Ján Jakubík

Weak isometries of lattice ordered groups

Mathematica Slovaca, Vol. 38 (1988), No. 2, 133--138

Persistent URL: http://dml.cz/dmlcz/128628

Math. Slovaca 38, 1988, No. 2, 133—138

WEAK ISOMETRIES OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK

K. L. Swamy [10] defined an isometry in an abelian lattice ordered group G to be a bijection $f\colon G \to G$ such that

(1) $$|f(x) - f(y)| = |x - y| \quad \text{for each} \quad x, y \in G.$$

This definition can be applied for non-abelian lattice ordered groups as well.

Isometries in abelian lattice ordered groups were investigated by Swamy [10], [11] and by W. B. Powell [8]; for the non-abelian case cf. W. Ch. Holland [2] and the author [3], [4].

Isometries for some types of abelian partially ordered groups were studied by J. Rachůnek [9], M. Jasem [6], M. Kolibiar and the author [5].

In [4] it was proved that for each isometry f we have

(2) $$f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)] \quad \text{for each} \quad x, y \in G.$$

In the present paper the following results will be established:

(A) *Let G be a representable lattice ordered group and let $f\colon G \to G$ be a mapping such that* (1) *is valid. Then f is a bijection.*

(B) *Let G be a lattice ordered group and let $f\colon G \to G$ be a mapping such that* (1) *and* (2) *are valid. Then f is a bijection.*

A mapping $f\colon G \to G$ which satisfies the condition (1) will be said to be a weak isometry in G.

1. Auxiliary lemmas

For the terminology and denotations concerning lattice ordered qroups cf. Conrad [1] and Kopytov [7].

1.1. Lemma. *Let f be a weak isometry in a lattice ordered group G. Then f is an injection.*

Proof. Let x and y be distinct elements of G. Then $|x - y| \neq 0$, hence in view of (1) we have $f(x) \neq f(y)$.

In the remaining part of this section G is a lattice ordered group and f is a weak isometry in G. In the lemmas 1.2—1.10 we assume that the condition (2) is satisfied. The method from [3], Section 1 will be applied (with the distinction that the bijectivity of G will not be assumed).

We denote by M_1 and M_2 the sets of all intervals $[r, s]$ of G such that $f(r) \leqq f(s)$ or $f(r) \geqq f(s)$, respectively.

1.2. Lemma. *Let $a, b, c \in G$, $a \leqq b \leqq c$, $i \in \{1, 2\}$. If $[a, c] \in M_i$, then both the intervals $[a, b]$ and $[b, c]$ belong to M_i.*

Proof. This is a consequence of (2).

Each interval belonging to $M_1 \cap M_2$ contains only one element; thus from 1.2 we obtain:

1.3. Lemma. *Let $[a, b] \in M_1$, $[a, c] \in M_2$. Then $a = b \wedge c$.*

The assertion dual to 1.3 is also valid.

1.4. Lemma. *Let $a, b \in G$, $a \leqq b$. There exist elements c, $d \in [a, b]$ such that*

(i) $[a, c]$, $[d, b] \in M_1$ *and* $[a, d]$, $[c, b] \in M_2$;

(ii) $c \wedge d = a$ *and* $c \vee d = b$;

(iii) $f(c) = f(a) \vee f(b)$, $f(d) = f(a) \wedge f(b)$.

Proof. According to (2) there exist elements c and d in $[a, b]$ such that (iii) is valid. Hence (i) holds. Thus in view of 1.3 and of its dual, the condition (i) is satisfied.

Let $x, y \in G$, $x \wedge y = u$, $x \vee y = v$.

1.5. Lemma. *Let $[u, x]$ and $[u, y]$ belong to M_1. Then $f(x) \wedge f(y) = f(u)$ and $f(x) \vee f(y) = f(v)$ (hence $[x, v]$, $[y, v] \in M_1$).*

Proof. Cf. [3], Proof of Lemma 1.5.

Similarly we have

1.6. Lemma. *Let $[u, x]$, $[u, y] \in M_2$. Then $f(x) \wedge f(y) = f(v)$ and $f(x) \vee f(y) = f(u)$ (hence $[x, v]$, $[y, v] \in M_2$).*

1.7. Lemma. *Let $[u, x] \in M_1$, $[u, y] \in M_2$. Then $[x, v] \in M_2$ and $[y, v] \in M_1$.*

Proof. According to Lemma 1.4 applied to the interval $[x, v]$ there exists $d \in [x, v]$ such that $[x, d] \in M_1$ and $[d, v] \in M_2$. Then we have $[u, d] \in M_1$, hence in view of 1.2, $[u, d \wedge y] \in M_1$. But from $[u, d \wedge y] \subseteq [u, y] \in M_2$ we obtain $[u, d \wedge \wedge y] \in M_2$, thus in view of Lemma 1.3 we have $d \wedge y = u$. Hence $d = x$ and therefore $[x, v] \in M_2$. Analogously we deduce that $[y, v] \in M_1$.

1.8. Lemma. *Let $[u, x] \in M_1$. Then $[y, v] \in M_1$.*

Proof. According to 1.4 there is $c \in [u, y]$ such that $[u, c] \in M_1$ and $[c, y] \in M_2$. Put $c_1 = x \vee c$. In view of 1.5 we have $[c, c_1] \in M_1$. Hence according to 1.3, $c_1 \wedge y = c$. Clearly $c_1 \vee y = v$. Now by applying 1.4 we obtain $[y, v] \in M_1$.

By duality we get $[y, v] \in M_1 \Rightarrow [u, x] \in M_1$. An analogous result holds for M_2; thus we conclude:

1.9. Lemma. *Let $i \in \{1, 2\}$. Then $[u, x] \in M_i$ if and only if $[y, v] \in M_i$.*

1.10. Lemma. *Let the assumptions of Lemma* 1.7 *be satisfied. Then we have $f(u) \wedge f(v) = f(y)$ and $f(u) \vee f(v) = f(x)$.*

Proof. In view of the assumptions we have $f(y) \leqq f(v)$ and $f(y) \leqq f(u)$, hence $f(y) \leqq f(u) \wedge f(v)$. On the other hand, from (2) we obtain $f(u) \wedge f(v) \leqq f(y)$. Thus $f(u) \wedge f(v) = f(y)$. Analogously we can verify that $f(u) \vee f(v) = f(x)$.

1.11. Lemma. *Let* $f(0) = 0$. Then

(a) $x \wedge f(x) \geqq 0 \Rightarrow f(x) = x$;
(b) $x \wedge (-f(x)) \geqq 0 \Rightarrow f(x) = -x$;
(c) $x \vee f(x) \leqq 0 \Rightarrow f(x) = x$;
(d) $x \vee (-f(x)) \leqq 0 \Rightarrow f(x) = -x$.

Proof. Cf. [3], the proof of 1.8.

1.12. Lemma. *Let* $f(0) = 0$. *Let* (2) *be valid and let* $0 \leqq x \in G$. *Then*

(a) $f(x) = x \Leftrightarrow f(-x) = -x$,
(b) $f(x) = -x \Leftrightarrow f(-x) = x$.

Proof. Cf. [3], the proof of 1.9.

Hence we arrived at the conclusion that if f is a weak isometry on a lattice ordered group G such that (2) is satisfied, then the assertions of the lemmas 1.3—1.9 of [3] remain valid.

2. Representable lattice ordered groups

Recall that a lattice ordered group is said to be representable if it can be embedded into a direct product of linearly ordered groups. Each abelian lattice ordered group is representable.

2.1. Lemma. *Let G be a lattice ordered group and let $f\colon G \to G$ be a mapping. Put $g(x) = f(x) - f(0)$ for each $x \in G$. Let $j \in \{1, 2\}$. Then the following conditions are equivalent:*

(i) *f satisfies the condition* (j);
(ii) *g satisfies the condition* (j).

The proof is immediate.

2.2. Lemma. *Let G be a linearly ordered group and let f be a weak isometry in G. Let g be as in* 2.1. *Then some of the following conditions is valid:*

(α) $g(x) = x$ *for each* $x \in G$;
(β) $g(x) = -x$ *for each* $x \in G$.

Proof. The assertion is trivial for the case $G = \{0\}$. Assume that $G \neq \{0\}$. Then there is $x \in G$, $x > 0$. Since G is linearly ordered, according to 1.11 we have either $g(x) = x$ or $g(x) = -x$.

Let $g(x) = x$ and $y \in G$. By way of contradiction, assume that $g(y) \neq y$. Then $y \neq 0$ and $g(y) = -y$. If $y > 0$, then $|g(x) - g(y)| > |x - y|$; if $y < 0$, then $|g(x) - g(y)| < |x - y|$. Since g is a weak isometry in G, we have arrived at a contradiction. The case $g(x) = -x$ is analogous.

In the rest of this section we assume that G is a representable lattice ordered group and that f is a weak isometry in G.

Without loss of generality we may suppose that G is a subgroup of the lattice ordered group $\Pi_{i \in I} G_i$, where

(a) all G_i are linearly ordered,

(b) for each $i \in I$, the natural projection of G into G_i is a surjection.

For $x \in G$ and $i \in I$ we denote by $x(i)$ the i-th component of x. Let g be as above.

2.3. Lemma. *Let $x, y \in G$ and $i \in I$. If $x(i) = y(i)$, then $g(x)(i) = g(y)(i)$.*

Proof. Let $x(i) = y(i)$. From 2.1 we infer that

$$|g(x) - g(y)|(i) = |x - y|(i),$$

hence

$$|g(x)(i) - g(y)(i)| = |x(i) - y(i)|.$$

Therefore $g(x)(i) = g(y)(i)$.

In view of (b), for each $i \in I$ and each $x_i \in G_i$ there is $x \in G$ with $x(i) = x_i$. We put $g_i(x_i) = g(x)(i)$. According to 2.3, g_i is a correctly defined mapping of G_i into G_i.

Since all operations in G are performed component-wise, from 2.1 we obtain that for each $i \in I$, g_i is a weak isometry in G_i.

2.4. Lemma. *Let $i \in I$. Then g_i satisfies the condition* (2).

Proof. Since G_i is linearly ordered, we can apply 2.2 (G and g are replaced by G_i and g_i) and then by a straigth-forward calculation we obtain that (2) holds.

In view of 2.4, g satisfies (2) as well; hence according to 2.1 we get

2.5. Corollary. *Let G be a representable lattice ordered group and let f be a weak isometry in G. Then the condition* (2) *is satisfied.*

3. Proofs of (A) **and** (B)

In view of 2.5, the assertion (A) is a consequence of (B).

Let G be a lattice ordered group and let $f\colon G \to G$ be a mapping which satisfies (1) and (2).

For the next procedure we have two alternatives.

a) As we have already remarked, we have verified in Section 1 above that the assertions of Lemmas 1.3—1.9, [3] remain valid if the assumption that f is an isometry is replaced by the assumption that f is a weak isometry satisfying (2). This assumption also suffices to carry out the proofs of [3], Section 2. In particular, from 2.5.1 in [3] we infer that $g^2(x) = x$ is valid for each $x \in G$, where g is as in Lemma 2.1. Hence in view of 1.1, g is a bijection. Therefore f is a bijection as well. Thus we have proved that (B) holds. According to 1.1 and 2.5, (A) is valid.

b) We can proceed directly without applying the results of Section 2 of [3] (concerning the direct product decomposition of G corresponding to the mapping f with $f(0) = 0$).

Let g be as in 2.1. The following assertion is obvious.

3.1. Lemma. *The mapping g^2 satisfies the conditions* (1) *and* (2).

3.2. Lemma. *Let $x \in G$, $0 \leqq x$. Then $g^2(x) = x$.*

Proof. We apply Lemma 1.4 for the interval $[0, x]$ and for g instead of f (in view of 2.1, this can be done). There are $a, b \in [0, x]$ such that $[0, a]$, $[b, x] \in M_1$ and $[0, b]$, $[a, x] \in M_2$ (where M_1 and M_2 are taken with respect to g). According to 1.10 we have

$$g(0) \wedge g(x) = g(b), \quad g(0) \vee g(x) = g(a),$$

whence

$$0 \wedge g(x) = -b, \quad 0 \vee g(x) = a.$$

Since $g(-b) = b$ (cf. 1.12), according to (2) we obtain

$$g(g(x)) \in [g(a) \wedge g(-b), g(a) \vee g(-b)] = [a \wedge b, a \vee b] = [0, x],$$

hence $g^2(x) \geqq 0$. Now in view of 3.1 and 1.11 (a) (applied to g^2) we infer that $g^2(x) = x$.

3.3. Lemma. *Let $x \in G$. Then $g^2(x) = x$.*

Proof. Put $0 \wedge x = u$, $0 \vee x = v$. In view of 1.12 and 3.2 we have $g^2(u) = u$ and $g^2(v) = v$. Hence $g^2(u) \leqq g^2(v)$. Thus according to 3.1 and 1.2, $g^2(u) \leqq g^2(x) \leqq g^2(v)$. Since g^2 satisfies (1) and $g^2(0) = 0$, we get $|g^2(x)| = |x|$. If either $g^2(x) \wedge 0 > u$ or $g^2(x) \vee 0 < v$, then we would have

$$|g^2(x)| = g^2(x) \vee 0 - g^2(x) \wedge 0 < v - x = |x|,$$

which is a contradiction. Hence $g^2(x) \wedge 0 = u$ and $g^2(x) \vee 0 = v$. Therefore $g^2(x) = x$.

Now we can apply the identity $g^2(x) = x$ in the same way as in a) to obtain that (A) and (B) hold.

3.4. Corollary. *Let G be a representable lattice ordered group and let $f\colon G \to G$ be a mapping. Then the following conditions are equivalent*:

(i) *f is an isometry in G.*

(ii) *f satisfies* (1).

3.5. Corollary. *Let G be a lattice ordered group and let $f\colon G \to G$ be a mapping. Then the following conditions are equivalent*:

(i) *f is an isometry in G.*

(ii) *f satisfies* (1) *and* (2).

The question whether (2) is a consequence of (1) remains open.

REFERENCES

[1] CONRAD, P.: Lattice Ordered Groups. Tulane University, 1970.

[2] HOLLAND, W. C.: Intrinsic metrics for lattice ordered groups. Algebra Univ. 19, 1984, 142—150.

[3] JAKUBÍK, J.: Isometries of lattice ordered groups. Czechoslovak Math. J. 30, 1980, 142—152.

[4] JAKUBÍK, J.: On isometries of non-abelian lattice ordered groups. Math. Slovaca 31, 1981, 171—175.

[5] JAKUBÍK, J.—KOLIBIAR, M.: Isometries of multilattice groups. Czechoslov. Math. J. 33, 1983, 602—612.

[6] JASEM, M.: Isometries in Riesz groups, Czechoslov, Math. J. 36, 1986, 35—43.

[7] КОПЫТОВ, В. М.: Решеточно упорядоченные группы, Москва 1984.

[8] POWELL, W. B.: On isometries in abelian lattice ordered groups. Preprint, Oklahoma State University.

[9] RACHŮNEK, J.: Isometries in ordered groups. Czechoslov. Math. J. 34, 1984, 334—341.

[10] SWAMY, K. L.: Isometries in autometrized lattice ordered groups. Algebra Univ. 8, 1977, 58—64.

[11] SWAMY, K. L.: Isometries in autometrized lattice ordered groups, II. Math. Seminar Notes, Kobe Univ. 5, 1977, 211—214.

Received August 11, 1986

Matematický ústav SAV
Dislokované pracovisko
Ždanovova 6
040 01 Košice

СЛАБЫЕ ИЗОМЕТРИИ РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

Ján Jakubík

Резюме

Пусть G — решеточно упорядоченная группа, и $f\colon G \to G$ такое отображение, что $|f(x) - f(y)| = |x - y|$ для всех $x, y \in G$. В статье доказано: если G является o-аппроксимируемой, тогда отображение f будет биекцией.