Ján Ohriska

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NONOSCILLATORY SOLUTIONS OF A SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATION

JÁN OHRISKA

In the delay differential equation

$$u''(t)+p(t)u^{\alpha}(\tau(t))=0 \tag{1}$$

for a function $u(t)$ we suppose throughout the paper that

(i) $0\leqslant p(t)\in C_{[t_0,\infty)}$, $p(t)$ is not identically zero in any neighbourhood $o(\infty)$,

(ii) $\tau(t)\in C_{[t_0,\infty)}$, $\tau(t)\leqslant t$, $\lim\limits_{t\to\infty}\tau(t)=\infty$

(iii) $\alpha=r/s$, where r and s are odd natural numbers.

Suppose that there exist solutions of equation (1) on an interval of the form $[b,\infty)$, where $b\geqslant t_0$. In the sequel we shall use the term "solution" only to denote a solution which exists on $[b,\infty)$. Moreover, we shall exclude from our considerations solutions of (1) with the property that $u(t)\equiv 0$ for $t\geqslant T\geqslant t_0$. A solution is said to be oscillatory if it has arbitrary large zeros, otherwise it is said to be nonoscillatory.

It is well known (cf., e.g., [2]) that nonoscillatory solutions of (1) can be only of the following three types:

(a) $|u(t)|\to c$, $u'(t)\to 0$ $\quad(0<c)$ for $t\to\infty$,
(b) $|u(t)|\to\infty$, $|u'(t)|\to c$ $\quad(0<c)$ for $t\to\infty$,
(c) $|u(t)|\to\infty$, $u'(t)\to 0$ $\quad$ for $t\to\infty$.

The purpose of this paper is to investigate nonoscillatory solutions of (1) of the types (a) and (c).

Remark 1. According to (iii) we have that with a solution $u(t)$ of (1) also $-u(t)$ is its solution. This enables us to consider, e.g. only positive nonoscillatory solutions of (1). Further, by (i) and (ii), if $u(t)$ is a nonoscillatory solution of (1) such that $u(t)>0$ for $t\geqslant t_1$ then there exists a number $t_2\geqslant t_1$ such that $u(\tau(t))>0$ for $t\geqslant t_2$ and now we see from (1) that $u''(t)\leqslant 0$ for $t\geqslant t_2$. However, this means that $u'(t)>0$ for $t\geqslant t_2$.

Theorem 1. *Suppose that* $\alpha > 0$ *and*

$$\int^{\infty} t \cdot \tau^{\alpha}(t) p(t) \, \mathrm{d}t < \infty. \tag{2}$$

Then (1) *has not a nonoscillatory solution of the type* (c).

In order to prove this theorem we need the following result which is proved in [3].

Lemma 1. *Let* $f(t)$ *be a continuous and non-negative function defined on some neighbourhood* $o(\infty)$. *Let* k *be a natural number. If*

$$\limsup_{t\to\infty} t^k \int_t^{\infty} f(x) \, \mathrm{d}x = \infty,$$

then

$$\int^{\infty} t^k f(t) \, \mathrm{d}t = \infty.$$

Proof of Theorem 1. Let $u(t)$ be a positive non-oscillatory solution of (1) of the type (c). Then, by Remark 1, there exists $t_1 \geq t_0$ such that $u(t) > 0$, $u(\tau(t)) > 0$, $u'(t) > 0$ and $u''(t) \leq 0$ for $t \geq t_1$.

Integrating equation (1) from t to $\infty (t \geq t_1)$ we have

$$u'(t) = \int_t^{\infty} p(x) u^{\alpha}(\tau(x)) \, \mathrm{d}x. \tag{3}$$

Since the function $u(t)$ is increasing and concave with $\lim_{t\to\infty} u'(t) = 0$, there exists a number $\beta > 0$ such that $u(x) \leq \beta x$ for sufficiently large x, e.g. for $x \geq t_2 \geq t_1$ and now by (ii) there is a point $t_3 \geq t_2$ such that $u^{\alpha}(\tau(x)) \leq \beta^{\alpha} \tau^{\alpha}(x)$ for $x \geq t_3$. Therefore for $t \geq t_3$, it follows from (3) that

$$u'(t) \leq \beta^{\alpha} \int_t^{\infty} p(x) \tau^{\alpha}(x) \, \mathrm{d}x.$$

Integrating the above inequality from t_3 to t ($t \geq t_3$), we have

$$u(t) \leq u(t_3) + \beta^{\alpha} [(t - t_3) \int_t^{\infty} \tau^{\alpha}(x) p(x) \, \mathrm{d}x + \int_{t_3}^{t} (x - t_3) \tau^{\alpha}(x) p(x) \, \mathrm{d}x].$$

As $\lim_{t\to\infty} u(t) = \infty$, from the last inequality, according to Lemma 1, we obtain

$$\int^{\infty} x \tau^{\alpha}(x) p(x) \, \mathrm{d}x = \infty,$$

which contradicts (2). This completes the proof.

Remark 2. It is clear that if $\int^\infty t\tau^\alpha(t)p(t)\,dt<\infty$, then $\int^\infty tp(t)\,dt<\infty$, $\int^\infty \tau^\alpha(t)p(t)\,dt<\infty$ and (for $\tau(t)\equiv t$) also $\int^\infty t^\alpha p(t)\,dt<\infty$. On the other hand, from paper [2] we know that

1° equation (1) has a nonoscillatory solution of the type (a) if and only if $\int^\infty tp(t)\,dt<\infty$ and it is true for $0<\alpha<1$ as well as for $\alpha>1$ and also for $\tau(t)\leqslant t$ as well as for $\tau(t)\equiv t$ (from the corollary 2.1 in [7] we know that this is true for $\alpha=1$ too),

2° equation (1) has, in the case $\tau(t)\leqslant t$ and $0<\alpha<1$ or $\alpha>1$, a nonoscillatory solution of the type (b) if and only if $\int^\infty \tau^\alpha(t)p(t)\,dt<\infty$,

3° equation (1) has, in the case $\tau(t)\equiv t$ and $0<\alpha<1$ or $\alpha>1$, a nonoscillatory solution of the type (b) if and only if $\int^\infty t^\alpha p(t)\,dt<\infty$. From this we see that (1) has not a nonoscillatory solution of the type (c) if $\int^\infty t\tau^\alpha(t)p(t)\,dt<\infty$ but it has nonoscillatory solutions of the types (a) and (b), which is true for $\tau(t)\leqslant t$ as well as for $\tau(t)\equiv t$ and also for $0<\alpha<1$ as well as for $\alpha>1$.

Theorem 2. *Let $\alpha\geqslant 1$ and $\int^\infty tp(t)\,dt=\infty$. Let there exist a number $\beta\leqslant 1$ such that the function $P(t)=p(t)\tau^\alpha(t)t^\beta$ is non-decreasing (for all sufficiently large t). Then every nonoscillatory solution of* (1) *is of the type* (c).

The following results from [4] and [7] will be used in the proof of Theorem 2.

Theorem 3. (Theorem 5 in [4]). *Let $\alpha>0$. Let there exist a number $\beta\leqslant 1$ such that the function $P(t)=p(t)\tau^\alpha(t)t^\beta$ is non-decreasing (for all sufficiently large t). Then for every nonoscillatory solution $u(t)$ of* (1) *the condition $\lim_{t\to\infty} u'(t)=0$ holds true.*

Theorem 4 (Corollary 2.1 in [7]). *Let $\alpha>0$. Equation* (1) *has a nonoscillatory solution $u(t)$ such that $\lim_{t\to\infty} u(t)=a\neq 0$ if and only if*

$$\int^\infty tp(t)\,dt<\infty.$$

Proof of Theorem 2. Let $u(t)$ be a non-oscillatory solution of (1). From the assumptions of Theorem 2 we see that by Theorem 3 the solution $u(t)$ must be of the type (a) or (c), and by Theorem 4 $u(t)$ must be of the type (b) or (c). Hence $u(t)$ is of the type (c) and the theorem is proved.

Remark 3. The above Theorems 3 and 4 allow to pronounce Theorem 2 for $\alpha>0$ but such an assertion would give empty information in the case $0<\alpha<1$. Namely, by Theorem 1 in [5] we know that for $0<\alpha<1$ all solutions of (1) are oscillatory if and only if $\int^{\infty}\tau^{\alpha}(t)p(t)\,dt=\infty$, and by [2] equation (1) has a nonoscillatory solution of the type (b) if and only if $\int^{\infty}\tau^{\alpha}(t)p(t)\,dt<\infty$. Then, just for $\alpha\geqslant 1$ equation (1) can have non-oscillatory solutions only of the type (c), or only of the type (a).

Example 1. Consider the equation

$$u''(t)+\frac{1}{4t^2}\,u^{7/3}(t^{3/7})=0.$$

It is easy to see that assumptions of Theorem 2 are satisfied for this equation $\left(P(t)=\frac{1}{4}\text{ if }\beta=1\right)$. Thus, by Theorem 2, every non-oscillatory solution of the above equation is of the type (c). One of such solutions is $u(t)=t^{1/2}$.

We start the following part of the paper by two preliminary lemmas.

Lemma 2. *Let* $u(t)\in C^2_{[T,\infty)}$ *and let*

$$u(t)>0,\ u'(t)>0,\ u''(t)\leqslant 0\ \text{ for }\ t\in[T,\infty).$$

Then for each $k\in(0,1)$ there is a $T_k\geqslant T$ such that

$$u(\tau(t))\geqslant k\,\frac{\tau(t)}{t}\,u(t),\quad t\geqslant T_k.$$

Proof of Lemma 2 may be found in [1].

Lemma 3. *Let* $u(t)\in C^2_{[T,\infty)}$ *and let*

$$u(t)>0,\ u'(t)>0,\ u''(t)\leqslant 0\ \text{ for }\ t\in[T,\infty).$$

Then for each $k\in(0,1)$ *there is a* $T_k\geqslant T$ *such that*

$$u(t)\geqslant ktu'(t),\quad t\geqslant T_k.$$

Proof of Lemma 3 may be found in [6].

Theorem 5. *Let* $\alpha>1$ *and let*

$$\int^{\infty}\left[\int^{s}\tau^{\alpha}(x)p(x)\,dx\right]^{1/1-\alpha}ds<\infty. \tag{4}$$

Then every solution of (1) is either oscillatory or of the type (a).

Proof. To prove Theorem 5 we show that if (1) has a non-oscillatory solution

$u(t)$ and the assumptions of Theorem 5 are fulfilled, so the solution $u(t)$ is of the type (a).

Let $u(t)$ be a non-oscillatory solution of (1), e.g. such that $u(t)>0$, $u(\tau(t))>0$ for $t \geqslant t_1 \geqslant t_0$. Then $u'(t)>0$ and $u''(t) \leqslant 0$ for $t \geqslant t_1$, and by Lemmas 2 and 3 we know that for any $k \in (0, 1)$ there is $t_2 \geqslant t_1$ such that for $t \geqslant t_2$ we have

$$u(\tau(t)) \geqslant k\tau(t)u'(t). \tag{5}$$

Now, if we estimate $u(\tau(t))$ in (1) by (5), we obtain

$$u''(t)+k^\alpha p(t)\tau^\alpha(t)(u'(t))^\alpha \leqslant 0, \quad t \geqslant t_2.$$

Since $u'(t)>0$ for $t \geqslant t_2$, from the last inequality we have

$$\frac{u''(t)}{(u'(t))^\alpha} \leqslant -k^\alpha \tau^\alpha(t)p(t). \tag{6}$$

If we put $u'(t)=z(t)$ and $z^{1-\alpha}(t)=v(t)$, then $v'(t)=(1-\alpha)z^{-\alpha}(t)z'(t)$ and we can write the inequality (6) in the form

$$v'(t) \geqslant c\tau^\alpha(t)p(t),$$

where $c=(\alpha-1)k^\alpha>0$.

Integrating this inequality from t_2 to $t (t \geqslant t_2)$ we have

$$v(t) \geqslant v(t_2)+c\int_{t_2}^{t} \tau^\alpha(x)p(x)\,\mathrm{d}x,$$

or

$$(u'(t))^{1-\alpha} \geqslant c\int_{t_2}^{t} \tau^\alpha(x)p(x)\,\mathrm{d}x, \tag{7}$$

because $v(t_2)=(u'(t_2))^{1-\alpha} \geqslant 0$. Since $\alpha>1$ and $u'(t)>0$, the inequality (7) gives

$$u'(t) \leqslant \left[c\int_{t_2}^{t} \tau^\alpha(x)p(x)\,\mathrm{d}x\right]^{\frac{1}{1-\alpha}}$$

and integration from t_3 to t $(t>t_3>t_2)$ yields

$$u(t) \leqslant u(t_3)+c^{\frac{1}{1-\alpha}}\int_{t_3}^{t}\left[\int_{t_2}^{s} \tau^\alpha(x)p(x)\,\mathrm{d}x\right]^{\frac{1}{1-\alpha}}\mathrm{d}s.$$

Now we see that the non-oscillatory solution $u(t)$ of (1) is bounded, i.e. $u(t)$ is of the type (a). This completes the proof.

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Katedra matematickej analýzy
Prírodovedeckej fakulty UPJŠ
nám. Februarového víťazstva 9
041 54 Košice

НЕОСЦИЛЛИРУЮЩИЕ РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ЗАПАЗДЫВАНИЕМ

Ján Ohriska

Резюме

В работе рассматривается дифференциальное уравнение

$$u''(t)+p(t)u^{\alpha}(\tau(t))=0, \quad \text{где} \quad p(t)\geq 0 \text{ и } \tau(t)\leq t.$$

Доказано несколько теорем об неосциллирующих решениях определенных типов