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INVARIANT MEASURES

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Introduction. Let (Ω, α, P) be a complete probability space. Let the Markov operator Q: $L_1(\Omega, \alpha, P) \to L_1(\Omega, \alpha, P)$ satisfy the following conditions:

$$Q(f \cdot Q(g)) = Q(f) \cdot Q(g) \text{ for all } f \in L_1(\Omega, \alpha, P) \text{ such that } f \cdot g \in L_1(\Omega, \alpha, P) \tag{1}$$

$$f \in L_p(\Omega, \alpha, P) \text{ implies } Q(f) \in L_p(\Omega, \alpha, P) \text{ for all } p \geq 1. \tag{2}$$

In the present paper we study the set M of all Q-invariant probability measures on (Ω, α). M clearly depends on P, as the operator Q works on $L_1(\Omega, \alpha, P)$.

A similar problem for Markov operators in the space of bounded α-measurable functions was solved by Dynkin in [2]. However, his results cannot be applied in our case.

Theorem 1 *(Dynkin, [2]). Let Q be a Markov operator on the space of bounded α-measurable functions with the property $Q(f \cdot Q(g)) = Q(f) \cdot Q(g)$ for all f, g bounded α-measurable functions. Let J_Q be the collection of all Q-invariant α-measurable sets. Then:*

i) *M is a simplex.*

ii) *P is an extreme point of M iff $P(A) = 0$ or $P(A) = 1$ for all $A \in J_Q$.*

iii) *The mapping $P \to P_Q$, where P_Q is the restriction of the probability measure P to J_Q, is the isomorphism of M onto $M(J_Q)$, the class of all probability measures on (Ω, J_Q).*

Throughout this paper let Q be a Markov operator satisfying conditions (1) and (2).

1. Q-invariant measures

Definition. *A probability measure m on (Ω, α) is Q-invariant if $L_1(\Omega, \alpha, P) \subset L_1(\Omega, \alpha, m)$ and if for all $f \in L_1(\Omega, \alpha, P)$ there holds:*

$$\int_\Omega f \, dm = \int_\Omega Q(f) \, dm.$$

Lemma 1. *Let* $m \in M$. *Then* $w = \frac{dm}{dP} \in L_\infty(\Omega, \alpha, P)$.

Proof. As $m \in M$ implies $L_1(\Omega, \alpha, P) \subset L_1(\Omega, \alpha, m)$, we have $m \ll P$. The Radon—Nikodym theorem [1] implies $w \in L_1(\Omega, \alpha, P)$. Define for $f \in L_1(\Omega, \alpha, P)$

$$L(f) = \int_\Omega f \, dm = \int_\Omega f \cdot w \, dP.$$

L is a continuous linear functional in the space $L_1(\Omega, \alpha, P)$, so that there exists $g \in L_\infty(\Omega, \alpha, P)$ with the property

$$L(f) = \int_\Omega f \cdot g \, dP, f \in L_1(\Omega, \alpha, P)$$

(see [1]).

Clearly $w = g(P - \text{a. e.})$.

Theorem 2. *Let* $m \in M$, $w = \frac{dm}{dP}$. *Then* $Q(w) = w(P - a. e.)$.

Proof. As Q is a Markov operator on $L_1(\Omega, \alpha, P)$ we have $\int_\Omega f \, dP = \int_\Omega Q(f) \, dP$ for all $f \in L_1(\Omega, \alpha, P)$. Then for every $f \in L_1(\Omega, \alpha, P)$ there holds:

$\int_\Omega f \cdot w \, dP = \int_\Omega f \, dm = \int_\Omega Q(f) \, dm = \int_\Omega Q(f) \cdot w dP = \int_\Omega Q(Q(f) \cdot w) \, dP$ $= \int_\Omega Q(f) \cdot Q(w) \, dP = \int_\Omega Q(f \cdot Q(w)) \, dP = \int_\Omega f \cdot Q(w) \, dP$, so that $\int_\Omega f \cdot (w - Q(w)) \, dP = 0$, for all $f \in L_1(\Omega, \alpha, P)$. Let $f = w - Q(w)$. Then $f \in L_1(\Omega, \alpha, P)$, so that $\int_\Omega (w - Q(w))^2 \, dP = 0$. This fact immediatelly implies $Q(w) = w(P - \text{a. e.})$.

Theorem 3. *Let* $m \ll P$ *be a probability measure on* (Ω, α), $\frac{dm}{dP} = w \in L_\infty(\Omega, \alpha, P)$, $w = Q(w)$ $(P - \text{a. e.})$. *Then* $m \in M$.

Proof. Let $f \in L_1(\Omega, \alpha, P)$. Then $\int_\Omega f \, dm = \int_\Omega f \cdot w \, d < \infty$, so that $L_1(\Omega, \alpha, P) \subset L_1(\Omega, \alpha, m)$. As $\int_\Omega f \, dm = \int_\Omega f \cdot w \, dP = \int_\Omega f \cdot Q(w) \, dP$ $= \int_\Omega Q(f) \cdot w \, dP = \int_\Omega Q(f) \, dm$, we have $m \in M$.

Lemma 2. *Denote* $J_Q = \{A \in \alpha,\ Q(\chi_A) = \chi_a\ (P - a.\ e.)\}$. *Then:*

i) *The system* J_Q *is a complete sub-σ-algebra of* α.

ii) $Q(f) = E(f \mid J_Q)$, *i. e. the operator* Q *is an operator of conditional expectation in* $L_1(\Omega, \alpha, P)$.

iii) $w = Q(w)\ (P - a.\ e.)$ *if and only if* w *is a* J_Q*-measurable integrable random variable, i. e.* $w \in L_1(\Omega, J_Q, P)$.

Proof. See [4].

Corollary 1. *The mapping* $m \to w = \dfrac{\mathrm{d}m}{\mathrm{d}P}$ *is an isomorphism of* M *onto the system* $M(P, Q)$ *of all nonnegative bounded* J_Q*-measurable radom variables of* $L_1(\Omega, \alpha, P)$ *whose integral with respect to* P *is equal to* 1, $M(P, Q) = \{w \in L_\infty(\Omega, J_Q, P),\ w \geqslant 0,\ \int_\Omega w\ \mathrm{d}P = 1\}$.

2. Extreme points

Theorem 4. *Let* $A \in J_Q$, $P(A) > 0$. *Then:*

i) $P(.\mid A) \in M$.

ii) $P(.\mid A)$ *is an extreme point of* M *iff* A *is an atom of* J_Q.

Proof.

i) $\dfrac{\mathrm{d}P(.\mid A)}{\mathrm{d}P} = \dfrac{\chi_A}{P(A)} \in M(P, Q)$, so that $P(.\mid A) \in M$.

ii) Let A be an atom of J_Q. Then for all sets $B \in J_Q$ there is $P(B \mid A) = 0$ or $P(B \mid A) = 1$. Let m_1 and m_2 be two Q-invariant probability measures and $P(.\mid A) = cm_1 + 1 - c)m_2$ for some $c \in (0, 1)$. Then evidently $m_1 = m_2 = P(.\mid A)$ on J_Q. As all these measures are Q-invariant, we have $m_1 = m_2 = P(.\mid A)$ on α, so that $P(.\mid A)$ is an extreme point of M.

Now let A don't be an atom of J_Q, so that $A = B \cup C$, B, $C \in J_Q$, $B \cap C = \emptyset$, $P(B) > 0$, $P(C) > 0$. Then $P(.\mid B) \neq P(.\mid C)$ and $P(.\mid A) = \dfrac{P(B)}{P(A)} P(.\mid B) + \left(1 - \dfrac{P(B)}{P(A)}\right) P(.\mid C)$, so that $P(.\mid A)$ is a convex combination of two different measures of M. Then clearly $P(.\mid A)$ is not an extreme point of M.

Corollary 2. *The mapping* $A \to P(.\mid A)$ *is an isomorphism of the system* K *of all atoms of* J_Q *onto the set* M_e *of all extreme points of* M.

Remark 1. For every $B \in J_Q$, $m \in M_e$ is $m(B) = 0$ or $m(B) = 1$. This fact fully agrees with the results of Dynkin (assertion ii) of Theorem 1 of this paper). However, there exist probability measures on (Ω, α) with this property which are not of M_e (as they are not absolutely continuous with respect to P).

R e m a r k 2. Lemma 2 implies the fact that the present paper solves the problem of invariant measures with respect to conditional expectations.

REFERENCES

[1] DUNFORD, N., SCHWARTZ, J. T.: Linear Operators. Part I, Interscience, New York, 1958.
[2] DYNKIN, E. B.: Sufficient statistics and extreme points. Ann. Prob. 6, 1978, 705—730.
[3] HALMOS, P. R.: Measure Theory. Van Nostrand, New York, 1950.
[4] NEVEU, J.: Martingales à temps discret. Masson et Cie, Paris, 1972.

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ИНВАРИАНТНЫЕ МЕРЫ

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Резюме

Пусть оператор Маркова Q является оператором условной вероятности. Множество M всех Q — инвариантных мер изоморфно множеству

$$M(P, Q)=\{f \in L_\infty(\Omega, J_Q, P), f \geq 0, \int_\Omega f \, dP = 1\}.$$

Множество M_e всех экстремальных точек множества M изоморфно множеству всех P-атомов σ-альгебры J_Q.