obviously $\psi(\tilde{\Gamma}) = \Gamma_g$. Similarly, if ϕ is a Grifone force on TM, then ψ maps the canonical Grifone connection corresponding to the kinetic energy T and Grifone's force ϕ to the semispray connection Γ of the mechanical system $(M, g, \psi(\phi))$. Computing the Lie derivative of E by the semispray ζ associated with the connection Γ we get

$$\partial_\zeta E = \dot{x}^i\left(\frac{\partial E}{\partial x^i} - \tilde{\Gamma}^j_i\frac{\partial E}{\partial \dot{x}^j}\right) = \dot{x}^i\phi^j_i\frac{\partial^2 T}{\partial\dot{x}^j\partial\dot{x}^k}\,\dot{x}^k = g_{jk}\phi^j_i\dot{x}^i\dot{x}^k = \phi_{ki}\dot{x}^k\dot{x}^i = 0\,,$$

i.e. the "principal energy" E (which is nothing but the Hamiltonian of the free particle (M, g)) is conserved. This interesting property, of course, will no longer last for a general (possibly time-independent) force in $\Omega^{1,1}_{\mathbb{R}\times M}(\mathbb{R}\times M)$ (it is sufficient to take a force $F_i = \phi_{ij}\dot{x}^j$, $\phi_{ij} + \phi_{ji} \neq 0$, which is not the image of a Grifone force).

(4) *Examples of variational metrics in classical and relativistic mechanics.* Consider the manifold $\mathbb{R}^3$ with the canonical global chart (x^i).

Putting $g = m\delta$, where $\delta = \delta_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$ is the Kronecker tensor and m is a positive constant, we get the semi-finslerian manifold $(\mathbb{R}^3, m\delta)$ which is a free particle of classical mechanics with mass m; in this case the canonical connection $\Gamma_g = 0$, and $E_g = (m\delta_{ij}\ddot{x}^i)\,\mathrm{d}x^j\wedge\mathrm{d}t$. Considering a force F on $(\mathbb{R}^3, m\delta)$ we get a classical particle of mass m in the force field F, described by the equations of motion $m\ddot{x}^i = F^i(t, x, \dot{x})$.

Put $g = f(t)\delta$, where f is a nowhere zero function. Then $(\mathbb{R}^3, f(t)\delta)$ is a semi-finslerian manifold. Computing the components of the canonical connection according to (4.6)–(4.8) we get

$$\ddot{x}^i\circ\Gamma_g = -\frac{1}{f(t)}\,\delta^{ij}\dot{x}^k\delta_{jk}\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{1}{f(t)}\frac{\mathrm{d}f}{\mathrm{d}t}\,\dot{x}^i\,.$$

Hence, the equations of motion of the free particle on $(\mathbb{R}^3, f(t)\delta)$ are the Newton equations of a classical free particle with nonconstant mass. Considering a force field F on this semi-finslerian manifold we get the mechanical system

$(\mathbb{R}^3, f(t)\delta, F)$ which is a classical particle with nonconstant mass moving in the force field F.

Let us define a semi-finslerian metric $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ on $\mathbb{R}^3$ by

$$g_{ij} = \frac{m\delta_{ij}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} + \frac{m}{c^2}\,\frac{\delta_{ip}\dot{x}^p\,\delta_{kq}\dot{x}^q}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}\,,$$

where m and c are positive constants, and $v^2 = \delta_{ij}\dot{x}^i\dot{x}^j$. Then $\Gamma_g = 0$, $E_g = (g_{ij}\ddot{x}^j)\,\mathrm{d}x^i \wedge \mathrm{d}t$, i.e. $(\mathbb{R}^3, g)$ is a free particle of special relativity theory. Let F be the Lorentz force on the semi-finslerian manifold $(\mathbb{R}^3, g)$, $F = \delta(\vec{F}, \cdot)$, where $\vec{F} = e\vec{E} + \frac{em}{c}(\vec{v} \times \vec{H})$. Since $\Gamma_g = 0$, we get the components of the connection Γ of the mechanical system $(\mathbb{R}^3, g, F)$ in the form

$$\Gamma^i = g^{ij}F_j = \frac{e}{m}\sqrt{1 - \frac{v^2}{c^2}}\left(E^i + \frac{1}{c}(\vec{v} \times \vec{H})^i - \frac{1}{c^2}v^i\vec{v}\vec{E}\right).$$

This connection obviously differs from that describing a classical particle in the Lorentz force field, i.e. the mechanical system $(\mathbb{R}^3, m\delta, F)$; in this case we have

$$\Gamma^i = \frac{e}{m}\left(E^i + \frac{1}{c}(\vec{v} \times \vec{H})^i - \frac{1}{c^2}v^i\right).$$

The difference between a mechanical system and the semispray connection describing the motion of this system can be demonstrated on the following easy example: the semispray connection $\Gamma^i = kx^i$ can describe a mechanical system $(\mathbb{R}^3, m\delta, F)$, where $F = \delta(\vec{F}, \cdot)$, $\vec{F} = (k\dot{x}^1, k\dot{x}^2, k\dot{x}^3)$, i.e. a classical particle of mass m moving in the dissipative force field, or a mechanical system $(\mathbb{R}^3, e^{kt}\delta)$, i.e. a classical free particle with the mass-accretion rule $f(t) = me^{kt}$, or some other mechanical system (according to the choice of a semi-finslerian metric on $\mathbb{R}^3$).

(5) *Metrizable linear connections on TM*. We shall show that the results on metrizability and variationality of semispray connections on $\mathbb{R} \times TM$ obtained in Sec. 5. generalize the known results on linear connections on TM (and on M), obtained by Krupka and Sattarov [5].

Let M be an m-dimensional manifold. Denote by ΓM the *bundle of linear connections over M*. Recall that by a *linear connection on TM* we mean a fibered morphism $\gamma\colon TM \to \Gamma M$ over id_M. Denote by ∇_γ the *covariant derivative*. If $g\colon TM \to T^0_2 M$ is a fibered morphism over id_M, then in any coordinates (x^i) on M, $\nabla_\gamma g \in T^0_3 M$ is expressed by

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial \dot{x}^p}\,\gamma^p_{qk}\dot{x}^q - g_{ip}\gamma^p_{jk} - g_{jp}\gamma^p_{ik}\,, \tag{6.2}$$

where $(x^i, \dot{x}^i)$ are coordinates on TM, associated with (x^i), and g_{ij}, γ^i_{jk}, $1 \le i, j, k \le m$, are the components of g and γ, respectively.

To any linear connection γ on TM we can assign a semispray connection Γ on $\mathbb{R} \times TM$, setting in each fiber chart

$$\Gamma^i = -\gamma^i_{jk} \dot{x}^j \dot{x}^k , \tag{6.3}$$

where γ^i_{jk}, $1 \le i, j, k \le m$, are the components of γ. The semispray connection Γ will be called *associated* to γ. Obviously, geodesics of Γ and graphs of geodesics of γ coincide.

A linear connection γ on TM will be called *variational* if there exists a *Finslerian metric* g on TM such that

$$\Gamma^* E_g = 0$$

for the semispray connection Γ, associated to γ and the canonical dynamical 2-form E_g of g. Using Proposition 6 we can see immediately that *if a linear connection γ on TM is variational, then the associated semispray connection Γ is metrizable, and there is a Finslerian metric g such that $\mathcal{D}_\Gamma g = 0$.*

A linear connection γ on TM is called *metrizable* if there exists a *Finslerian metric* g such that $\nabla_\gamma g = 0$.

PROPOSITION 7. *Let γ be a linear connection on TM, Γ the semispray connection on $\mathbb{R} \times TM$ associated to γ. Let g be a Finslerian metric. If $\nabla_\gamma g = 0$, then $\mathcal{D}_\Gamma g = 0$, and $\Gamma = \Gamma_g$.*

Proof. Suppose that γ is metrizable, $\nabla_\gamma g = 0$. Then, by (6.1) and (6.2), the components of γ and of the canonical connection Γ_g of the Finslerian metric g satisfy the relation

$$2\Gamma_{ijk}(g) - 2\gamma_{ijk} - \frac{\partial g_{ij}}{\partial \dot{x}^p} \gamma^p_{qk} \dot{x}^q - \frac{\partial g_{ik}}{\partial \dot{x}^p} \gamma^p_{qj} \dot{x}^q + \frac{\partial g_{jk}}{\partial \dot{x}^p} \gamma^p_{qi} \dot{x}^q = 0 ,$$

where $\Gamma_{ijk}(g) = g_{ip} \Gamma^p_{jk}(g)$ and $\gamma_{ijk} = g_{ip} \gamma^p_{jk}$. Hence, using the homogeneity of g, we obtain

$$(\Gamma_{ijk}(g) - \gamma_{ijk}) \dot{x}^j \dot{x}^k = \Gamma_{ijk}(g) \dot{x}^j \dot{x}^k - \Gamma_i = 0 ,$$

i.e. the associated connection Γ of γ is the canonical connection of the Finslerian metric g. Hence, $\mathcal{D}_\Gamma g = \mathcal{D}_{\Gamma_g} g = 0$.

Now, by Corollary to Proposition 6, we get (cf. [5])

COROLLARY. (Krupka-Sattarov theorem) *Every metrizable linear connection on TM is variational, and it is the Cartan connection of the corresponding Finslerian metric.*

The situation is further simplified if we consider a *linear connection* γ *on* M, i.e. $\gamma \in \Gamma M$. In this case, while speaking of variationality or metrizability, we shall naturally have the existence of a metric on M in mind. Similarly as above, we assign to γ a semispray connection Γ on $\mathbb{R} \times TM$ by (6.3). Now, however, this mapping is one-to-one, and we have

$$\gamma^i_{jk} = -\frac{1}{2}\frac{\partial^2 \Gamma^i}{\partial \dot{x}^j \partial \dot{x}^k} . \tag{6.4}$$

For a metric g on M we get $(\mathcal{D}_\Gamma g)_{ij} = g_{ij;k}\dot{x}^k$, hence $\mathcal{D}_\Gamma g = 0 \iff \nabla_\gamma g = 0$. As a direct consequence of this property and of Proposition 7, we obtain (cf. [5], [6])

PROPOSITION 8. *Let γ be a linear connection on M, let Γ be the semispray connection associated to γ. The following four conditions are equivalent:*

(1) *γ is variational.*
(2) *γ is metrizable.*
(3) *Γ is variational, and there exists a metric g on M such that $\Gamma = \Gamma_g$.*
(4) *There exists a metric g on M such that $\mathcal{D}_\Gamma g = 0$.*

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