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Mathematica Slovaca, Vol. 36 (1986), No. 4, 367--386

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ON FACE VECTORS OF TRIVALENT MAPS

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Dedicated to Professor E. Jucovič on his sixtieth birthday

1. Introduction

A map M on T_g will be understood to mean a 2-dimensional topological complex whose union of cells forms an orientable surface T_g of genus g. (For the definition of a complex see Grünbaum [4]; cells of a topological complex are topological analogues of those of a complex). It can be easily shown that the graph of M is 3-connected. The 2-cells (1-cells, 0-cells) will be called faces (edges, or vertices, respectively). A face (vertex) of M incident with i vertices is called an i-gon (i-valent vertes) of M. Let $p_i(M)$ (or $v_i(M)$) denote the number of i-gonal faces (or i-valent vertices) of M. A sequence of integers

$$(p_i(M) \mid i \geq 3) \tag{A}$$

associated in a natural way with a map M on T_g is the *face vector* of M. The *vertex vector* of M is defined analogously. (We note that from the definition of M there follows $p_1(M) = p_2(M) = v_1(M) = v_2(M) = 0$.)

The present paper is concerned with face vectors of *trivalent* maps (i. e. maps for which $v_i(M) = 0$ for $i \neq 3$) on an orientable surface T_g of genus g for all nonnegative values of g.

The well-known Euler formula as applied to elements of (A) for trivalent maps leads to the following condition

$$\sum_{i \geq 3} (6 - i)\, p_i(M) = 12(1 - g). \tag{1}$$

An interesting property of (1) is that it gives no information about the values of $p_6(M)$. This brings up the following

Problem: *We have a sequence of nonnegative integers*

$$p = (p_i \mid 3 \leq i \neq 6)$$

and a nonnegative integer g such that

$$\sum_{i\geq 3}(6-i)\,p_i = 12(1-g). \tag{2}$$

The sequence p and the number g determine the set $P(p, g)$ of all nonnegative integers such that the sequence p with any element of $P(p, g)$ added as p_6 is the face vector of a trivalent map on T_g. The problem is one of the characterization of $P(p, g)$ for all pairs (p, g).

For $g=0$ the above problem is equivalent to the same problem for trivalent convex polyhedra (cf. [4], [9], [15]). Various aspects of solutions of this problem have been investigated by several authors, e. g. Eberhard [2], Fisher [3], Grünbaum [4, 5, 6], Grünbaum—Motzkin [7], Jendroľ [9], Jucovič [12, 15], Kraeft [16], Malkevitch [17].

The author of the present paper has obtained the following result for $g=0$ (cf. [9]):

Theorem 1. *Let $p=(p_i|3\leq i\neq 6)$ be a sequence of nonnegative integers satisfying (2) for $g=0$.*

(i) *If* $\sum_{3\leq i\not\equiv 0 \pmod 3} p_i \leq 2$ *and* $\sum_{3\leq j\neq 6} p_j \equiv 0 \pmod 2$,
then there exists a nonnegative integer d such that the set $P(p, 0)$ contains every even integer $\geq d$ and no odd numbers.

(ii) *If* $\sum_{3\leq i\not\equiv 0 \pmod 3} p_i \leq 2$ *and* $\sum_{3\leq j\neq 6} p_j \equiv 1 \pmod 2$,
then there exists a nonnegative integer d such that the set $P(p, 0)$ contains every odd integer $\geq d$ and no even numbers.

(iii) *If* $\sum_{3\leq i\not\equiv 0 \pmod 3} p_i \geq 3$, *then there exists a nonnegative integer d such that the set $P(p, 0)$ contains every integer $\geq d$.*

Certain properties of $P_6(p, 1)$ are discussed in Fisher [3], Grünbaum [5] and Jendroľ—Jucovič [10]. Barnette [1] and Jucovič [13] have found two distinct lower bounds for the number $m=\min\{p_6\colon\ p_6\in P(p, g)\}$ as functions of the elements of p and the number g for any pair (p, g).

In [11], the authors treat the above problem in a more general way. For trivalent maps their result yields

Theorem 2. *The set $P(p, g)$ is empty if and only if $g=1$ and $p=(p_i|p_5=p_7=1,$ $p_i=0$ for $i\neq 5, 7)$.*

For a somewhat related result see Jucovič [14, 15].

The aim of the present paper is to find a characterization of $P(p, g)$ for all pairs of (p, g) with $g\geq 2$ and to prove that (as opposed to the case $g=0$, see Theorem 1) the number of nonnegative integers which are not members of $P(p, g)$ is *finite* for each pair (p, g) where $g\geq 2$. The result is contained in

Theorem 3. *For every sequence of nonnegative integers $p=(p_i|3\leqslant i\neq 6)$ and every $g\geqslant 2$ satisfying the condition* (2) *there exists a nonnegative integer d such that the set $P(p, g)$ contains every integer $\geqslant d$.*

The situation for *toroidal* maps (i. e. for $g=1$) is given by

Theorem 4. *Suppose that the sequence of nonnegative integers $p=(p_i|3\leqslant i\neq 6)$ satisfies the condition* (2) *with $g=1$.*

(i) *If $\sum_{3\leqslant i\neq 6} p_i\neq 2$, then there exists a nonnegative integer d such that $P(p, 1)$ contains every integer $\geqslant d$.*

(ii) *If $p_5=p_7=1$, $p_i=0$* for *$i\neq 5, 7$, then $P(p, 1)$ is empty.*

(iii) *If $p_4=p_8=1$, $p_i=0$ for $i\neq 4, 8$ or $p_3=p_9=1$, $p_i=0$ for $i\neq 3, 9$, then there exists a nonnegative integer d such that $P(p, 1)$ contains every even number $\geqslant d$.*

2. Basic construction elements and certain existential lemmas

In this chapter we shall prove certain existential lemmas valid for all maps and an orientable surface of genus g for every nonnegative g (i. e. not only for trivalent maps).

Consider a map M with the sequences $p=(p_i|i\geqslant 3)$ and $v=(v_i|i\geqslant 3)$ as the face vector and the vertex vector, respectively. The trivial equation $\sum_{i\geqslant 3} iv_i=\sum_{i\geqslant 3} ip_i$ yields a useful relationship

$$v_3=\frac{1}{3}\left(\sum_{i\geqslant 3} ip_i-\sum_{i\geqslant 4} iv_i\right). \tag{3}$$

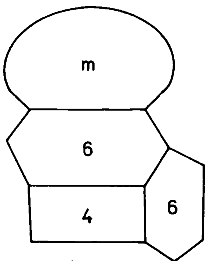

Fig. 1a

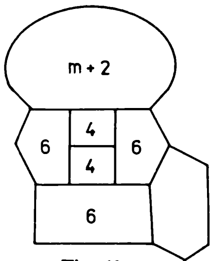

Fig. 1b

Basic construction elements: The face-aggregate of a map M as in Fig. 1a (or its mirror image), or 2a, or 3a called an A_m *configuration,* or a B_m *configuration,* or a C_m *configuration* consists of an m-gon, $m\geqslant 6$, two hexagons and one quadrangle, or of an m-gon, $m\geqslant 6$, two hexagons and two quadrangles, or of an m-gon, $m\geqslant 6$, two hexagons and three quadrangles, respectively; the mentioned m-gon will be called a *basic face* of a configuration. (We note that in the sequel i, j, k, m, n, t, w mean nonnegative integers. We shall denote in the figures the size of every nonhexagonal face excluding faces of the X configurations, $X \in \{A_m, B_m, C_m, D, E, F, G\}$, bounded by heavy lines; hexagons are to be denoted only in more important cases.)

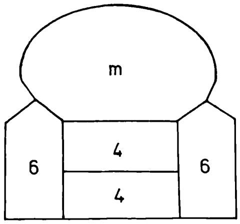

Fig. 2a

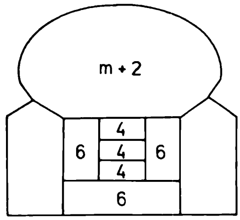

Fig. 2b

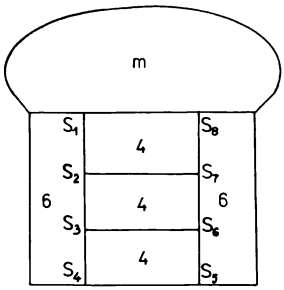

Fig. 3a

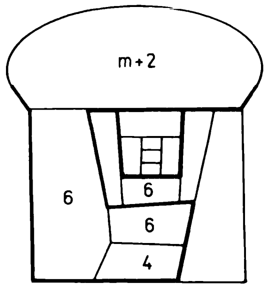

Fig. 3b

Basic construction steps: A basic construction step transforms a starting map M into a map M'; it uses the presence of the X_m configuration, $X \in \{A, B, C\}$, in M (see Figs. 1, 2, 3). For the map M' we have $p_4(M') = p_4(M) + 1$, $p_{m+2}(M') = p_{m+2}(M) + 1$, $p_j(M') = p_j(M)$, $j \neq 4, 6, m, m+2$ and $p_6(M') = p_6(M') + z$ ($z = 2,3$ or 7 for X = A, B or C, respectively), $p_m(M') = p_m(M) - 1$ (if $m \neq 6$) or $p_6(M') = p_6(M) + z - 1$ (if $m = 6$). For continuing the construction it is important that transforming an A_m configuration (a B_m or a C_m configuration) we get a B_{m+2} configuration (a C_{m+2} or an A_{m+2} configuration) and a B_6 configuration (a C_6 or an A_6 configuration, respectively) (differing only in their basic faces). If an $(m+2)$-gon is needed, we use the basic construction step to the X_6 configuration; if not, use the X_{m+2} configuration producing an $(m+4)$-gon. Note that the transformation of a C_m configuration yields a new C_6 configuration face-disjoint from A_{m+2} and A_6 configurations (see Fig. 3b); this C_6 configuration is not used in basic construction steps.

Let $M = M(q, v, g, a, b, c)$ be a map on an orientable surface T_g of genus g with the following properties:

(i) The sequences $q=(q_i \mid i\geqslant 3)$ and $v=(v_i \mid i\geqslant 3)$ are the face vector and vertex vector of M respectively.

(ii) M contains as submaps at least a A_6 configurations, $a\geqslant 0$, b B_6 configurations, $b\geqslant 0$, and c C_6 configurations, $c\geqslant 0$, such that all mentioned configurations are pairwise face-disjoint.

A very useful transformation of a map M into a map M' called the *replacing of edges by hexagons* will now be described (cf. [4], [11], [15]). In this transformation every edge of M is replaced by a hexagon in such a way that a pair of neighbouring faces in M consisting of a k-gon K and an l-gon L is replaced by a k-gon K' and an l-gon L' in M' which are separated by a hexagon. The vertices of K' and L' are trivalent and at the same time to every *w-valent vertex of M* there corresponds in M' a w-valent vertex in the same position which is incident with w hexagons (cf. Fig. 4a where the original map is drawn by dashed lines). This transformation will be designated in the sequel as an "*ε-transformation*".

The ε-transformation changes configurations A_m, B_m, C_m, $m\geqslant 6$, into configurations which will be designated as $\varepsilon(A_m)$, $\varepsilon(B_m)$ and $\varepsilon(C_m)$ respectively. Fig. 4b shows an $\varepsilon(C_6)$ configuration.

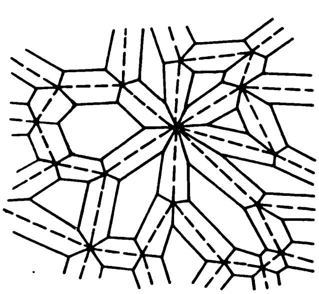

Fig. 4a

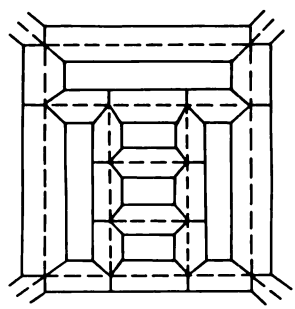

Fig. 4b

Auxiliary construction elements: The configurations whown in Fig. 5 will play an important role together with the basic construction elements. The configuration shown in Fig. 5a will be designated as a D *configuration* and its mirror image as a D′ *configuration.* Figs. 5b, 5c and 5d show configurations which will henceforth be designated as E, F and G *configurations* respectively.

Note that the map $\varepsilon(M)$, which is a result of using the ε-transformation on *a map M* containing a C_6 configuration, will contain an E configuration as a part of $\varepsilon(C_6)$. Analogously, $\varepsilon(M)$ will contain an F configuration if the original map M has contained a pair of adjacent quadrangles.

Lemma 1. α ($\alpha \in \{1, 2, \ldots, 27\}$; cf. [9, p. 172, Lemma 3α]). *Let* $f = (f_i \mid i \geq 7)$ *be a sequence of nonnegative integers with a finite number of nonzero elements and let*

$$j = 6 + \sum_{i \geq 7} (i-6) f_i.$$

If there is a map $M = M(q, v, g, a, b, c)$ *with* $a + b + c \neq 0$, *then there is a map* $M' = M(q', v', g, a', b', c')$ *with*

$$q' = (q'_i \mid q'_3 = q_3 + s_3,\ q'_4 = q_4 + s_4,\ q'_5 = q_5 + s_5,\ q'_6 = q_6 + s_6,$$
$$q'_i = q_i \ \text{ for all } i \geq 7)$$
$$v' = (c'_i \mid v'_i = v_i \ \text{ for all } i \neq 3;\ v'_3 = \frac{1}{3}\Big(\sum_{i \geq 3} i q'_i - \sum_{i \geq 4} i v'_i\Big)\Big);$$

for the values s_3, s_4, s_5, a', b', c' *see* Table 1, $\alpha \in \{1, 2, \ldots, 9\}$ *if* $a \neq 0$; $\alpha \in \{10, \ldots, 18\}$ *if* $b \neq 0$, $\alpha \in \{19, \ldots, 27\}$ *if* $c \neq 0$. *The value* s_6 *is a constant depending on the sequence* f.

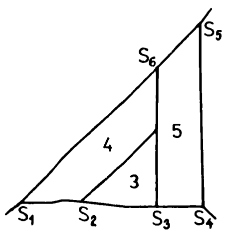

Fig. 5a

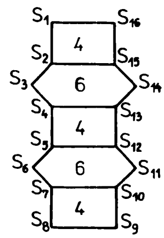

Fig. 5b

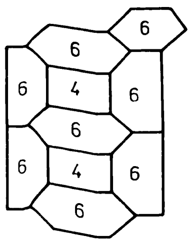

Fig. 5c

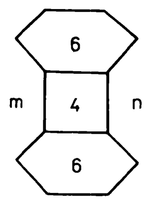

Fig. 5d

Lemma 2. α ($\alpha \in \{1, 2, 3, 4\}$) *If there is a map* $M = M(q, v, g, a, b, c)$ *with* $c \neq 0$, *then there is a map* $M' = M(q', v', g, a', b', c')$ *where for*

$$\alpha = 1.\ \ q' = (q'_i \mid q'_i = q_i \ \text{ for all } i \neq 6,\ q'_6 = q_6 + 2t),$$
$$v' = (v'_i \mid v'_i = v_i \ \text{ for all } i \neq 3,\ v'_3 = v_3 + 4t)$$

Table 1

α	j	s_3	s_4	s_5	a'	b'	c'
1.	$6k$	0	$3k-3$	0	a	b	$c+k-1$
2.	$6k+1$	1	$3k-4$	0	$a-1$	b	$c+k-1$
3.	$6k+1$	0	$3k-3$	1	$a-1$	b	$c+k-1$
4.	$6k+2$	0	$3k-2$	0	$a-1$	$b+1$	$c+k-1$
5.	$6k+3$	1	$3k-3$	0	a	b	$c+k-1$
6.	$6k+3$	0	$3k-2$	1	$a-1$	$b+1$	$c+k-1$
7.	$6k+4$	0	$3k-1$	0	$a-1$	b	$c+k$
8.	$6k+5$	1	$3k-2$	0	$a-1$	$b+1$	$c+k-1$
9.	$6k+5$	0	$3k-1$	1	$a-1$	b	$c+k-1$
10.	$6k$	0	$3k-3$	0	a	b	$c+k-1$
11.	$6k+1$	1	$3k-4$	0	$a+1$	$b-1$	$c+k-1$
12.	$6k+1$	0	$3k-3$	1	a	$b-1$	$c+k-1$
13.	$6k+2$	0	$3k-2$	0	a	$b-1$	$c+k$
14.	$6k+3$	1	$3k-2$	0	a	b	$c+k-1$
15.	$6k+3$	0	$3k-2$	1	a	$b-1$	$c+k-1$
16.	$6k+4$	0	$3k-1$	0	$a+1$	$b-1$	$c+k$
17.	$6k+5$	1	$3k-2$	0	a	$b-1$	$c+k$
18.	$6k+5$	0	$3k-1$	1	a	$b-1$	$c+k$
19.	$6k$	0	$3k-3$	0	a	b	$c+k-1$
20.	$6k+1$	1	$3k-4$	0	a	$b+1$	$c+k-2$
21.	$6k+1$	0	$3k-3$	1	a	b	$c+k-2$
22.	$6k+2$	0	$3k-2$	0	$a+1$	b	$c+k-1$
23.	$6k+3$	1	$3k-3$	0	a	b	$c+k-1$
24.	$6k+3$	0	$3k-2$	1	a	b	$c+k-1$
25.	$6k+4$	0	$3k-1$	0	a	$b+1$	$c+k-1$
26.	$6k+5$	1	$3k-2$	0	$a+1$	b	$c+k-1$
27.	$6k+5$	0	$3k-1$	1	a	b	$c+k-1$

where t is a nonnegative integer and $a'=a,\ b'=b,\ c'=c$, or for

$$\alpha=2.\ \ q'=(q'_i \mid q'_3=q_3+2,\ q'_4=q_4-3,\ q'_i=q_i \text{ for all } i\geqslant 5),$$
$$v'=(v'_i \mid v'_3=v_3-2,\ v'_i=v_i \text{ for all } i\geqslant 4)$$

and $a'=a,\ b'=b,\ c'=c-1$ or for

$$\alpha=3.\ \ q'=(q'_i \mid q'_3=q_3+1,\ q'_4=q_4-2,\ q'_5=q_5+1,\ q'_6=q_6-1,$$
$$q'_i=q_i \text{ for all } i\geqslant 7),$$
$$v'=(v'_i \mid v'_3=v_3-2,\ v'_i=v_i \text{ vor all } i\geqslant 4)$$

and $a'=a,\ b'=b,\ c'=c-1$ or for

$$\alpha=4.\ \ q'=(q'_i \mid q'_3=q_3+1,\ q'_4=q_4-3,\ q'_5=q_5+3,\ q'_6=q_6-3,$$

$q'_i = q_i$ for all $i \geqslant 7)$

$v' = (v'_i | v'_3 = v_3 - 4,\ v'_i = v_i$ for all $i \geqslant 4)$

and $a' = a,\ b' = b,\ c' = c - 1.$

Proof. For $\alpha = 1, 2, 3$ cf. [9, p. 173]. For $\alpha = 4$, the modification of C_6 necessary to obtain the map satisfying the statement of the lemma is shown in Fig. 6 (cf. also Fig. 3a).

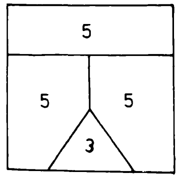

Fig. 6

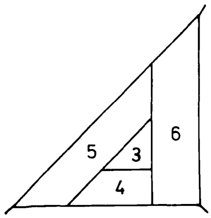

Fig. 7

Lemma 3. (cf. [9, p. 174]). *If there is a map* $M = M(q, v, g, a, b, c)$ *with* $b \neq 0$, *then there is a map* $M' = M(q', v', g, a, b-1, c)$, *where*

$$q' = (q'_i | q'_3 = q_3 + 1,\ q'_4 = q_4 - 2,\ q'_5 = q_5 + 1,\ q'_6 = q_6 - 1,\ q'_i = q_i \text{ for all } i \geqslant 7) \text{ and}$$

$$v' = (v'_i | v'_3 = v_3 - 2,\ v'_i = v_i \text{ for all } i \neq 3).$$

Lemma 4. *If there is a map* $M = M(q, v, g, a, b, c)$ *with at least one* D (*or* D′) *configuration, then there is a map* $M' = M(q', v', g, a, b, c)$ *with the same number of* D (*or* D′) *configurations and such that*

$$q' = (q'_i | q'_i = q_i \text{ for all } i \neq 6,\ q'_6 = q_6 + t, \text{ where } t = 0, 1, 2, \ldots)$$

and

$$v' = (v'_i | v'_i = v_i \text{ for all } i \geqslant 4,\ v'_3 = v_3 + 2t).$$

Proof. Fig. 7 shows how to obtain one hexagon by inserting one edge into a D configuration. Note that no other changes outside the D configuration occur in the map.

Lemma 5. *If there is a map* $M = M(q, v, g, a, b, c)$ *with at least one* E *configuration face-disjoint from* A_6 *configurations of* M, *then there is a map* $M' = M(q', v', g, a, b, c)$ *with the same number of* E *configurations such that*

$$q' = (q'_i | q'_i = q_i \text{ for all } i \neq 6,\ q'_6 = q_6 + 4t, \text{ where } t = 0, 1, 2, \ldots)$$

$$v' = (v'_i | v'_i = v_i \text{ for all } i \neq 3,\ v'_3 = v_3 + 8t).$$

Proof. Fig. 8 shows the modifications made inside the E configuration which add four hexagons so that another E configuration results. Repeating this t times leads to M'.

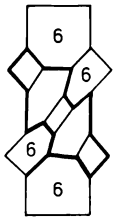

Fig. 8

Lemma 6. α ($\alpha \in \{1, 2\}$) *If there is a map* $M = M(q, v, g, a, b, c)$ *with at least one* F *configuration face-disjoint from a* A_6 *configurations, then there is a map* $M' = M(q', v', g, a, b, c)$ *with one less* F *configuration and such that for*

$$\alpha = 1. \quad q' = (q'_i | q'_i = q_i \text{ for all } i \neq 4, 5, 6, \; q'_4 = q_4 - 2, \; q'_5 = q_5 + 4,$$

$$q'_6 = q'_6 - z, \text{ where } z = 3, 4, 5 \text{ or } 6) \text{ and}$$

$$v' = (v'_i | v'_i = v_i \text{ for all } i \neq 3, \; v_3 = \frac{1}{3}\left(\sum_{i \geq 3} i q'_i - \sum_{i \geq 4} i v'_i\right))$$

or for

$$\alpha = 2. \quad q' = (q'_i | q'_i = q_i \text{ for all } i \neq 4, 5, 6, \; q'_4 = q_4 - 1, \; q'_5 = q_5 + 2$$

$$q'_6 = q_6 - z \text{ where } z = 1, 2, 3, 4) \text{ and}$$

$$v' = (v'_i | v'_i = v_i \text{ for all } i \geq 4, \; v'_3 = \frac{1}{3}\left(\sum_{i \geq 3} i q'_i - \sum_{i \geq 4} i v'_i\right)).$$

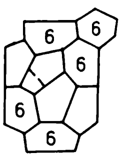

Fig. 9a

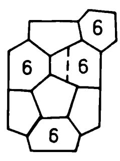

Fig. 9b

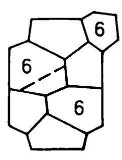

Fig. 9c

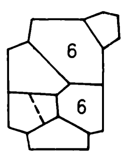

Fig. 9d

Proof. Figs. 9a to 9d show the modifications made to an F configuration of M leading to M' for $\alpha = 1$ (without the dashed lines) or $\alpha = 2$ (with them).

Lemma 7. *If there is a map* $M = M(q, v, g, a, b, c)$ *with at least one* G *configuration face-disjoint from a* A_6, *b* B_6 *and c* C_6 *configurations, then there is a map* $M' = M(q', v', g, a, b, c)$ *with one less* G *configurations such that*

$$q' = (q'_i \mid q'_i = q_i \text{ for all } i \neq 4, 5, 6,\ q'_4 = q_4 - 1,\ q'_5 = q_5 + 2,\ q'_6 = q_6 - 2)$$

$$v' = (v'_i \mid v'_i = v_i \text{ for all } i \geqslant 4,\ v'_3 = v_3 - 2).$$

Proof. Note the difference between the G configuration in Fig. 5d and its modification in Fig. 10.

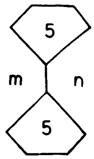

Fig. 10

Lemma 8. *If there is a map $M = M(q, v, g, a, b, c)$ containing at least two face-disjoint* D *configurations face-disjoint from a* A_6, *b* B_6 *and c* C_6 *configurations, then there is a map $M' = M(q', v', g+1, a, b, c)$ containing two less* D *configurations such that*

$$q' = (q'_i \mid q'_3 = q_3 - 2,\ q'_4 = q_4 - 2,\ q'_5 = q_5 - 2,\ q'_i = q_i \text{ for all } i \geqslant 6) \text{ and}$$

$$v' = (v'_i \mid v'_i = v_i,\ i \geqslant 4,\ v'_3 = v_3 - 8).$$

Proof. Designate the vertices of the first (second) D configuration $S_1, S_2, \ldots, S_6$ (or $S'_1, S'_2, \ldots, S'_6$ respectively) as shown in Fig. 5a. By cutting out two regions bounded by graph circuits $S_1 S_2 \ldots S_6$ and $S'_1 S'_2 \ldots S'_6$ we obtain two holes in T_g. "Gluing" together these holes by identifying vertices S_i and S'_{7-i}, $i = 1, 2, \ldots, 6$, and the corresponding edges we obtain the required map M' on T_{g+1} for which $q'_i(M) = q_i$ for all $i \geqslant 6$. Note that the gluing together neither disturbs the structure of the remaining faces nor creates new ones.

Lemma 9. *If there is a map $M = M(q, v, g, a, b, c)$ with $c \geqslant 2$, then there is a map $M' = M(q', v', g+1, a, b, c-2)$ such that*

$$q' = (q'_i \mid q'_i = q_i \quad \text{for all} \quad i \neq 4,\ q'_4 = q_4 - 6) \quad \text{and}$$

$$v' = (v'_i \mid v'_i = v_i \quad \text{for all} \quad i \geqslant 4,\ v'_3 = v_3 - 8).$$

Proof. Choose two face-disjoint C_6 configurations in the map M on T_g. Label the vertices incident with the quadrangles of the first (second) one $S_1, S_2, \ldots, S_8$ and $S'_1, S'_2, \ldots, S'_8$, respectively. This is shown in Fig. 3a. Cutting out the regions bounded by the graph circuits $S_1 S_2 \ldots S_8$ and $S'_1 S'_2 \ldots S'_8$ (regions consisting of three quadrangles each) we obtain two holes. By identifying the vertices S_s and S'_t, where $s = 1, 2, \ldots, 8$ and $t = s + 6 \pmod 8$ as well as the corresponding edges we obtain the required map $M' = M(q', v', g+1, a, b, c-2)$ on T_{g+1}. The gluing together destroys six quadrangles; no other face changes its type.

Lemma 10. *If there exists a map $M = M(q, v, g, a, b, c)$ with at least two face-disjoint* E *configurations face-disjoint from a* A_6, $b\,B_6$ *and* $c\,C_6$ *configurations, then there exists a map* $M' = M(q', v', g+1, a, b, c)$ *with two less* E *configurations and such that*

$$q' = (q_i' \mid q_i' = q_i \text{ for all } i \neq 4, 6,\ q_4' = q_4 - 6,\ q_6' = q_6 - 4)$$

$$v' = (v_i' \mid v_i' = v_i \text{ for all } i \geq 4,\ v_3' = v_3 - 16).$$

Proof. Choose two face-disjoint E configurations in M. Label the vertices of the first (second) one $S_1, \ldots, S_{16}$ and $S_1', \ldots, S_{16}'$, respectively, as in Fig. 5b. Cut two holes in T_g bounded by circuits $S_1S_2 \ldots S_{16}$ and $S_1'S_2' \ldots S_{16}'$. Identify vertices S_s and S_t', where $s = 1, 2, \ldots, 16$, $t = s + 4 \pmod{16}$, as well as the corresponding edges. The result is the required map M' on T_{g+1} with $q_i'(M) = q_i(M)$ for all $i \neq 4, 6$, $q_4'(M') = q_4(M) - 6$, $q_6'(M) = q_6(M) - 4$.

3. Proof of Theorems 3 and 4

First we shall prove two lemmas.

Lemma 11. *Let $p = (p_i \mid 3 \leq i \neq 6)$ be a sequence of nonnegative integers satisfying condition* (2) *with some g, $g \geq 0$, and condition*

$$p_5 \geq 4, \text{ or } p_5 \geq 2 \text{ and } p_4 \geq 1. \tag{4}$$

Then there exists a number d such that $P(p, g)$ contains all integers $\geq d$.

Proof. First we shall prove the lemma for those sequences $p = (p_i \mid 3 \leq i \neq 6)$ for which

$$3p_3 + 2p_4 + p_5 \neq 5. \tag{5}$$

Put

$$j = 6 + \sum_{i \geq 7} (i-6)\,p_i = 6k + r \tag{6}$$

where k, r are nonnegative integers such that $0 \leq r \leq 5$.

We shall now consider the following nine cases:

1. $r = 0$	2. $r = 1$ and $p_3 \neq 0$	3. $r = 1$ and $p_3 = 0$
4. $r = 2$	5. $r = 3$ and $p_3 \neq 0$	6. $r = 3$ and $p_3 = 0$
7. $r = 4$	8. $r = 5$ and $p_3 \neq 0$	9. $r = 5$ and $p_3 = 0$

Note that by (2) and (6) p_5 is odd in the cases 3, 6 and 9.

We shall prove Lemma 11 for one case only; for the other eight cases the proof is analogous and we shall limit our consideration of these cases to the numbers of lemmas which must be used.

Consider, e. g., case 2, i. e. $r = 1$, $p_3 \neq 0$. The proof of the existence of the corresponding maps starts with the map M_0 shown in Fig. 11a, where $M_0 =$

$M(q^0, v^0, 0, 1, 0, 1)$. This map contains a pair of adjacent quadrangles face-disjoint from C_6 and A_6 configurations. Its face vector is $q^0 = (q_i^0 \mid q_i^0 = 0$ for all $i \neq 4, 6$; $q_4^0 = 6$, $q_6^0 = 10)$. We shall not write the vertex vector since we are dealing with trivalent maps and for them $v_3 = \frac{1}{3}\left(\sum_{i \geq 3} iq_i\right)$. By Lemma 1.2, since there exists the

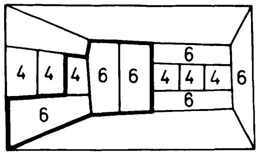

Fig. 11a

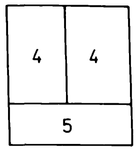

Fig. 11b

map M_0 (with its A_6 configuration), there also exists a map $M_1 = M(q^1, v^1, 0, 0, 0, k)$ such that $q^1 = (q_3^1 = 1, q_4^1 = 3k + 2, q_5^1 = 0, q_6^1, q_i^1 = p_i$ for all $i \geq 7)$. For our case

$$k = \frac{3p_3 + 2p_4 + p_5 + 12g - 7}{6} \tag{7}$$

by (2) and (6). Lemma 2.2 is now applied $\left[\frac{p_3 - 1}{2}\right]$ times to the map M_1; if p_3 is even (so that by (2) and (6) p_5 is odd), Lemma 2.3 is applied afterwards. We obtain a map $M_2 = M(q^2, v^2, 0, 0, 0, c^2)$ such that it contains a pair of adjacent quadrangles (outside any of the c^2 C_6 configurations) and if p_3 is odd (and therefore p_5 even), then $q^2 = (q_i^2 \mid q_3^2 = p_3, q_4^2 = 3c^2 + 2, q_5^2 = 0, q_6^2, q_i^2 = q_i^1 = p_i$ for all $i \geq 7)$ where $c^2 = \frac{1}{6}(2p_4 + p_5 + 12g - 4)$ or if p_3 is even (and therefore p_5 odd), then

$$q^2 = (q_i^2 \mid q_3^2 = p_3,\ q_4^2 = 3c^3 + 3,\ q_5^2 = 1,\ q_6^2,\ q_i^2 = p \text{ for all } i \geq 7)$$

and $c^2 = \frac{1}{6}(2p_4 + p_5 + 12g - 7)$.

The ε-transformation is now used on the map M_2, i. e. every of its edges is replaced by a hexagon. This gives us a map $M_3 = \varepsilon(M_2) = M(q^3, v^3, 0, 0, 0, 0)$ which contains one F configuration obtained from the pair of adjacent quadrangles which have been contained in M_2 and c^2 $\varepsilon(C_6)$ configurations formed from the C_6 configurations of M_2. The face vector of M_3 is
$q^3 = (q_i^3 \mid q_i^3 = q_i^2$ for all $i \geq 3$, $i \neq 6$, $q_6^3 = d^3$, where d^3 is a constant).

If $p_5 \geq 4$ (or $p_5 \geq 2$ and $p_4 \geq 1$), we change M_3 further by applying Lemma 6.1 (or 6.2, respectively) to its F configuration and Lemma 5 to one of its E configurations. This gives us a map $M_4 = M(q^4, v^4, 0, 0, 0, 0)$ with c^2 $\varepsilon(C_6)$ configurations whose face vector is

$q^4 = (q_i^4 \mid q_i^4 = p_i$ for all $i \neq 4, 5, 6$, $i \geq 3$; $q_4^4 = q_4^3 - 2$ (or $q_4^4 = q_4^3 - 1$);

$q_5^4=q_5^3+4$ (or $q_5^4=q_5^3+2$, respectively); $q_6^4=d^4+t$, $t=0, 1, 2, \dots$; d^4 is a constant).

Applying Lemma 10 g times to g pairs of E configurations in M_4 (they are contained in the $\varepsilon(C_6)$ configurations — cf. Fig. 4b and Fig. 5b) leads to a map $M_5=M(q^5, v^5, g, 0, 0, 0)$ on T_g which contains c^2-2g $\varepsilon(C_6)$ configurations and whose face vector is
$q^5=(q_i^5 \mid q_i^5=p_i$ for all $i\geqslant 3$, $i\neq 4, 5, 6$; $q_4^5=q_4^4-6g$, $q_5^5=q_5^4$, $q_6^5=d^5+t$, $t=0, 1, 2, \dots$, d^5 is a constant).

Note that three pairwise face-disjoint G configurations are contained in an $\varepsilon(C_6)$ configuration. Applying Lemma 7 to $\frac{p_5-q_5^5}{2}$ G configurations of M_5, we obtain a map $M_6=M(q, v, g, 0, 0, 0)$ where $q=(q_i \mid q_i=p_i$ for all $i\geqslant 3$, $i\neq 6$, $q_6=d+t$, $t=0, 1, 2, \dots$; d is a constant). This map satisfies the statement of our Lemma 11.

The remaining eight cases are handled analogously. To prove the existence of a map with the required properties we start with the map shown in Fig. 11a using its A_6 configuration and a pair of adjacent quadrangles. For the case α, $\alpha\in\{1, 3, 4, \dots, 9\}$, we use Lemma 1. α. In all cases we continue by applying Lemma 2.2, then at most one of Lemmas 2.3 and 3, the ε-transformation, Lemma 6.1 (or 6.2 if $p_5\geqslant 2$ and $p_4\geqslant 1$), Lemma 5, Lemma 10 and finally Lemma 7.

Suppose that $3p_3+2p_4+p_5=5$. To prove the existence of a map with the required properties we again start with the map in Fig. 11a, regarding it now as a map with two C_6 configurations, i. e. a map $M_0=M(q^0, v^0, 0, 0, 0, 2)$ where

$$q^0=(q_i^0 \mid q_i^0=0 \text{ for all } i\neq 4, 6;\ q_4^0=6;\ q_6^0=10).$$

Applying Lemma 1.27, we obtain the map $M_1=M(q^1, v^1, 0, 0, 0, 2g)$ such that $q^1=(q_i^1 \mid q_i^1=p_i$ for all $i\geqslant 3$, $i\neq 4, 5, 6$; $q_4^1=6g+2$, $q_5^1=1$, $q_6^1=d^1$, where d^1 is a constant). This map will also contain a configuration such as that shown in Fig. 11b (cf. [9]). Replacing every edge of M_1 by a hexagon (i. e. by using ε-transformation), we obtain a map $M_2=M(q^2, v^2, 0, 0, 0, 0)$ with $2g$ E configurations and one F configuration and with $q^2=(q_i^2 \mid q_i^2=q_i^1$ for all $i\geqslant 3$, $i\neq 6$, $q_6^2)$. Using Lemma 6.1 (or 6.2) together with Lemma 5 and then Lemma 10 (g times), we obtain the desired map.

Lemma 12. *Consider an integer $g\geqslant 1$ and a sequence of nonnegative integers $(p_i \mid 3\leqslant i\neq 6)$ satisfying (2) and the conditions*
(i) *$p_5=0, 1$, or $p_5=2, 3$ and $p_4=0$*
(ii) *if $g=1$, then the sequence does not satisfy the condition*

$$\sum_{3\leqslant i\neq 6} p_i=2.$$

Then there exists a constant d such that $(P(p, g)$ contains every integer $\geqslant d$.

Proof. To prove our Lemma it is necessary to consider a large number of possibilities. We shall limit our description to the construction of the initial maps M_0 to which we then apply Lemmas 1.1 to 10 described above. The map $M_0 = M(q^0, v^0, 0, a^0, b^0, c^0)$ will contain two face-disjoint D configurations and will be such that $a^0 + b^0 \leqslant 1$, $q^0 = (q_i^0 | q_i^0 \leqslant p_i$ for all $i \geqslant 7$, $q_3^0 \leqslant p_3$, $q_5^0 \leqslant p_5$, $q_6^0 = d^0$ where d^0 is a constant, $q_4^0 = 6 + \frac{1}{2} \sum_{3 \leqslant i \neq 4} (i-6) q_i^0 \Big)$.

To this map M_0 we apply a suitable Lemma 1. α, $\alpha = 1, 2, \ldots, 27$ in such a way that for the elements of the sequence $(f_i | i \geqslant 7)$ mentioned in these lemmas $f_i = p_i - q_i^0$, $i \geqslant 7$. We obtain a map M_1 to which Lemma 4 is applied. For the resulting map $M_2 = M(q^2, v^2, 0, a^2, b^2, c^2)$ we have $q^2 = \Big(q_i^2 | q_i^2 = p_i$ for all $i \geqslant 7$, $q_3^2 \leqslant p_3$, $q_5^2 \leqslant p_5$, $q_6^2 = d^2 + t$, $t = 0, 1, 2, \ldots$; d^2 is a constant, $q_4^2 = 6 + \frac{1}{2} \sum_{3 \leqslant i \neq 4} (i-6) q_i^2 \Big)$. Applying Lemma 8 and then Lemma 9 $g-1$ times leads to a map $M_3 = M(q^3, v^3, g, a^3, b^3, c^3)$ on $T_g \cdot \left[\frac{p_3}{2}\right]$ applications of Lemma 2.2 and an application of at most one of the Lemmas 2.3, 2.4, 3 and 7 leads to a map on T_g which will have the required properties. The whole method of proof is analogous to that for Lemma 11.

Consider the following eight cases:

1. $p_5 = 0$, $p_4 \geqslant 0$, $p_3 \equiv 0 \pmod 2$
2. $p_5 = 0$, $p_4 \geqslant 0$, $p_3 \equiv 1 \pmod 2$
3. $p_5 = 1$, $p_4 \geqslant 0$, $p_3 \equiv 0 \pmod 2$
4. $p_5 = 1$, $p_4 \geqslant 0$, $p_3 \equiv 1 \pmod 2$
5. $p_5 = 2$, $p_4 = 0$, $p_3 \equiv 0 \pmod 2$
6. $p_5 = 2$, $p_4 = 0$, $p_3 \equiv 1 \pmod 2$
7. $p_5 = 3$, $p_4 = 0$, $p_3 \equiv 0 \pmod 2$
8. $p_5 = 3$, $p_4 = 0$, $p_3 \equiv 1 \pmod 2$

Note that by (2) the number of odd-gonal faces with $\geqslant 7$ edges is even for cases 1, 4, 5 and 8 and odd for the remaining cases.

Case 1. Let $\sum_{i \geqslant 3} p_{2i+1} \neq 0$; therefore $\sum_{i \geqslant 3} p_{2i+1} \geqslant 2$. For $p_7 \neq 0$ we obtain the starting map from the map shown in Fig. 12a. This map contains a 7-gon and an A_7 configuration which we shall use to obtain another odd-gon with at least 7 edges by the elementary step method described in Chapter 2 of the present paper. If still another 7-gon is required, it is easy to find an A_6 configuration in the map of Fig. 12a. In this case the map in Fig. 12a is already the initial map M_0.

If $p_7 = 0$ and $\sum_{i \geqslant 2} p_{6i+1} \neq 0$, then the starting map is obtained from the map of Fig. 12b. This map contains C_{13} and A_7 configurations. These will be used to obtain the required pair of odd-gons in such a way that the C_{13} configuration is used to obtain the required $(6i+1)$-gon, $i \geqslant 2$, and the A_7 configuration for the other odd-gon.

Suppose that $\sum_{i\geqslant 1} p_{6i+1}=0$ and $\sum_{i\geqslant 1} p_{6i+3}\neq 0$. If $p_9\neq 0$, we start with the map shown in Fig. 12c, if $p_9=0$, then Fig. 12d shows the starting map.

If $\sum_{i\geqslant 1} p_{6i+1}=\sum_{i\geqslant 1} p_{6i+3}=0$, then $\sum_{i\geqslant 1} p_{6i+5}\geqslant 2$ and we start the construction of the initial map from the map in Fig. 12e. If no odd-gons with $\geqslant 7$ edges are required, then Fig. 12f shows the initial map, provided $p_{10}\neq 0$. If $\sum_{i\geqslant 7} p_{2i}\neq 0$, then we start

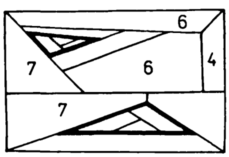

Fig. 12a

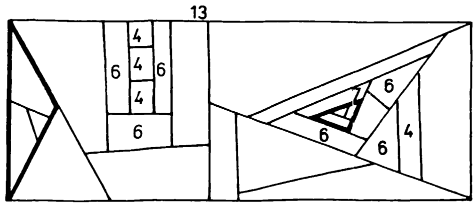

Fig. 12b

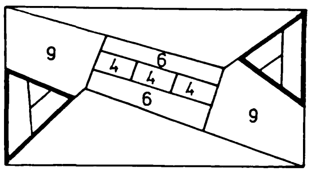

Fig. 12c

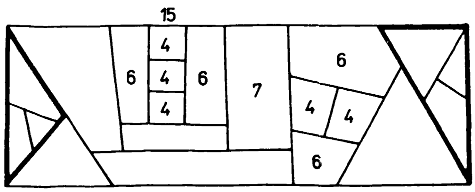

Fig. 12d

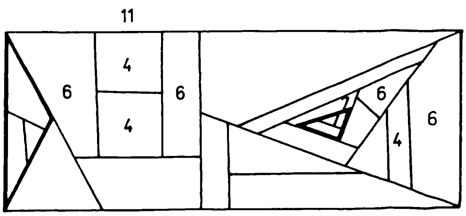

Fig. 12e

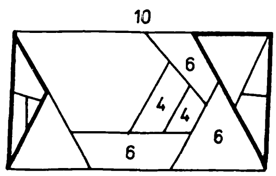

Fig. 12f

with the map in Fig. 13a. For $\sum_{5\leqslant i\neq 6} p_{2i}=0$ the condition (2) together with the conditions of the lemma allows only the following possibilities: $\sum_{i\geqslant 4} p_{2i}=0$ — the initial map is shown in Fig. 13b, for $p_8\geqslant 2$ we use Fig. 14a, for $p_8=1$, $p_{12}\geqslant 1$ Fig. 14b; for $p_8=0$, $p_{12}\geqslant 2$ Fig. 14c. The possiblity $p_8=0$, $p_{12}=1$ applies only to a map on T_1 with a sequence $p=(p_i|p_3=2,\ p_{12}=1,\ p_i=0$ for $i\neq 3, 6, 12)$. The initial map for this case is shown in Fig. 14d.

Cases 4, 5 and 8. With the exception of the possibility of $p_{12}=1$, $p_i=0$ for all $i \geqslant 7$, $i \neq 12$ the starting maps are the same as in the case 1. For cases 4 and 8 and the above possibility Figs. 15a and 15b, respectively, show the starting maps. Case 5 excludes the above possibility owing to (2) but allows the possibility of $p_8=1$, $p_i=0$ for all $i \geqslant 7$, $i \neq 8$ with Fig. 15c showing the starting map.

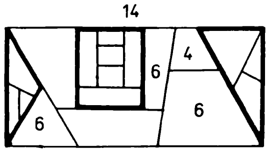

Fig. 13a

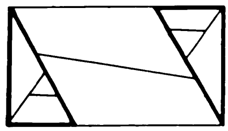

Fig. 13b

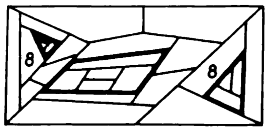

Fig. 14a

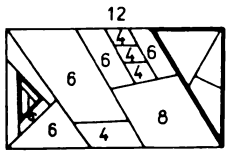

Fig. 14b

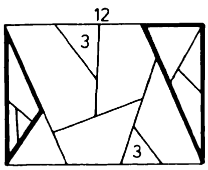

Fig. 14c

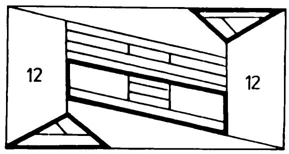

Fig. 14d

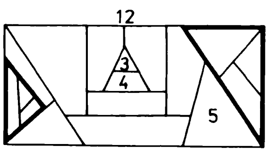

Fig. 15a

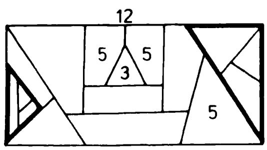

Fig. 15b

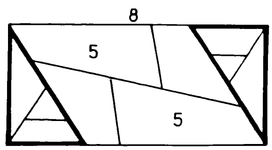

Fig. 15c

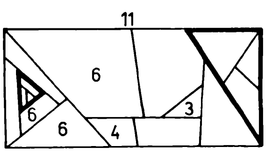

Fig. 16

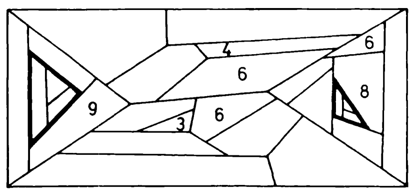

Fig. 17

Case 2. In this case we need an odd number of odd-gonal faces with ≥ 7 edges. If $\sum_{i \geq 5} p_{2i+1} \neq 0$, we use a map in Fig. 16 to obtain the starting map. The map shown contains two face-disjoint D configurations, one C_{11} (or C_6 if an 11-gon is required) and one of the necessary triangles. If $\sum_{i \geq 5} p_{2i+1} = 0$, $p_9 = 1$ and $\sum_{i \geq 4} p_{2i} = 0$, we start with a map in Fig. 17. If $p_9 \geq 2$, then Fig. 12c shows the starting map. The case $p_3 = p_9 = 1$, $p_i = 0$ for $i \neq 3, 9$ could occur for $q = 1$ but is excluded by the conditions. Fig. 12a will be used to obtain the initial map if $p_7 \geq 1$.

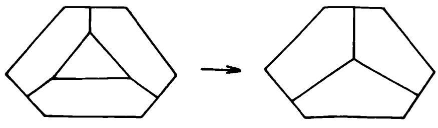

Fig. 18a Fig. 18b

Case 7. The construction of initial maps is analogous to the case 2 except for the case $p_9 = 1$. Every map used to obtain the starting maps in case 2 contains a triangle surrounded by hexagons as in Fig. 18a. Changing the structure of this configuration as shown in Fig. 18b, we obtain the necessary maps for case 7 except for the case $p_9 = 1$ and $p_i = 0$ for all $i \geq 7$, $i \neq 9$. Fig. 19 shows the maps necessary in this case.

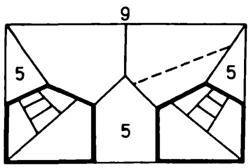

Fig. 19

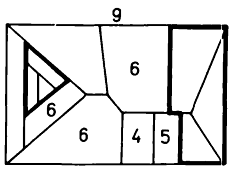

Fig. 20

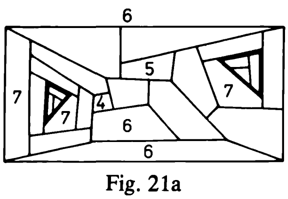

Fig. 21a

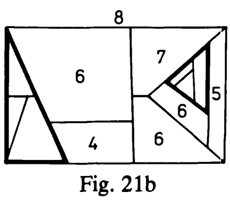

Fig. 21b

Case 3. Here again an odd number of odd-gons with ≥ 7 edges is necessary. If $\sum_{i \geq 4} p_{2i+1} \neq 0$, then the initial map is obtained from that shown in Fig. 20. If $\sum_{i \geq 4} p_{2i+1} = 0$, then $p_7 \neq 0$. If $p_7 \geq 3$, then the initial map is shown in Fig. 21a. If

$p_7=1$, then $\sum\limits_{i\geqslant 4} p_{2i}\neq 0$ since by [10] it is impossible that $p_5=p_7=1$, $p_i=0$ for all $i\neq 5, 6, 7$. The starting map for this case is shown in Fig. 21b.

Case 6. Here also $\sum\limits_{i\geqslant 3} p_{2i+1}\neq 0$. If $p_{2i+1}\neq 0$ for some $i\geqslant 6$, then the initial map is obtained from that shown in Fig. 22. If $p_{11}=1$ and $\sum\limits_{7\leqslant i\neq 11} p_i\neq 0$, then we use Fig. 16. If $p_{11}=1$ and $\sum\limits_{7\leqslant i\neq 11} p_i=0$, then Fig. 23 shows the starting map. If $\sum\limits_{i\geqslant 5} p_{2i+1}=0$ and $p_9\geqslant 1$, then we start with the map shown in Fig. 20. If $p_{2i+1}=0$ for all $i\geqslant 4$, then $p_7\neq 0$. If $p_7\neq 0$ and $p_{2i}\neq 0$ for some $i\geqslant 4$, then we start with the map in Fig. 21b; if $p_7\geqslant 3$, then with the map in Fig. 21a.

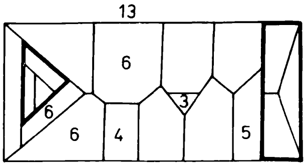

Fig. 22

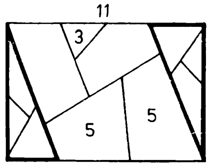

Fig. 23

Proof of Theorems 3 and 4. Theorem 3 is an immediate consequence of Lemmas 11 and 12. To complete the proof of Theorem 4, we must consider the sequences such that $\sum\limits_{3\leqslant i\neq 6} p_i=2$. This comprises the sequences

(i) $p_5=p_7=1$, $p_i=0$ for $i\neq 5, 6, 7$,
(ii) $p_4=p_8=1$, $p_i=0$ for $i\neq 4, 6, 8$,
(iii) $p_3=p_9=1$, $p_i=0$ for $i\neq 3, 6, 9$.

By [10] there exists no map on T_1 with one 5-gon, one 7-gon and hexagons. Thus $P(p, 1)$ is empty for the corresponding sequence. To prove that for sequences (ii) and (iii) $P(p, 1)$ contains all even numbers, starting with a certain number it is sufficient to start with maps in Figs. 24a and 24b, respectively, then apply Lemma 2.1 and finally Lemma 9.

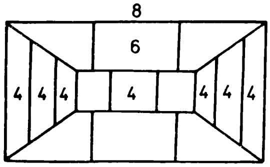

Fig. 24a

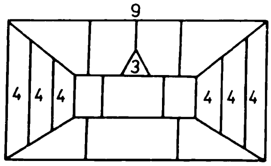

Fig. 24b

For (ii) we have also a decomposition of T_1 with an odd number of hexagons, but this decomposition is not a map. For (iii) we conjecture that there exists no decomposition of T_1 containing one triangle, one 9-gon and an odd number of hexagons.

Remark. The results in the present paper have been presented to the mathematical public in [8].

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Received June 21, 1984

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О ГРАНЕВЫХ ВЕКТОРАХ КАРТ С РЕГУЛЯРНЫМ ГРАФОМ ТРЕТЬЕЙ СТЕПЕНИ

Stanislav Jendroľ

Резюме

Пусть M — клеточный комплекс (карта) с регулярным графом третьей степени на ориентируемой поверхности рода g. Граневым вектором карты M называется последовательность $(p_i(M))$, где $p_i(M)$ — число граней, ограниченных i ребрами.

Каждой последовательности неотрицательных целых чисел $p=(p_i|3\leq i\neq 6)$ и неотрицательному числу g, удовлетворяющим следствию теоремы Эйлера (2), ставится в соответствие множество $P(p, g)$, где для любого $p_6\in P(p, g)$ последовательность p, доплненная p_6, является граневым вектором некоторого клеточного комплекса с графом третьей степени на ориентированной поверхности рода g. В работе найдены некоторые свойства множеств $P(p, g)$ для всех пар (p, g).