Vikramaditya Singh

On the $|E, q|$ summation of the Fourier series

Mathematica Slovaca, Vol. 30 (1980), No. 1, 13--18

Persistent URL: http://dml.cz/dmlcz/128639

Math. Slovaca 30, 1980, No. 1, 13—18

ON THE $|E, q|$ SUMMATION OF THE FOURIER SERIES

VIKRAMADITYA SINGH

1. Definition *If*

$$\sum_{0}^{\infty}(q+1)^{-n}b_n = S,$$

where

$$b_n = \sum_{k=0}^{n}\binom{n}{k} q^{n-k}a_k \ (q \geq 0),$$

we say that the series $\sum_{0}^{\infty} a_n$ *is summable* (E, q) *to S.*

If

$$\sum_{0}^{\infty}(q+1)^{-n}b_n$$

is an absolutely convergent series, then the series is said to be summable $|E, q|$[1].

It is easy to see that an absolutely convergent series is summable $|E, q|$.

2. Let $f(t)$ be integrable L in $(-\pi, \pi)$, periodic with period 2π, and let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{2.1}$$

The allied series of (2.1) at $t = x$ is

$$\sum_{1}^{\infty}(b_n \cos nx - a_n \sin nx). \tag{2.2}$$

We write

$$\Phi(t) = \frac{1}{2}[f(x+t) + f(x-t)]$$

$$h(t) = \Phi(t) \log\log\frac{k}{t}, \quad k > \pi.$$

In 1968 Mohanty and Mohapatra[2] proved the following Theorems.

[1] Hardy [1]

[2] Mohanty and Mohapatra [3]

Theorem MM. *If $g(t)$ is of bounded variation in $(0, \delta)$, where $0<\delta<1$, then the series (2.1) is summable $|E, q|$ $(0<\delta<1)$ at $t=x$.*

Our object in this paper is to generalize the above Theorem MM by using a less strict condition. In fact we prove

Theorem. *If $h(t)$ is of bounded variation in $(0, \delta)$, where $0<\delta<1$, then the series (2.1) is summable $|E, q|$ $(0<q<1)$ at $t=x$.*

Notation.

$$P(n, t)=\int_0^t \left(\log\log\frac{k}{u}\right)^{-1}(1+q^2+2q\cos u)^{n/2}\cos n\left(\operatorname{arctg}\frac{\sin u}{q+\cos u}\right)du,$$

$$Q(n, t)=\int_t^\delta \left(\log\log\frac{k}{u}\right)^{-1}(1+q^2+2q\cos u)^{n/2}\cos n\left(\operatorname{arctg}\frac{\sin u}{q+\cos u}\right)du.$$

3. For the proof of our Theorem we need the following Lemma.

Lemma.

$$\sum_0^\infty (q+1)^{-n}|P(n, \delta)|<\infty.$$

Proof of the Lemma. It can be proved that the series $\sum_0^\infty M_n$, where $M_n = \int_0^\delta \left(\log\log\frac{k}{u}\right)^{-1}\cos nu\, du$, is summable $|E, q|$.

Integrating by parts we have for $n\geqslant 1$

$$\int_0^\delta \left(\log\log\frac{k}{u}\right)^{-1}\cos nu\, du =$$

$$=\left(\log\log\frac{k}{\delta}\right)^{-1} n^{-1}\sin n\delta-\frac{1}{n}\int_0^\delta\left(\log\log\frac{k}{u}\right)^{-2}\left(\log\frac{k}{u}\right)^{-1}\frac{\sin nu}{u}du=$$

$$=\left(\log\log\frac{k}{\delta}\right)^{-1} n^{-1}\sin n\delta+[O\{n^{-1}(\log n)^{-1}(\log\log n)^{-2}\}]^{3)}.$$

Thus

$$(\sin n\delta)(n+1)^{-1}\left(\log\log\frac{k}{\delta}\right)^{-1}+O\{n^{-1}(\log n)^{-1}(\log\log n)^{-2}\}=$$

$$=M_n^1+O\{n^{-1}(\cos n)^{-1}(\log\log n)^{-2}\}.$$

To prove that ΣM_n^1 is summable $|E, q|$, we need only to show that

$$\sum_0^\infty (q+1)^{-n}\left|\sum_0^n \binom{n}{k} q^{n-k}(\sin k\delta)(k+1)^{-1}\right|<\infty$$

[3] Haslam Jones [2]

Now

$$\sum_{0}^{\infty}(q+1)^{-n}\left|\sum_{0}^{n}\binom{n}{k} q^{n-k} (\sin k\delta)(k+1)^{-1}\right| =$$

$$= \sum_{0}^{\infty}(q+1)^{-n}\left|(n+1)^{-1}\left\{(1+q^2+2q\cos\delta)^{n/2}\sin n\left[\left(\operatorname{arctg}\frac{\sin\delta}{q+\cos\delta}\right)-\delta\right]+\right.\right.$$

$$\left.\left. + q^{n+1}\sin\delta\right\}\right| \overset{4)}{\leq} \sum_{0}^{\infty}(q+1)^{-n}(n+1)^{-1}(1+q^2+2q\cos\delta)^{n/2}+$$

$$+\sum_{0}^{\infty}(q+1)^{-n}q^{n+1}(n+1)^{-1}=$$

$$=\sum_{0}^{\infty}(n+1)^{-1}\left[1-\frac{4q}{(1+q)^2}\sin\frac{\delta}{2}\right]^{n/2}+q\sum_{0}^{\infty}\left(\frac{q}{q+1}\right)^n(n+1)^{-1}<\infty.$$

Hence $\sum_{0}^{\infty}M_n$ is summable $|E, q|$, i.e.

$$\sum_{0}^{\infty}(q+1)^{-n}\left|\int_0^\delta\left(\log\log\frac{k}{u}\right)^{-1}(1+q^2+2q\cos u)^{n/2}\cos n\cdot\right.$$

$$\left.\cdot\left(\operatorname{arctg}\frac{\sin u}{q+\cos u}\right)du\right| = \sum_{0}^{\infty}(q+1)^{-n}|P(n,\delta)|<\infty,$$

which proves the Lemma.

4. It will be helpful in proving the Theorem to use the following inequalities, satisfied by the function defined in § 2. These can be obtained easily by applying the Second Mean Value Theorem:

(4.1) $$P(n,t)=O\left\{(q+1)^n\left(\log\log\frac{k}{t}\right)^{-1}n^{-1}\right\}$$

(4.2) $$Q(n,t)=O\{n^{-1}(1+q^2+2q\cos t)^{n/2}\}.$$

5. Proof of the Theorem.

We have

$$A_n(x)=\frac{2}{\pi}\int_0^\pi \Phi(t)\cos nt\,dt$$

$$=\frac{2}{\pi}\int_0^\delta \Phi(t)\cos nt\,dt+\frac{2}{\pi}\int_\delta^\pi \Phi(t)\cos n\,dt$$

$$=R_n+S_n,\quad \text{say.}$$

$\sum_{0}^{\infty}S_n$ is summable $|E, q|$ if

[4] Mohanty and Mohapatra [3]

$$(5.1)\quad \sum_0^\infty (q+1)^{-n}\left|\int_\delta^\pi \Phi(t)(1+q^2+2q\cos t)^{n/2}\cos\left(\operatorname{arctg}\frac{\sin u}{q+\cos u}\right)du\right|<\infty.$$

The expression on the left-hand side of (5.1)

$$\leq \sum_0^\infty (q+1)^{-n}\int_\delta^\pi |\Phi(t)|(1+q^2+2q\cos t)^{n/2}dt$$

$$=\int_\delta^\pi |\Phi(t)|\,dt\sum_0^\infty (q+1)^{-n}(q+1)^n\left[1-\frac{4q}{(1+q)^2}\sin^2 t-2\right]^{n/2}$$

$$=\int_\delta^\pi |\Phi(t)|\,dt\sum_0^\infty \left[1-\frac{4q}{(1+q)^2}\sin^2 t/2\right]^{n/2}$$

$$=\int_\delta^\pi |\Phi(t)|\,dt\sum_0^\infty [1-\sin^2\tau/2]^{n/2},$$

where $\sin\frac{\tau}{2}=\frac{2\sqrt{q}}{(1+q)}\sin t/2$

$$(5.2)\qquad =\int_\delta^\pi \frac{|\Phi(t)|\,dt}{t^2}{}^{5)}$$

Thus $\sum_0^\infty S_n$ is summable $|E, q|$

$$R_n=\frac{2}{\pi}\int_0^\delta \Phi(t)\cos nt\,dt.$$

$$=\frac{2}{\pi}\int_0^\delta h(t)\left(\log\log\frac{k}{t}\right)^{-1}\cos nt\,dt$$

$$=\frac{2}{\pi}\left[h(t)\int_0^t\left(\log\log\frac{k}{u}\right)^{-1}\cos nu\,du\right]_0^\delta-$$

$$-\frac{2}{\pi}\int_0^\delta dh(t)\int_0^t\left(\log\log\frac{k}{u}\right)^{-1}\cos nu\,du$$

$$=\frac{2}{\pi}h(\delta)\int_0^\delta\left(\log\log\frac{k}{u}\right)^{-1}\cos nu\,du-$$

$$-\frac{2}{\pi}\int_0^\delta dh(t)\int_0^t\left(\log\log\frac{k}{u}\right)^{-1}\cos nu\,du$$

$$=\frac{2}{\pi}h(\delta)R_n^1-R_n^{11}.$$

5) Singh [4]

R_n^{1} is summable $|E, q|$ by the Lemma above. R_n^{11} is summable $|E, q|$ if

$$\begin{aligned} I &= \sum_0^\infty (q+1)^{-n} \left| \int_0^\delta \mathrm{d}h(t) P(n,t) \right| < \infty \\ &\leqslant \sum_0^\infty (q+1)^{-n} \int_0^\delta |\mathrm{d}h(t)|\ |P(n,t)| \\ &= \int_0^\delta |\mathrm{d}h(t)| \sum_0^\infty (q+1)^{-n} |P(n,t)|. \end{aligned}$$

Since $\int_0^\delta |\mathrm{d}h(t)|$ is finite, it is enough to show that

$$\sum_0^\infty (q+1)^{-n} |P(n,t)| < \infty$$

Let $m = \left[\frac{1}{t^2}\right]$ writing

$$\sum_0^\infty (q+1)^{-n} |P(n,t)| = \sum_0^{m-1} (q+1)^{-n} |P(n,t) + \sum_m^\infty (q+1)^{-n} |P(n,t)|.$$

We have using (4.1)

$$\begin{aligned} \sum_0^{m-1} (q+1)^{-n} |P(n,t)| &= O\left\{ \sum_0^{m-1} (q+1)^{-n} (q+1)^n n^{-1} \left(\log\log\frac{k}{t}\right)^{-1} \right\} \\ &= O\left\{ \left(\log\log\frac{k}{t}\right)^{-1} \sum_0^{m-1} \frac{1}{n} \right\} \\ &= O(1). \end{aligned}$$

Again

$$\begin{aligned} \sum_m^\infty (q+1)^{-n} |P(n,t)| &\leqslant \sum_m^\infty (q+1)^{-n} |P(n,\delta)| + \sum_m^\infty (q+1)^{-n} |Q(n,t)| \\ &< \sum_0^\infty (q+1)^{-n} |P(n,\delta)| + \sum_m^\infty (q+1)^{-n} |Q(n,t)|. \end{aligned}$$

Since $\sum_0^\infty (q+1)^{-n} |P(n,\delta)| < \infty$, by the Lemma above and using (4.2)

$$\sum_m^\infty (q+1)^{-n} |Q(n,t)| < A \sum_m^\infty (q+1)^{-n} n^{-1} (1+q^2+2q\cos t)^{n/2}$$

$$A m^{-1} \sum_0^\infty (q+1)^n (q+1)^{-n} \left[1 - \frac{4q}{(1+q)^2} \sin t/2\right]^{n/2}$$

$$Am^{-1}\sum_{0}^{\infty}\left[1-\frac{4q}{(1+q)^2}\sin^2 t/2\right]^{n/2}$$

$$=Am^{-1}\sum_{0}^{\infty}[1-\sin^2\tau/2]^{n/1}$$

where $\sin\tau/2=\dfrac{2\sqrt{q}}{(1+q)^2}\sin t/2$

$$=\frac{Am^{-1}}{1-\cos t/2}$$

$$=O(1).$$

This proves the Theorem.

The author takes this opportunity to express his deep gratitude to Dr. S. R. Sinha for his kind advice and encouragement during the preparation of this paper.

REFERENCES

[1] HARDY, G. H.: Divergent Series. Oxford, 1949.

[2] HASLAM JONES, U. S.: A note on the Fourier coefficient of unbounded functions. JLMS, 2, 1927, 151—154.

[3] MOHANTY, R. & MOHAPATRA, S.: On the $|E, q|$ Summation of Fourier series and its allied Series, J. Indian Math. Soc., 32, 1968.

[4] SINGH, V.: On the $|E, q|$ Summation of Derived Fourier Series its conjugate derived Fourier series, abstracted in Part III of the Proceedings of the Indian Science Session, Bangalore, 1971, pp. 15—16.

Received September 24, 1976

Department of Mathematics
University of Allahabad
Allahabad 2

ОБ $|E, q|$ СУММИРОВАНИИ РЯДА ФУРЬЕ

Викрамадитиня Синг

Резюме

В работе доказано достаточное условие для того чтобы (2.1) был $|E, q|$ суммируемый.