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K-CONVERGENCE ENTAILS ABSOLUTE K-CONVERGENCE IN QUASI-NORMED GROUPS

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ABSTRACT. "K-convergence" in a topological group entails absolute K-convergence in each of its continuous quasi-norms. This leads to improvements in matrix convergence and boundedness theorems proved by "hump" techniques.

The author of [S] has proposed making a survey of the manifold versions of the Uniform Boundedness Principle as derived by the "sliding hump" technique, on the basis of the "Antosik-Mikusinski Matrix Theorem". Although assembling a truly remarkable bibliography [S; pp. 197 209] on the subject, he omitted to cite or to take account of the references [FT], [T]. It is true that these deal with the non-commutative case, but even specialized they yield a more powerful guide in abstracting the essence of the divers derivations. This will be documented below, following an *ab initio* presentation of the commutative specialization of the source result from [FT].

The kernel of the "hump" method is to deduce that the diagonal of an infinite matrix (with entries from a topological group) converges to zero when each of the columns does and moreover can be "thinned out" to a submatrix whose rows "sum" to a convergent sequence. In the "A-M Matrix" method this summation is the usual one of convergence of the partial sums; what its proponents failed to notice is that when every subseries has a convergent subseries indeed, just one whose terms converge to zero then it has a subseries which sums absolutely in any given continuous quasi-norm (thus absolutely in a quasi-normed group). So there is no loss of generality in having the thinned out rows converge in this stronger sense. In [FT] the generality is pushed further by requiring only convergence of the absolute remainder sums under increase of the row index.

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1. Convergence

With this in view, the following two results will be seen to generalize, respectively, the "Antosik Diagonal Lemma" ([FPV; Corollary 1]) and the "Basic Matrix Theorem" ([FPV; Corollary 2]) by obtaining their conclusions under weaker hypotheses.

LEMMA. *Let x_{ij} be an infinite matrix from a (commutative) quasi-normed group whose columns converge to zero and every infinite subset of whose indices has, for every $\varepsilon > 0$, an infinite subset J of indices for which*

$$\limsup_{i\in J} \limsup_{\text{finite } F\uparrow J} \sup_{\text{finite } F'\subset J} \Big| \sum_{j\in F'\setminus F} x_{ij} \Big| < \varepsilon .$$

Then the diagonal x_{ii} converges to zero.

Remark. Since only elements in the same row are being added (and will be, in the proof) the result holds for matrices whose rows could come from different groups, each with its own quasi-norm (e.g. it includes [S; Theorem 2.2.8]).

P r o o f. The hypotheses are inherited by every "square" infinite submatrix: i.e. with diagonal contained in the original's diagonal. We'll choose these to have, for every m, subdiagonal horizontal sums, $\sum_{n<m} |y_{mn}|$, $< \varepsilon$ as well as superdiagonal, $\sup_{\text{finite } n>m} |\sum_n y_{mn}|$, $< \varepsilon$. Then it would not have been possible to have started with an infinite submatrix having all $|x_{ii}| \geq 3\varepsilon$.

Both inequalities can be attained by recursion: If y_{mm} has been chosen, choose for the next diagonal element an index i beyond the largest j in an F_m making $|\sum_{F\setminus F_m} y_{ij}| < \varepsilon$ for all finite $F \subset J$ and so large that $\sum_{n\leq m} |y_{in}| < \varepsilon$. □

COROLLARY. *Let x_{ij} be an infinite matrix from a topological group whose columns are Cauchy and every infinite subset of indices has, for every quasi-norm $|\,|$ and $\varepsilon > 0$, an infinite subset J for which*

$$\limsup_{i,i'\in J} \limsup_{\text{finite } F\uparrow J} \sup_{\text{finite } F'\subset J} \Big| \sum_{j\in F'\setminus F} x_{ij} - x_{i'j} \Big| < \varepsilon .$$

Then the columns are uniformly Cauchy.

P r o o f. If not, there would be sequences of strictly increasing indices j_n, $i_n < i'_n$ such that $|x_{i_n j_n} - x_{i'_n j_n}|$ was bounded away from zero. Apply the lemma to $x_{i_m j_n} - x_{i'_m j_n}$. □

The hypotheses do not entail any convergence along the rows. If for some infinite index set the rows converge to zero, then one can find (by recursion) an infinite subset for which they converge rapidly, i.e. are absolutely summable (for a given quasi-norm) and then any subsequence which sums does so absolutely. Finally, if these infinite sums are Cauchy, then (since the above lim sup will be a lim whose value is $\left|\sum_{j=1}^{\infty} x_{ij} - x_{i'j}\right|$) the displayed inequality is verified. Thus, the hypotheses of the Basic Matrix Theorem yield those of this Corollary.

2. Function spaces

For application to group-valued function spaces, two options are available. One could take the rows to be arguments and the columns functions — e.g. a sequence of continuous functions on a convergent sequence yields a matrix with convergent columns. These will converge uniformly when every subsequence (of the functions) has a subsequence whose large finite sums eventually reach a specified level of closeness. This includes (the essence of) the "Uniform Convergence Principles" ([S; pp. 25–26]).

The other option is to make the columns arguments and the rows functions. Convergence of the columns then comes to pointwise convergence of the sequence of functions; this will be uniform if every infinite subset (of arguments) contains another on whose large finite subsets the functions sum to pairs eventually within specified closeness.

There appears to be no "K-" version of this result in [S]: i.e. which would infer the uniformity of convergence under the stronger hypothesis that every subsequence has a subsequence on which the f_i sum to a Cauchy (in i) sequence — except implicitly in the "General Banach-Steinhaus Theorem" (to be discussed below).

This option is particularly applicable when the arguments are endowed with an "addition" which the functions preserve: i.e. the functions are homomorphisms to the group. A pointwise convergent sequence of such will converge uniformly under the above hypotheses, which may now be formulated alternatively as requiring the images of the sums of large finite subsets to be close.

3. Boundedness

As a consequence of Lemma and Remark:

Corollary. *A double sequence x_{ij}, from quasi-normed groups G_i, is bounded finally in i uniformly in j (hence bounded in i and j if also bounded in j for each i) if it is bounded in i for each j and each infinite set of j's contains* *an infinite* J *such that*

$$\limsup_{i\in J}\ \limsup_{F\uparrow J}\ \sup_{\text{finite } F'\subset J}\Big|\sum_{j\in F'\setminus F} x_{ij}\Big|_i < \varepsilon .$$

Proof. If there were strictly increasing sequences i_n, j_n for which $|x_{i_n j_n}|_{i_n} \uparrow \infty$ — hence $t_n|x_{i_n j_n}|_{i_n}$ not convergent to zero for some real $t_n \downarrow 0$ — then $x_{i_m j_n}$ quasi-normed by $t_m|\cdot|_{i_m}$ would contradict the Lemma. □

This justifies:

Let f_i be a sequence of bounded functions on a common domain S to (possibly varying) quasi-normed groups G_i. If $|f_i(\cdot)|_i$ is pointwise bounded and the sums of each f_i are finally in i bounded on large finite subsets of some infinite subset in every countable subset of S, then $|f_i(\cdot)|_i$ is uniformly bounded on S.

This yields a generalization to groups of the "General UBP" ([S; p. 42], [AS; p. 22]). Observe first that a subset of a TVS is bounded just when it is so for every continuous quasi-norm, hence it is enough to consider seminormed spaces, which are in particular quasi-normed groups. (In a "semiconvex TVS" ([S; p. 51]) the two notions of boundedness coincide). As seen above, a K-convergent sequence may be defined as one every subsequence of which contains an absolutely summable subsequence. If the f_i are continuous homomorphisms, they will send this on absolutely summable sequences in the G_i which sum to their bounded values on the sum in S.

Similarly, the previously developed conditions for uniform convergence of pointwise converging functions will yield the "General Banach-Steinhaus Theorem" ([S; p. 59], [AS; p. 30]) for groups: Convergence is determined by quasi-norms hence K-convergence of a sequence can be construed as absolute summability of arbitrarily sparse subsequences and so a pointwise converging sequence of continuous homomorphisms converges uniformly on a K-convergent sequence.

4. Uniform convergence for (possibly infinite) sums

From Cauchy convergence along the columns follows the same for the horizontal sums indexed by any finite set of indices; to have this uniform in these finite sets (hence, by passage to the limit, also in any infinite sets on which the rows sum) it suffices to strengthen the hypothesis of the Corollary, which serves to rule out non-convergence to zero of summed differences on increasing index pairs, so as to apply not just to individual x_{ij} but also to finite horizontal sums over non-overlapping index sets. (For group-valued homomorphisms this is not a strengthening.) Indeed, from a sequence of finite F's for which

$\left|\sum_{j\in F} x_{ij} - \sum_{j\in F} x_{i'j}\right| \geq 2\varepsilon$ for $i' > i \uparrow \infty$, one can extract a subsequence of non-overlapping subsets (by recursion: for the one following F take the part $> \max F$ of a finite G whose i is so large that all sums over $j's \leq \max F$ have all summed differences of norm $< \varepsilon$) with some difference $\geq \varepsilon$.

In a more general formalism, let f_j be group-valued functions on a domain containing the entries y_{ij} of the infinite matrix and formulate the above for the image matrix $f_j(y_{ij})$. There is uniformity also in multiple f_j's if the $\{f_j(y_{ij})\}$ are columnwise (i.e. for each j) Cauchy uniformly in them. This leads to a substantial extension of the "abstract Hahn-Schur Theorem" [S; 9.3.3] (columnwise multiplier uniformity follows from 9.3.(1)) — itself extending the preceding 8.1.1 and 8.2.6.

5. Supplementary remarks

Theorem 2.2.4, given a proof of a full page, and Theorem 3.2.7 follow immediately from the fact that quasi-norms are the same for elements and their negatives, and more generally their images under additive isometries. Theorem 3.2.9 has been lifted verbatim (including notation) from his Ref [BKL] without crediting. The connection between subseries and unordered convergence of series developed in 8.1 would be simpler and clearer if it were done for Cauchy convergence, for which they are immediately recognized as equivalent. This impacts on the formulations of Phillips' Lemma in 8.4 which can be freed from the requirement of sequential completeness by asserting uniformity just for existent sums, as in [T; Sect. 6, Theorem (2)] — which also establishes the result in a considerably more general setting.

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