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CONTROL OF A SYSTEM GOVERNED BY DIRICHLET PROBLEM

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ABSTRACT. In this article we shall prove uniqueness of the solution for the class of symbols considered in [BERZANSKI YU. M.: *Eigenfunction Expansion of Self-Adjoint Operators*. Transl. Math. Monogr. 17, Amer. Math. Soc., Providence, RI, 1968], [BEREZANSKI YU. M.: *Self-Adjoint Operators in Spaces of Functions of Infinitely Many Variables*. Transl. Math. Monogr. 63, Amer. Math. Soc., Providence, RI, 1986]. We shall consider the existence of solutions for operators with conditionally exponential convex functions with symbols satisfying some boundedness condition. In order to prove uniqueness we shall apply the localization procedure.

1. A class of pseudodifferential operators generating Dirichlet forms

In this section we recall some of our results [1]. For $1 \le j \le n$ let $a_j^2 \colon \mathbb{R} \to \mathbb{R}$ be a continuous conditionally exponential convex function, see [3], [4], [7], [8], and let $b_j \in L^\infty(\mathbb{R}^n)$ be independent of x_j. By $\tilde{x}_j$ we denote the $(n-1)$-tuple $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and we identify $\mathbb{R}^{n-1}$ with a subspace of $\mathbb{R}^n$. Thus, we shall write $b_j(\tilde{x}_j)$ instead of $b_j(x)$, $x \in \mathbb{R}^n$.

We consider the operator

$$L(x, \mathrm{D}) = \sum_{j=1}^{n} b_j(\tilde{x}_j) a_j^2(\mathrm{D}_j)$$

defined on $C_0^\infty(\mathbb{R}^n)$ by

$$L(x, \mathrm{D})u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{x \cdot \xi} \left(\sum_{j=1}^{n} b_j(\tilde{x}_j) a_j^2(\xi_j) \right) \tilde{u}(\xi)\, \mathrm{d}\xi\,, \tag{1.1}$$

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where

$$\tilde{u}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-x\cdot\xi}\, u(x)\ \mathrm{d}x\,.$$

The bilinear form B associated with $L(x,\mathrm{D})$ is given on $C_0^\infty(\mathbb{R}^n)$ by

$$\begin{aligned} B(u,v) &= \big(L(x,\mathrm{D})u,v\big)_0 \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(\tilde{x}_j)a_j(\mathrm{D}_j)u(x)\cdot a_j(\mathrm{D}_j)v(x)\ \mathrm{d}x\,. \end{aligned} \tag{1.2}$$

Let us define $a^2\colon \mathbb{R}^n \to R$ by

$$a^2(\xi) = \sum_{j=1}^{n} a_j^2(\xi)\,.$$

Thus a^2 is a continuous conditionally exponential convex function on $\mathbb{R}^n$.

Let us introduce some Sobolev spaces related to a continuous conditionally exponential convex function $a^2\colon \mathbb{R}^n \to R$ and $S \geq 0$ a real number. We define $H^{a^2,S}(\mathbb{R}^n)$ by

$$H^{a^2,S}(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n):\ \int_{\mathbb{R}^n} \big(1+a^2(\xi)\big)^{2S}|\tilde{u}(\xi)|^2\ \mathrm{d}\xi < \infty\right\}. \tag{1.3}$$

On $H^{a^2,S}(\mathbb{R}^n)$ we have the norm

$$\|u\|^2_{S,a^2} = \int_{\mathbb{R}^n} \big(1+a^2(\xi)\big)^{2S}|\tilde{u}(\xi)|^2\ \mathrm{d}\xi\,. \tag{1.4}$$

With this norm the space $H^{a^2,S}(\mathbb{R}^n)$ is a Hilbert space and $C_0^\infty(\mathbb{R}^n)$ is a dense subspace.

Moreover by [6], we can construct a chain

$$H^{S,a^2}(\mathbb{R}^n) \subseteq L_2(\mathbb{R}^n) \subseteq H^{-S,a^2}(\mathbb{R}^n)\,. \tag{1.5}$$

In [1] we have obtained the following:

THEOREM 1.1. *Suppose that $L(x,\mathrm{D})$ is given as above and with some $d_0 > 0$ we have*

$$b_j(\tilde{x}_j) \geq d_0 \qquad \text{for all} \quad j = 1,\dots,n\,, \tag{1.6}$$

then for all $u,v \in C_0^\infty(\mathbb{R}^n)$ the following estimates hold

$$|B(u,v)| \leq C\|u\|_{\frac{1}{2},a^2}\|v\|_{\frac{1}{2},a^2}\,, \tag{1.7}$$

$$B(u,u) \geq 0\,, \tag{1.8}$$

$$B(u,u) \geq d_0\|u\|^2_{\frac{1}{2},a^2} - d_0\|u\|^2_0\,, \tag{1.9}$$

and

$$B(v,v) \le B(u,u) \qquad \textit{for any} \quad v = (0 \vee u) \wedge 1 \in D(B)\,. \tag{1.10}$$

The proof of this theorem is given in [1], [2].

We introduce the Friedrichs mollifier $J_\varepsilon \colon L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$, $\varepsilon > 0$, defined by $J_\varepsilon u = j_\varepsilon * u$ where

$$j_\varepsilon = \varepsilon^{-n} j\big(\tfrac{x}{\varepsilon}\big)\,, \qquad x \in \mathbb{R}^n\,,$$

and

$$j(x) = \begin{cases} C_0 \exp\big(|x^2| - 1\big)^{-1} & \text{for } |x| < 1\,, \\ 0 & \text{for } |x| \ge 1 \end{cases} \tag{1.11}$$

and C_0 is chosen such that $\int_{\mathbb{R}^n} j(x)\,\mathrm{d}x = 1$.

Obviously, we have:

a) For $u \in L^2(\mathbb{R}^n)$ we have

$$\|J_\varepsilon(u)\|_0 \le \|u\|_0 \tag{1.12}$$

and

$$\lim_{\varepsilon \to 0} \|J_\varepsilon(u) - u\|_0 = 0\,. \tag{1.13}$$

b) Let $a^2 \colon \mathbb{R}^n \to \mathbb{R}$ be a continuous conditionally exponential convex function and $u \in H^{a^2,S}$. Then it follows that

$$\|J_\varepsilon(u)\|_{S,a^2} \le \|u\|_{S,a^2} \tag{1.14}$$

and for some $S \ge 0$ we have $\|J_\varepsilon(u)\|_{S,a^2} \le C$ with a constant C independent of ε.

Now, let $L \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for fixed $x \in \mathbb{R}^n$ the function $L(x,\cdot) \colon \mathbb{R}^n \to \mathbb{R}$ is conditionally exponential convex.

Further suppose that for some $x_0 \in \mathbb{R}^n$ we have

$$\begin{aligned} L(x,\xi) &= L(x_0,\xi) + \big(L(x,\xi) - L(x_0,\xi)\big) \\ &= L_1(\xi) + L_2(x,\xi) \end{aligned}$$

where L_1 and L_2 satisfy the following assumptions:

L.1. There exists a continuous conditionally exponential convex function $a^2 \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$|L_1(\xi)| \le \gamma_1\big(1 + a^2(\xi)\big) \tag{1.15}$$

holds for all $\xi \in \mathbb{R}^n$.

L.2. Let a^2 be as in L.1. For some $q \in \mathbb{N}$ the function $L_2(\cdot,\xi)\colon \mathbb{R}^n \to \mathbb{R}$ is q-times continuously differentiable and that for any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le q$, there exists a function $Q_\alpha \in L^2(\mathbb{R}^n)$ such that

$$|\partial_x^\alpha L_2(x,\xi)| \le Q_\alpha(x)\big(1 + a^2(\xi)\big) \tag{1.16}$$

$\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$, where $\partial_j = \frac{\partial}{\partial x_j}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1!\dots\alpha_n!$.

L.3. For all $\xi \in \mathbb{R}^n$, $|\xi| \ge \sigma \ge 0$,

$$L_1(\xi) \ge \gamma_0 a^2(\xi)\,. \tag{1.17}$$

L.4. Define

$$\gamma_b = C_a\tilde{\gamma}_q \sum_{|\alpha|\le q} \|Q_\alpha\|_{L^1} \int_{\mathbb{R}^n} \big(1 + |\tau|^2\big)^{(1-q)/2}\, \mathrm{d}\tau\,, \tag{1.18}$$

$q > n+1$, where $\tilde{\gamma}_q$ is a constant such that

$$\big(1 + |\xi|^2\big)^{q/2} \le \tilde{\gamma}_q \sum_{|\alpha|\le q} |\xi^\alpha|\,.$$

Then for some ε, $0 < \varepsilon < 1$, we require

$$\gamma_b \le (1-\varepsilon)\gamma_0\,.$$

In [2] we proved that: If L satisfies L.1 – L.4 with q sufficiently large, then we have for all $u \in H^{a^2,S+1}(\mathbb{R}^n)$ and $t > 0$

(i) $\|L(x,\mathrm{D})\|_{t,a^2} \le C\|u\|_{S+1,a^2}$;

(ii) if $B(u,v) = \big(L(x,\mathrm{D})u,v\big)_0$, then we have for all $u,v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$

$$|B(u,v)| \le C\|u\|_{\frac{1}{2},a^2}\|v\|_{\frac{1}{2},a^2}$$

and

(iii) $B(u,u) \ge \varepsilon\gamma_0\|u\|^2_{\frac{1}{2},a^2} - C_0\|u\|_0^2$.

Beside the operator $L(x,\mathrm{D})$ we also will often consider the operator $L^\lambda(x,\mathrm{D}) = L(x,\mathrm{D}) + \lambda$, $\lambda \in \mathbb{R}$.

2. On weak solutions of $L^\lambda(x,\mathrm{D})u = f$

We want to solve the equation

$$L^\lambda(x,\mathrm{D})u = f\,, \qquad f \in L^2(\mathbb{R}^n)\,, \tag{2.1}$$

where $L^\lambda(x,\mathrm{D}) = L(x,\mathrm{D}) + \lambda$, $\lambda \in \mathbb{R}$ and $L(x,\mathrm{D})$ fulfills L.1 – L.4 with q sufficiently large.

DEFINITION 2.1. We say that $u \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$ is a *weak solution to* (2.1) if

$$B^\lambda(u,v) = B(u,v) + \lambda(u,v)_0 = (f,v)_0 \tag{2.1}$$

holds for all $v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$

PROPOSITION 2.1. *The operator $L^\lambda(x,\mathrm{D})$, $\lambda \in \mathbb{R}$, has a continuous extension onto $H^{a^2,1}(\mathbb{R}^n)$, i.e $L^\lambda(x,\mathrm{D})\colon H^{a^2,1}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a continuous operator.*

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$ we find, using that b_j is bounded and continuous,

$$\begin{aligned} \|L^\lambda(x,\mathrm{D})u\|_0 &\le \left\| \sum_{j=1}^n b_j(\cdot)a_j^2(\mathrm{D}_j)u \right\|_0 + |\lambda|\|u\|_0 \\ &\le C'\|u\|_{1,a^2}. \end{aligned}$$

□

COROLLARY 2.1. *Let us consider $L^\lambda(x,\mathrm{D})$ as an operator on $L^2(\mathbb{R}^n)$ with domain $H^{a^2,1}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Then $L^\lambda(x,\mathrm{D})$ is a closed operator.*

The bilinear form associated with $L^\lambda(x,\mathrm{D})$ is denoted by B_λ.

Obviously (1.7) holds for B_λ, $\lambda \in \mathbb{R}$. Thus B_λ has a continuous extension on $H^{a^2,1}(\mathbb{R}^n)$, which is again denoted by B_λ.

Also for all $u \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$ and a constant d_0 we have

$$\begin{aligned} B_\lambda(u,u) &= \big(L^\lambda(x,\mathrm{D})u,u\big)_{L^2(\mathbb{R}^n)} = \big(L(x,\mathrm{D})u,u\big)_{L^2(\mathbb{R}^n)} + \lambda(u,u)_{L^2(\mathbb{R}^n)} \\ &\ge \sum_{j=1}^n \big(b_j(\tilde{x}_j)a(\mathrm{D}_j)u, a(\mathrm{D}_j)u\big)_{L^2(\mathbb{R}^n)} + \lambda(u,u)_{L^2(\mathbb{R}^n)} \\ &= \sum_{j=1}^n b_j(\tilde{x}_j)\|a(\mathrm{D}_j)u\|^2_{L^2(\mathbb{R}^n)} + \lambda\|u\|^2_{L^2(\mathbb{R}^n)} \\ &\ge d_0 \int_{\mathbb{R}^n} \sum_{j=1}^n a_j(\xi_j)|\tilde{u}(\xi)|^2 \,\mathrm{d}\xi + \lambda\|u\|^2_{L^2(\mathbb{R}^n)} \\ &= d_0\|u\|^2_{\frac{1}{2},a^2} + (\lambda - C_0)\|u\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.3}$$

3. Formulation of the problem

By (2.3) the bilinear form B_λ is continuous and coercive. Thus by Lax-Milgram Theorem ([9; p. 92]) for each $f \in L^2(\mathbb{R}^n)$ and $L^\lambda(x,\mathrm{D})$, $\lambda \in \mathbb{R}$, where

$L(x,\mathrm{D})$ satisfies the assumption above, we find a weak solution $u \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$ of the equation $L^\lambda(x,\mathrm{D})u = f$, i.e. there is a unique $u \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$ such that

$$B^\lambda(u,v) = (f,v)_0 \qquad \text{for all} \quad v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\,.$$

We may now formulate our theorem. For this purpose we use the statement of the control problem. The space $L^2(\mathbb{R}^n)$ being the space of controls and $L^\lambda(x,\mathrm{D})$ is given as in (2.1). For a control u the state of the system $y(u)$ is given by the solution of

$$\begin{aligned} &L^\lambda(x,\mathrm{D})y(u) = f + u\,, \qquad y(u) \in H_0^{a^2,1}(\mathbb{R}^n)\,,\\ &H_0^{a^2,1}(\mathbb{R}^n) = \{\phi:\ \phi \in H^{a^2,1}(\mathbb{R}^n)\,,\ \phi = 0 \ \text{on}\ \Gamma\} \end{aligned} \tag{3.1}$$

(hence $y(u) = 0$ on Γ).

We are also given an observation equation

$$Z(u) = y(u)\,.$$

Finally, we are given $N \in \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$, N is Hermitian positive definite, i.e

$$(Nu,u)_{L^2(\mathbb{R}^n)} \geq \gamma \|u\|^2_{L^2(\mathbb{R}^n)}\,. \tag{3.2}$$

With every control u we associate the cost function

$$\begin{aligned} J(u) &= \|y(u) - Z_d\|^2_{L^2(\mathbb{R}^n)} + (Nu,u)_{L^2(\mathbb{R}^n)}\\ &= \int\limits_{\mathbb{R}^n} \big(y(u) - Z_d\big)^2 \,\mathrm{d}x + (Nu,u)_{L^2(\mathbb{R}^n)} \end{aligned} \tag{3.3}$$

where Z_d is a given element in $L^2(\mathbb{R}^n)$.

Let U_{ad} (set of admissible controls) be a closed convex subset of $L^2(\mathbb{R}^n)$. The control problem is then to find $\inf\limits_{v \in U_{ad}} J(v)$.

THEOREM 3.1. *Assume that* (2.3) *holds, the cost function being given by* (3.3). *A necessary and sufficient condition for* $u \in L^2(\mathbb{R}^n)$ *to be an optimal control is that the following equations and inequalities are satisfied*

$$\begin{aligned} L^\lambda(x,\mathrm{D})y(u) &= f + u &&\text{in } \mathbb{R}^n\,,\quad y(u) = 0 \ \text{on}\ \Gamma\,,\\ L^\lambda p(u) &= y(u) - Z_d &&\text{in } \mathbb{R}^n\,,\quad p(u) = 0 \ \text{on}\ \Gamma\,,\\ \int\limits_{\mathbb{R}^n} \big(p(u) + Nu\big)(v-u)\,\mathrm{d}x &\geq 0 &&\text{for all}\quad u,v \in U_{\mathrm{ad}}\,, \end{aligned} \tag{3.4}$$

where $p(u)$ *is the adjoint state of* $y(u)$.

Proof. Let $J(u)$ be written in the form

$$J(u) = \big\|\big(y(u) - y(0)\big) + \big(y(0) - Z_d\big)\big\|^2_{L^2(\mathbb{R}^n)} + (Nu,u)_{L^2(\mathbb{R}^n)}\,. \tag{3.5}$$

If we set

$$\begin{aligned}\pi(u,v) &= \big(y(u)-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)} + (Nu,u)_{L^2(\mathbb{R}^n)}\,,\\ S(v) &= \big(Z_d-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)}\,,\end{aligned} \tag{3.6}$$

the form $\pi(u,v)$ is a continuous bilinear form and $S(v)$ is a continuous linear form on $L^2(\mathbb{R}^n)$.

So we have

$$\begin{aligned}J(v) &= \Big(\big(y(v)-y(0)\big)+\big(y(0)-Z_d\big), \big(y(v)-y(0)\big)+\big(y(0)-Z_d\big)\Big)_{L^2(\mathbb{R}^n)}\\ &\qquad + (Nv,v)_{L^2(\mathbb{R}^n)}\\ &= \big(y(v)-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)} + \big(y(v)-y(0), y(0)-Z_d\big)_{L^2(\mathbb{R}^n)}\\ &\qquad + \big(y(0)-Z_d, y(v)-y(0)\big)_{L^2(\mathbb{R}^n)} + \big(y(0)-Z_d, y(0)-Z_d\big)_{L^2(\mathbb{R}^n)}\\ &\qquad + (Nv,v)_{L^2(\mathbb{R}^n)}\\ &= \big(y(v)-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)} + (Nv,v)_{L^2(\mathbb{R}^n)}\\ &\qquad - \big(Z_d-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)} - \big(Z_d-y(0), y(v)-y(0)\big)_{L^2(\mathbb{R}^n)}\\ &\qquad + \big(y(0)-Z_d, y(0)-Z_d\big)_{L^2(\mathbb{R}^n)}\\ &= \pi(v,v) - 2S(v) + \|Z_d - y(0)\|^2_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

Since

$$\pi(v,v) = \|y(v) - y(0)\|^2_{L^2(\mathbb{R}^n)} + (Nv,v)_{L^n(\mathbb{R}^n)},$$

from (3.2) we have

$$\pi(v,v) \geq \gamma\|v\|^2\,, \qquad v \in L^2(\mathbb{R}^n)\,.$$

Therefore, we have reduced our problem to a form where Theorems of [5], [6] can be applied, i.e., there exists a unique element $u \in U_{\mathrm{ad}}$ such that

$$J(u) = \inf_{v\in U_{\mathrm{ad}}} J(v)$$

and this element is characterized by:

$$J'(u)(v-u) \geq 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,. \tag{3.7}$$

Since $L^\lambda(x,\mathrm{D})$ is a canonical isomorphism from $H_0^{a^2,1}(\mathbb{R}^n)$ onto $H_0^{a^2,-1}(\mathbb{R}^n)$, we may write

$$y(u) = (L^\lambda)^{-1}(f+u)$$

and hence

$$\begin{aligned} y'(u)\cdot(v-u) &= (L^\lambda)^{-1}(v-u) \\ &= (L^\lambda)^{-1}(f+v-f-u) \\ &= (L^\lambda)^{-1}\big((f+v)-(f+u)\big) \\ &= (L^\lambda)^{-1}(f+v)-(L^\lambda)^{-1}(f+u) \\ &= y(v)-y(u)\,. \end{aligned} \tag{3.8}$$

Since

$$J(u)(v-u) = 2\big[\pi(u,v{-}u) - S(v-u)\big]\,,$$

from (3.6) we have

$$\begin{aligned} \pi(u,v{-}u) &= \big(y(u){-}y(0), y(v{-}u){-}y(0)\big)_{L_2(\mathbb{R}^n)} + (Nu, v{-}u)_{L_2(\mathbb{R}^n)}\,, \\ S(v-u) &= \big(Z_d{-}y(0), y(v{-}u){-}y(0)\big)_{L_2(\mathbb{R}^n)}\,, \end{aligned}$$

and

$$J'(u)(v-u) = 2\Big[\big(y(u){-}Z_d, y(v{-}u){-}y(0)\big)_{L_2(\mathbb{R}^n)} + (Nu, v{-}u)_{L_2(\mathbb{R}^n)}\Big]\,.$$

But $y(u) = (L^\lambda)^{-1}(f+u)$, hence

$$\begin{aligned} y(v-u) &= (L^\lambda)^{-1}(f+v-u) = (L^\lambda)^{-1}f + (L^\lambda)^{-1}(v-u) \\ &= y(0) + y(v) - y(u) \qquad \text{(from (3.8) and } (L^\lambda)^{-1}f = y(0)), \end{aligned}$$

$$\begin{aligned} y(v-u)-y(0) &= y(v)-y(u)\,, \\ y'(u)(v-u) &= 2\Big[\big(y(u){-}Z_d, y(v){-}y(u)\big)_{L_2(\mathbb{R}^n)} + (Nu, v{-}u)_{L_2(\mathbb{R}^n)}\Big]\,. \end{aligned}$$

Therefore, after division by 2, (3.7) is equivalent to:

$$\big(y(u){-}Z_d, y(v){-}y(u)\big)_{L_2(\mathbb{R}^n)} + (Nu, v{-}u)_{L_2(\mathbb{R}^n)} \geq 0\,. \tag{3.9}$$

For the control $u \in L_2(\mathbb{R}^n)$ the adjoint state $p(u) \in H_0^{a^2,1}(\mathbb{R}^n)$ is defined by:

$$L^\lambda p(u) = y(u) - Z_d\,,$$

$$\begin{aligned} \big(y(u){-}Z_d, y(v){-}y(u)\big)_{L_2(\mathbb{R}^n)} &= \big(L^\lambda p(u), y(v){-}y(u)\big)_{L_2(\mathbb{R}^n)} \\ &= \big(p(u), L^\lambda y(v){-}L^\lambda y(u)\big)_{L_2(\mathbb{R}^n)} \\ &= \big(p(u), v{-}u\big)_{L_2(\mathbb{R}^n)} \end{aligned}$$

and hence (3.9) is equivalent to

$$\big(p(u){+}Nu, v{-}u\big)_{L_2(\mathbb{R}^n)} \geq 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,,$$

i.e

$$\int_{\mathbb{R}^n} (p(u) + Nu)(v - u)\,\mathrm{d}x \geq 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,,$$

which completes the proof. □

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