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# ON STATIONARY SETS FOR CERTAIN GENERALIZATIONS OF CONTINUITY

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Let $X$, $Y$ be two topological spaces. Let $\mathscr{F}$ be a class of functions $f: X \to Y$. A subset $A$ of $X$ with the property that whenever $f \in \mathscr{F}$ is constant on $A$, then $f$ must be constant on $X$, is said to be a stationary set for $\mathscr{F}$. Observe that if $\mathscr{F}_1 \subset \mathscr{F}_2$, then each stationary set for $\mathscr{F}_2$ is also a stationary set for $\mathscr{F}_1$. (See [2], p. 199.)

For the basic properties of stationary sets see [1]. A survey of results of stationary sets for derivatives is in [3].

In the present paper we give a complete characterization of stationary sets for certain generalizations of continuity (for somewhat continuous functions and quasi-continuous functions).

Let $X$, $Y$ be two topological spaces. A function $f: X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in $Y$ such that $f^{-1}(V) \neq \emptyset$ there exists a nonempty open set $U \subset X$ so that $U \subset f^{-1}(V)$ (see [5]).

A function $f: X \to Y$ is said to be quasi-continuous at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point $x_0$ (in $X$) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in $Y$) there exists a nonempty open set $U \subset U(x_0)$ such that $f(U) \subset V(f(x_0))$. A function $f: X \to Y$ is said to be quasi-continuous on $X$ if it is quasi-continuous at each point $x \in X$. (See [6] and [9].)

In the sequel $S(X, Y)$ and $Q(X, Y)$ denote the sets of all functions $f: X \to Y$ which are somewhat continuous and quasicontinuous on $X$, respectively.

In the paper it is supposed that $Y$ is a Hausdorff space which has at least two elements.

## 1. Stationary sets for the class of somewhat continuous functions

In the following theorem we give a characterization of the family of all stationary sets for the class of somewhat continuous functions.

**1.1. Theorem.** *Let $A$ be a nonempty subset of a topological space $X$. Then $A$ is a stationary set for the class $S(X, Y)$ if and only if every nonempty open subset of the set $X - \mathrm{Cl}\, A$ is dense in $X$.*

Proof. Sufficiency. By contradiction. Let every nonempty open subset of the set $X - \mathrm{Cl}\,A$ be dense in $X$. Let $f \in S(X, Y)$, $f(x) = a$ for $x \in A$, $f(b) \neq a$ for some $b \in X - A$. Choose open disjoint neighbourhoods $U$ and $V$ of the points a and $f(b)$, respectively. Then $G = \mathrm{Int}\,f^{-1}(U)$ and $H = \mathrm{Int}\,f^{-1}(V)$ are nonempty and disjoint. Since $A \subset X - H$ and $H$ is open, we have $H \subset X - \mathrm{Cl}\,A$. Then, by the assumption, $H$ is dense in $X$. Therefore $G \cap H \neq \emptyset$, a contradiction.

Necessity. Deny. Suppose that there exists a nonempty open set $W \subset X - \mathrm{Cl}\,A$ which is not dense in $X$. Choose $u, v \in Y$ such that $u \neq v$. Define the function $f\colon X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in W, \\ v & \text{otherwise.} \end{cases}$$

Then $f$ is somewhat continuous function which is constant on $A$, but $f$ is not constant on $X$. Hence the set $A$ is not stationary for the class $S(X, Y)$. The proof is complete.

**1.2. Remark.** Let $X$ be a topological space. Then every dense subset of $X$ is a stationary set for the class $S(X, Y)$.

**1.3. Definition.** An open almost-base for a space $X$ is a family $\mathscr{A}$ of open subsets of $X$ such that every nonempty open subset of $X$ contains some nonempty $A \in \mathscr{A}$ (see [4]).

**1.4. Theorem.** *Let $X$ be a topological space which is not antidiscrete. Then every stationary set for the class $S(X, Y)$ is dense in $X$ if and only if the family of all open subsets of $X$ which are not dense in $X$ is an almost-base for $X$.*

Proof. Sufficiency. Deny. Let $A$ be a stationary set for the class $S(X, Y)$ which is not dense in $X$. Evidently $X - \mathrm{Cl}\,A$ is a nonempty open set and by Theorem 1.1 we have that every nonempty open subset of the set $X - \mathrm{Cl}\,A$ is dense in $X$.

Necessity. Deny. Suppose that there exists a nonempty open set $U$ in $X$ such that every nonempty open subset of this set is dense in $X$. Let $V$ be a nonempty open proper subset of $X$. Since $U$ is dense in $X$, the set $W = U \cap V$ is nonempty. By Theorem 1.1 the set $X - W$ is stationary for the class $S(X, Y)$. But $X - W$ is not dense in $X$. The proof is complete.

By. 1.4 we have the following theorem.

**1.5. Theorem.** *Let $A$ be a subset of a Hausdorff space $X$. Then $A$ is a stationary set for the class $S(X, Y)$ if and only if $A$ is dense in $X$.*

**1.6. Definition.** A space $X$ is said to be hyperconnected if every nonempty open set is dense in $X$ (see [7]).

**1.7. Theorem.** *Let $X$ be a topological space. Then $X$ is hyperconnected if and only if each somewhat continuous function $f\colon X \to Y$ is constant on $X$.*

Proof. It is sufficient to take into account the following reason. If $X$ is not hyperconnected, then there is a set $A \subset X$ such that $\operatorname{Int} A \neq \emptyset \neq$ $\neq \operatorname{Int}(X - A)$.

**1.8. Corollary.** *Let $X$ be a hyperconnected space. Then every nonempty subset of $X$ is a stationary set for the class $S(X, Y)$.*

The following example shows that the assumption "Hausdorff space" in the Theorem 1.5 cannot be replaced by the assumption "$T_1$-space".

**1.9. Example.** Let $X$ be an infinite countably set with the cofinite topology. Evidently $X$ is a $T_1$-space. Since $X$ is hyperconnected, by 1.8 each nonempty finite set is a stationary set for the class $S(X, Y)$, but it is not dense in $X$.

## 2. Stationary sets for the class of quasi-continuous functions

**2.1. Definition.** Let $A$ be a subset of a topological space $X$. The set $A$ is regular open if $A = \operatorname{Int} \operatorname{Cl} A$ (see [8]).

In the following theorem we give a characterization of the family of all stationary sets for the class of quasi-continuous functions.

**2.2. Theorem.** *Let $A$ be a nonempty subset of a topological space $X$. Then $A$ is a stationary set for the class $Q(X, Y)$* if *and only if the set $X - \operatorname{Cl} A$ has not a nonempty regular open subset.*

Proof. Sufficiency. By contradiction. Suppose that the set $X - \operatorname{Cl} A$ has not a nonempty regular open subset. Let $f \in Q(X, Y)$, $f(x) = a$ for $x \in A$, $f(b) \neq a$ for some $b \in X - A$. Choose open disjoint neighbourhoods $U$ and $V$ of the points $a$ and $f(b)$, respectively. Since $f$ is quasi-continuous at the point $b$, there exists a nonempty open set $W$ in $X$ such that $f(W) \subset V$. Since $\operatorname{Int} \operatorname{Cl} W$ is nonempty regular open, by the assumption we have $A \cap \operatorname{Int} \operatorname{Cl} W \neq \emptyset$. Choose a point $c$ in this intersection. Since $f$ is quasi-continuous at the point $c$, there exists a nonempty open set $G \subset \operatorname{Int} \operatorname{Cl} W$ such that $f(G) \subset U$. Since $G \subset Cl\, W$, we have $G \cap W \neq \emptyset$. Thus $\emptyset \neq f(G \cap W) \subset U \cap V = \emptyset$, a contradiction.

Necessity. Deny. Suppose that the set $X - \operatorname{Cl} A$ has a nonempty regular open subset $W$. Choose $u$, $v \in Y$ such that $u \neq v$. Define the function $f\colon X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in W, \\ v & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $f \in Q(X, Y)$. Evidently $f$ is constant on $A$, but it is not constant on $X$. The proof is complete.

In the following we give some corollaries of this theorem.

**2.3. Theorem.** *Let $X$ be a topological space. Then every stationary set for the class $Q(X, Y)$ is dense in $X$ if and only if the family of all regular open sets is an almost-base for $X$.*

Proof. Sufficiency. Let $A$ be a stationary set for the class $Q(X, Y)$. Then by Theorem 2.2 the set $X - \mathrm{Cl}\, A$ has not a nonempty regular open subset. Hence by hypothesis we have $X - \mathrm{Cl}\, A = \emptyset$. Thus $A$ is dense in $X$.

Necessity. Let $U$ be a nonempty open proper subset of $X$. Evidently the set $X - U$ is not dense in $X$, then by hypothesis $X - U$ is not stationary for the class $Q(X, Y)$. By Theorem 2.2 we have that the set $U = X - \mathrm{Cl}(X - U)$ has a nonempty regular open subset. Hence the desired property is ensured.

**2.4. Theorem.** *Let $A$ be a subset of a regular space $X$. Then $A$ is a stationary set for the class $Q(X, Y)$ if and only if $A$ is dense in $X$.*

Proof. Let $U$ be a nonempty open set. Take an arbitrary $u \in \mathrm{U}$. Since $X$ is regular, there exists an open set $V$ such that $u \in V \subset \mathrm{Cl}\, V \subset U$. Put $W = \mathrm{Int}\, \mathrm{Cl}\, V$. Then $W$ is a nonempty regular open set such that $W \subset U$. By 2.3 every stationary set for $Q(X, Y)$ is dense. The converse is trivial.

The following example shows that the assumption "regular space" in Theorem 2.4 cannot be replaced by the assumption "Hausdorff space".

**2.5. Example.** Let $X$ be the set of all real numbers. Denote by $D$ the set of all rational numbers. Define a topology $\tau$ on $X$ generated from the Euclidean topology on $R$ by the addition of all sets of the form $D \cap U$ where $U$ is an open set in the Euclidean topology on $R$. By Theorem 2.3 there exists a stationary set for the class $Q(X, Y)$ which is not dense in $X$, but $X$ is a Hausdorff space.

The following example shows that there exists a Hausdorff space $X$ such that the statement of Theorem 2.4 is true, but $X$ is not regular.

**2.6. Example.** Let $X$ be the set of all real numbers and $A = \{1/n; n = 1, 2, 3, \ldots\}$. Define a topology $\tau$ on $X$ by letting $G \in \tau$ if $G = U - B$ where $B \subset A$ and $U$ is an open set in the Euclidean topology on $R$. Then by Theorem 2.3 the desired property is ensured. But $X$ is a Hausdorff space which is not regular.

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## ОБ СТАЦИОНАРНЫХ МНОЖЕСТВАХ ДЛЯ НЕКОТОРЫХ ОБОБЩЕНИЙ НЕПРЕРЫВНОСТИ

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Резюме

В настоящей работе мы предлагаем характеризацию стационарных множеств для некоторых обобщений непрерывности (для классов немножко-непрерывных функций и квазинепрерывных функций).