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Jan Lang; Aleš Nekvinda

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TRACES OF A WEIGHTED SOBOLEV SPACE IN A SINGULAR CASE

J. LANG, A. NEKVINDA, Praha

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1. INTRODUCTION

In this paper a characterization of traces of the Sobolev space $W^{1,p}(\Omega, d^{\varepsilon})$ is given for a singular value of ε. Let N be an integer, $N \geqslant 2$. Let ε, p be real numbers, $1 < p < \infty$. Denote by p' the conjugate exponent of p, i.e. $p' = \frac{p}{p-1}$. Let Ω be a domain in $\mathbb{R}^N$ taken such that the origin belongs to the boundary $\partial\Omega$ of Ω. The symbol $|x|$ will stand for the Euclidean norm of $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, that is $|x| = \left(\sum\limits_{i=1}^{N} x_i^2\right)^{1/2}$.

By $C^{\infty}(\overline{\Omega})$ we mean a set of all infinitely many times differentiable functions which together with all derivatives can be continuously extended to $\overline{\Omega}$. The set of all functions $u \in C^{\infty}(\overline{\Omega})$ such that $\operatorname{supp} u$ does not meet the origin will be denoted by $C_0^{\infty}(\overline{\Omega})$. We shall define a weighted Sobolev space $H^{1,p}(\Omega, d^{\varepsilon})$ as a set of all functions with a finite norm $\|u|H^{1,p}(\Omega, d^{\varepsilon})\| = \left(\int_{\Omega} |u(x)|^p |x|^{\varepsilon - p} \,\mathrm{d}x + \int\limits_{\Omega} \sum\limits_{i=1}^{N} |D_i u(x)|^p |x|^{\varepsilon} \,\mathrm{d}x\right)^{1/p}$, where the symbol $D_i u$ stands for generalized derivatives of u. The space $W^{1,p}(\Omega, d^{\varepsilon})$ is defined as the closure of $C_0^{\infty}(\overline{\Omega})$ for $\varepsilon \leqslant -N$ and as the closure of $C^{\infty}(\overline{\Omega})$ for $\varepsilon > -N$ with respect to the norm $\|u|W^{1,p}(\Omega, d^{\varepsilon})\| = \left(\int\limits_{\Omega} |u(x)|^p |x|^{\varepsilon} \,\mathrm{d}x + \int\limits_{\Omega} \sum\limits_{i=1}^{N} |D_i u(x)|^p |x|^{\varepsilon} \,\mathrm{d}x\right)^{1/p}$.

Let us recall the frequently used concept of a domain with a Lipschitz boundary (see e.g. [4, Definition 4.3]):

Definition 1.1. A bounded domain Ω is said to be of the class $C^{0,1}$ (notation: $\Omega \in C^{0,1}$) if its boundary can locally be described as a graph of a Lipschitz function in a neighborhood of each of its points.

For $\Omega \in C^{0,1}$, we shall introduce the space $H^{1-1/p,p}(\partial\Omega, d^\varepsilon)$ as the set of all functions defined on $\partial\Omega$ with a finite norm

$$\|u|H^{1-1/p,p}(\partial\Omega, d^\varepsilon)\| = \Bigg(\int\limits_{\partial\Omega} |u(x)|^p |x|^{\varepsilon-p+1} \, \mathrm{d}S_{N-1}(x) + \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{\big|u(x)|x|^{\varepsilon/p} - u(y)|y|^{\varepsilon/p}\big|^p}{|x-y|^{N+p-2}} \, \mathrm{d}S_{N-1}(x)\, \mathrm{d}S_{N-1}(y)\Bigg)^{1/p},$$

where S_{N-1} is the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. In the sequel we shall assume $\Omega \in C^{0,1}$.

Proposition 1.2 (see [1]). *$C_0^\infty(\overline{\Omega})$ is dense in $H^{1,p}(\Omega, d^\varepsilon)$.*

Proposition 1.3 (see [2]). *There exists a unique linear bounded (trace) operator $T\colon H^{1,p}(\Omega, d^\varepsilon) \to H^{1-1/p,p}(\partial\Omega, d^\varepsilon)$ such that $Tu = u|_{\partial\Omega}$ for all $u \in C_0^\infty(\overline{\Omega})$, and there exists a corresponding extension operator $R\colon H^{1-1/p,p}(\partial\Omega, d^\varepsilon) \to H^{1,p}(\Omega, d^\varepsilon)$ such that $TRu = u$ for all $u \in H^{1-1/p,p}(\partial\Omega, d^\varepsilon)$.*

Proposition 1.4. (see [1] and [3]). *Let $\varepsilon \neq p-N$. Then the direct decomposition $W^{1,p}(\Omega, d^\varepsilon) = H^{1,p}(\Omega, d^\varepsilon) \oplus X$ holds, where X is the trivial space for $\varepsilon \leqslant -N$ or $\varepsilon > p-N$ and X is the space of constant functions in the case $-N < \varepsilon < p-N$.*

The last three propositions give a characterization of traces of $W^{1,p}(\Omega, d^\varepsilon)$ in the case $\varepsilon \neq p-N$.

2. Density

The following three assertions show that we cannot expect a similar direct decomposition in the singular case $\varepsilon = p-N$.

Lemma 2.1. *Let $\varepsilon > p-N$. Then the imbedding*

$$W^{1,p}(\Omega, d^\varepsilon) \hookrightarrow L^p(\Omega, d^{\varepsilon-p})$$

holds and the norm of the imbedding is majorized by $c\frac{p}{\varepsilon-p+N}$ with c independent of ε.

Proof. The existence of this imbedding is proved in [4] (Theorem 8.15) and the bound for this norm follows from its proof. □

Theorem 2.2. *The space $H^{1,p}(\Omega, d^{p-N})$ is dense in $W^{1,p}(\Omega, d^{p-N})$.*

Proof. Define a sequence of real functions on $[0,\infty]$ by

$$\varphi_n(t) = \begin{cases} (nt)^{1/n} & \text{for } t \in \left[0, \frac{1}{n}\right], \\ 1 & \text{for } t \in \left(\frac{1}{n}, \infty\right). \end{cases}$$

Let $u \in W^{1,p}(\Omega, d^{p-N})$. Let $u_n(x) = u(x)\varphi_n(|x|)$ and denote $\Omega_n = \Omega \cap B\big(0, \frac{1}{n}\big)$, where $B\big(0, \frac{1}{n}\big)$ stands for the ball with the center at the origin and with the radius $\frac{1}{n}$. Note that for every $i = 1, 2, \ldots, N$ we have $\big|D_i\varphi_n(|x|)\big| \leqslant n^{\frac{1}{n}-1}|x|^{\frac{1}{n}-1}$ for a.e. $x \in \Omega_n$. First we shall prove that $u_n \in H^{1,p}(\Omega, d^{p-N})$ for each positive integer n. An easy calculation gives

$$\begin{aligned} \big\|u_n|H^{1,p}(\Omega, d^{p-N})\big\|^p &\leqslant \int_{\Omega\setminus\Omega_n} |u(x)|^p|x|^{-N}\,\mathrm{d}x + 2^{p-1}\int_\Omega \sum_{i=1}^N |D_iu(x)|^p|x|^{p-N}\,\mathrm{d}x \\ &\quad + n^{p/n}(1 + N2^{p-1}n^{-p})\int_{\Omega_n} |u(x)|^p|x|^{p/n-N}\,\mathrm{d}x \\ &= I_1 + 2^{p-1}I_2 + n^{p/n}(1 + N2^{p-1}n^{-p})I_3. \end{aligned}$$

Since $n|x| \geqslant 1$ in $\Omega \setminus \Omega_n$, we get

$$I_1 \leqslant n^p \int_\Omega |u(x)|^p|x|^{p-N}\,\mathrm{d}x \leqslant n^p\big\|u\big|W^{1,p}(\Omega, d^{p-N})\big\|^p.$$

Evidently,

$$I_2 \leqslant \big\|u\big|W^{1,p}(\Omega, d^{p-N})\big\|^p.$$

According to Lemma 2.1 and because $|x|^{p/n} \leqslant 1$ in Ω_n, there exists a positive constant c_1 such that

$$\begin{aligned} I_3 &\leqslant c_1\Big(\int_{\Omega_n} |u(x)|^p|x|^{p/n+p-N}\,\mathrm{d}x + \int_{\Omega_n}\sum_{i=1}^N |D_iu(x)|^p|x|^{p/n+p-N}\,\mathrm{d}x\Big) \\ &\leqslant c_1\big\|u\big|W^{1,p}(\Omega, d^{p-N})\big\|^p. \end{aligned}$$

Thus $\{u_n\} \in H^{1,p}(\Omega, d^{p-N})$; note that the sequence $\big\|u_n\big|H^{1,p}(\Omega, d^{p-N})\big\|$ may be unbounded.

Now, we shall prove that $\|u - u_n|W^{1,p}(\Omega, d^{p-N})\| \to 0$ for $n \to \infty$. Obviously,

$$\begin{aligned}\|u - u_n|W^{1,p}(\Omega, d^{p-N})\|^p &\leqslant \int_{\Omega_n} |u(x)|^p (1 - \varphi_n(|x|))^p |x|^{p-N}\,dx \\ &\quad + 2^{p-1}\Big(\int_{\Omega_n} \sum_{i=1}^N \big|D_i u(x)\big|^p (1 - \varphi_n(|x|))^p |x|^{p-N}\,dx \\ &\quad + n^{p/n} \int_{\Omega_n} |u(x)|^p |x|^{p/n-N}\,dx\Big) \\ &= I_1(n) + 2^{p-1}\big(I_2(n) + n^{p/n-p} I_3(n)\big).\end{aligned}$$

Since $(1-\varphi_n(|x|))^p \leqslant 1$ and $|\Omega_n| \to 0$, we have $I_1(n) \to 0$ and $I_2(n) \to 0$. According to Lemma 2.1 we get

$$\begin{aligned}n^{p/n-p} I_3(n) &\leqslant c n^{p/n-p}\Big(\frac{p}{p/n}\Big)^p \|u|W^{1,p}(\Omega_n, d^{p/n+p-N})\|^p \\ &\leqslant c_3 n^{p/n} \|u|W^{1,p}(\Omega_n, d^{p-N})\|^p.\end{aligned}$$

The facts $n^{p/n} \to 1$ and $|\Omega_n| \to 0$ give $n^{p/n-p} I_3(n) \to 0$ which completes the proof. □

As an easy consequence we obtain the following theorem.

Theorem 2.3. *The set $C_0^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega, d^{p-N})$.*

3. Direct Decomposition

Theorem 2.2 implies that in the case $\varepsilon = p - N$ there is no space X such that $W^{1,p}(\Omega, d^{p-N}) = H^{1,p}(\Omega, d^{p-N}) \oplus X$. Our idea of characterization of traces is to find a space X such that $W^{1,p}(\Omega, d^{p-N}) = H^{1,p}(\Omega, d^{p-N}) + X$, but now, the sum on the right hand side is not direct.

In what follows we will use the following notation:

$$B_r = \{x \in \mathbb{R}^N : |x| < r, x_N > 0\}, \quad S_r = \{x \in \mathbb{R}^N : |x| = r, x_N > 0\}.$$

Let σ_N stand for the $(N-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^N$.

We shall prove the decomposition theorem only for the special case $\Omega = B_1$.

Lemma 3.1. *Let α be a real number, $0 < \alpha < 1$. Then there exists a positive constant c independent of α such that*

$$\iint\limits_{S_1 S_1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-2}}\,\mathrm{d}S_{N-1}(x)\,\mathrm{d}S_{N-1}(y) \leqslant c \int\limits_{S_1} \sum_{i=1}^{N} |D_i u(x)|^p\,\mathrm{d}S_{N-1}(x)$$

for all functions $u \in C^\infty(\overline{B_{1+\alpha}})$.

Proof. Fix $\lambda > 0$, $0 < \lambda < \alpha$ and $u \in C^\infty(\overline{B_{1+\alpha}})$. For $x \in B_2 - B_1$ we define a function v by $v(x) = u\big(x(\lambda + \frac{1-\lambda}{|x|})\big)$. Obviously, $v \in C^\infty(\overline{B_2 - B_1})$. According to the classical trace theorem in [5] there exists a positive constant c such that

$$\begin{aligned}(3.1)\qquad &\iint\limits_{S_1 S_1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-2}}\,\mathrm{d}S_{N-1}(x)\,\mathrm{d}S_{N-1}(y)\\ &\quad = \iint\limits_{S_1 S_1} \frac{|v(x)-v(y)|^p}{|x-y|^{N+p-2}}\,\mathrm{d}S_{N-1}(x)\,\mathrm{d}S_{N-1}(y)\\ &\quad \leqslant c \int\limits_{B_2\setminus B_1} \sum_{i=1}^{N} |D_i v(x)|^p\,\mathrm{d}x = cI.\end{aligned}$$

The direct calculation gives

$$D_i v(x) = \sum_{j=1}^{N} D_j u\Big(x\Big(\lambda + \frac{1-\lambda}{|x|}\Big)\Big)\Big(\delta_{ij}\Big(\lambda + \frac{1-\lambda}{|x|}\Big) + (\lambda - 1)\frac{x_i x_j}{|x|^3}\Big),$$

where $\delta_{i,j}$ stands for Kronecker's symbol. Consequently,

$$|D_i v(x)| \leqslant 2 \sum_{j=1}^{N} \Big|D_j u\Big(x\Big(\lambda + \frac{1-\lambda}{|x|}\Big)\Big)\Big|$$

which yields

$$(3.2)\qquad I \leqslant (2N)^p \int\limits_{B_2 - B_1} \sum_{j=1}^{N} \Big|D_j u\Big(x\Big(\lambda + \frac{1-\lambda}{|x|}\Big)\Big)\Big|^p\,\mathrm{d}x.$$

Now, we use the substitution $y = x\big(\lambda + \frac{1-\lambda}{|x|}\big)$, i.e.

$$(3.3)\qquad x = \frac{1}{\lambda}\Big(y - (1-\lambda)\frac{y}{|y|}\Big).$$

This transform is a composition of two transforms, the first being given by $z = y(1 - \frac{1-\lambda}{|y|})$ and the second by $x = \frac{z}{\lambda}$. The first transform is radial and transfers S_r on $S_{r(1-\frac{1-\lambda}{r})}$ for all $r \in (1,2)$. Since in the radial direction this transform is a shift, the Jacobian is equal to $\left(1 - \frac{1-\lambda}{|y|}\right)^{N-1}$, therefore the Jacobian $J(y)$ of the transform (3.3) is $J(y) = \frac{(|y|-\lambda-1)^{N-1}}{\lambda^N |y|^{N-1}}$. For $y \in B_{1+\lambda} \setminus B_1$ we have $0 \leqslant J(y) \leqslant \frac{2^{N-1}}{\lambda}$, which together with (3.1) and (3.2) gives

$$\iint\limits_{S_1 S_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+p-2}} \, \mathrm{d}S_{N-1}(x) \, \mathrm{d}S_{N-1}(y) \leqslant \frac{c(2N)^p 2^{N-1}}{\lambda} \int\limits_{B_{1+\lambda} \setminus B_1} \sum_{j=1}^{N} |D_j u(y)|^p \, \mathrm{d}y.$$

Due to the smoothness of u, letting $\lambda \to 0_+$ we obtain the assertion of the lemma. □

Define two linear integral operators K, L by

$$(Ku)(x) = \frac{2}{\sigma_N |x|^{N-1}} \int\limits_{S_{|x|}} u(y) \, \mathrm{d}S_{N-1}(y)$$

and

$$(Lu)(x) = u(x) - (Ku)(x).$$

Lemma 3.5. *The operator K is bounded from $W^{1,p}(B_1, d^{p-N})$ into $W^{1,p}(B_1, d^{p-N})$ and the operator L is bounded from $W^{1,p}(B_1, d^{p-N})$ into $H^{1,p}(B_1, d^{p-N})$.*

Proof. According to Theorem 2.3 we can consider $u \in C^\infty(\overline{B}_1)$. Denote $v(x) = (Ku)(x)$. Hölder's inequality and Fubini's theorem give

$$\begin{aligned}
\int\limits_{B_1} |v(x)|^p |x|^{p-N} \, \mathrm{d}x &= \int\limits_{B_1} \left| \frac{2}{\sigma_N |x|^{N-1}} \int\limits_{S_{|x|}} u(y) \, \mathrm{d}S_{N-1}(y) \right|^p |x|^{p-N} \, \mathrm{d}x \\
&\leqslant \int\limits_0^1 \frac{2}{\sigma_N r^{N-1}} \iint\limits_{S_r S_r} |u(y)|^p \, \mathrm{d}S_{N-1}(y) \, \mathrm{d}S_{N-1}(x) r^{p-N} \, \mathrm{d}r \\
&= \int\limits_0^1 \int\limits_{S_r} |u(y)|^p \, \mathrm{d}S_{N-1}(y) r^{p-N} \, \mathrm{d}r \leqslant \left\| u | W^{1,p}(\Omega_1, d^{p-N}) \right\|^p,
\end{aligned}$$

which yields

$$\left\| Ku | L^p(B_1, d^{p-N}) \right\| \leqslant \left\| u | W^{1,p}(B_1, d^{p-N}) \right\|. \tag{3.4}$$

Obviously,

$$v(x) = \frac{2}{\sigma_r} \int\limits_{S_1} u(z|x|)\, \mathrm{S}_{N-1}(z)$$

and consequently,

$$D_i v(x) = \frac{2}{\sigma_N} \int\limits_{S_1} \sum_{j=1}^{N} D_j u(z|x|) z_j \frac{x_i}{|x|} \,\mathrm{d}S_{N-1}(z) = \frac{2x_i}{\sigma_N |x|^N} \int\limits_{S|x|} \frac{\partial u}{\partial n}(y) \,\mathrm{d}S_{N-1}(y),$$

where $\frac{\partial u}{\partial n}$ stands for the derivative with respect to the outer normal. Since $\frac{x_i}{|x|^N} \leqslant \frac{1}{|x|^{N-1}}$, we obtain in an analogous way the estimate

$$\big\|D_i v|L^p(B_1, d^{p-N})\big| \leqslant c\big\|u|W^{1,p}(B_1, d^{p-N})\big\| \tag{3.5}$$

which together with (3.3) gives the first assertion of the lemma.

Now, we shall prove the boundedness of the operator L. The inequality (3.5) implies

$$\big\|D_i u - D_i v|L^p(B_1, d^{p-N})\big\| \leqslant (1+c)\big\|u|W^{1,p}(B_1, d^{p-N})\big\|.$$

It remains to estimate $\big\|u - v|L^p(B_1, d^{-N})\big\|$. We have

$$\big\|u - v|L^p(B_1, d^{-N})\big\|^p = \int\limits_0^1 \int\limits_{S_r} \left| \frac{2}{\sigma_N r^{N-1}} \int\limits_{S_r} (u(x) - u(y)) \,\mathrm{d}S_{N-1}(y) \right|^p \mathrm{d}S_{N-1}(x) r^{-N} \,\mathrm{d}r$$

$$\leqslant \frac{2}{\sigma_N} \int\limits_0^1 \frac{1}{r^{N-1}} \int\limits_{S_r}\int\limits_{S_r} |u(x) - u(y)|^p \,\mathrm{d}S_{N-1}(y) \,\mathrm{d}S_{N-1}(x) r^{-N} \,\mathrm{d}r.$$

The substitutions $x = r\xi$, $y = r\eta$ give

$$\int\limits_{S_r}\int\limits_{S_r} |u(x) - u(y)|^p \,\mathrm{d}S_{N-1}(y) \,\mathrm{d}S_{N-1}(x)$$

$$= r^{2N-2} \int\limits_{S_1}\int\limits_{S_1} |u(r\xi) - u(r\eta)|^p \,\mathrm{d}S_{N-1}(\eta) \,\mathrm{d}S_{N-1}(\xi).$$

For $r \in (0,1)$ denote the right hand side by $J(r)$ and set $w(\xi) = u(r\xi)$, $x \in S_i$. Lemma 3.1 yields

$$J(r) \leqslant c_1 r^{2N-2} \int\limits_{S_1}\int\limits_{S_1} \frac{|w(\xi) - w(\eta)|^p}{|\xi - \eta|^{N+p-2}} \,\mathrm{d}S_{N-1}(\xi) \,\mathrm{d}S_{N-1}(\eta)$$

$$\leqslant c_2 r^{2N-2} \int\limits_{S_1} \sum_{i=1}^{N} |D_i w(\xi)|^p \,\mathrm{d}S_{N-1}(\xi) = c_2 r^{p+N-1} \int\limits_{S_r} \sum_{i=1}^{N} |D_i u(x)|^p \,\mathrm{d}S_{N-1}(x).$$

Thus,

$$\begin{aligned}\big\|u-v\big|L^p(B_1,d^{-N})\big\|^p &\leqslant \frac{c_2}{\sigma_N}\int\limits_0^1 r^{p-N}\int\limits_{S_r}\sum_{i=1}^N |D_iu(x)|^p \,\mathrm{d}S_{N-1}(x)\,\mathrm{d}r\\ &\leqslant \frac{c_2}{\sigma_N}\big\|u\big|W^{1,p}(B_1,d^{p-N})\big\|^p\end{aligned}$$

and we are done. □

A function u is said to be radial if and only if u has a constant value on each sphere S_r, $0 < r$, i.e. $u(x) = u(|x|, 0, 0, \dots, 0)$ for all $x \in B_1$.

Definition 3.3. Let us define spaces $V^{1,p}(B_1, d^{p-N})$ and $X(B_1, d^{p-N})$. The space $V^{1,p}(B_1, d^{p-N})$ is defined as the closure of all radial functions from $C^\infty(\overline{B}_1)$ in the space $W^{1,p}(B_1, d^{p-N})$, the norm of a function u is equal to the norm of u in $W^{1,p}(B_1, d^{p-N})$. The space $X(B_1, d^{p-N})$ is the set of all functions $u = u_1 + u_2$, where $u_1 \in H^{1,p}(B_1, d^{p-N})$ and $u_2 \in V^{1,p}(B_1, d^{p-N})$. The norm in this space is given by

$$\big\|u\big|X(B_1,d^{p-N})\big\| = \inf_{u=u_1+u_2}\left(\big\|u_1\big|H^{1,p}(B_1,d^{p-N})\big\| + \big\|u_2\big|V^{1,p}(B_1,d^{p-N})\big\|\right).$$

Let us prove the basic decomposition theorem.

Theorem 3.4. *The spaces $W^{1,p}(B_1, d^{p-N})$ and $X(B_1, d^{p-N})$ coincide and the norms are equivalent.*

Proof. Let $u \in W^{1,p}(B_1, d^{p-N})$. From Lemma 3.2 we immediately obtain

$$\begin{aligned}\big\|u\big|X(B_1,d^{p-N})\big\| &\leqslant \big\|Lu\big|H^{1,p}(B_1,d^{p-N})\big\| + \big\|Ku\big|W^{1,p}(B_1,d^{p-N})\big\|\\ &\leqslant c\big\|u\big|W^{1,p}(B_1,d^{p-N})\big\|.\end{aligned}$$

The reverse inequality is a direct consequence of the obvious imbeddings $H^{1,p}(B_1, d^{p-N}) \hookrightarrow W^{1,p}(B_1, d^{p-N})$ and $V^{1,p}(B_1, d^{p-N}) \hookrightarrow W^{1,p}(B_1, d^{p-N})$.

Note that it is possible to prove a similar decomposition theorem for more general domains. However, for the characterization of traces in Theorems 4.11 and 5.4 we shall need only the special case $\Omega = B_1$. □

4. Direct Theorem

Definition 4.1. Let $G \subset \partial\Omega$ and $0 < s < 1$. Define the space $\widetilde{W}^{s,p}(G, d^\varepsilon)$ as the set of all functions u defined on G with a finite norm

$$\|u|\widetilde{W}^{s,p}(G, d^\varepsilon)\| = \left(\int_G |u(x)|^p |x|^\varepsilon \, \mathrm{d}S_{N-1}(x) + \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{N-1+sp}} |x|^\varepsilon \, \mathrm{d}S_{N-1}(y) \, \mathrm{d}S_{N-1}(x) \right)^{1/p}.$$

Now, our aim is to prove the direct trace theorem. By P_r for $r > 0$ we will denote the set $P_r = \{x \in \mathbb{R}^N : x = (x', 0), |x'| < r\}$. We make use of the weighted Sobolev space $H^{1,p}_{(1)}(B_1, d^{p-N})$ which is defined as the space of all functions u on B_1 with a finite norm

$$\|u|H^{1,p}_{(1)}(B_1, d^{p-N})\| = \left(\int_{B_1} |u(x)|^p |x|^{-N} \left(\ln \frac{2}{|x|} \right)^{-p} \mathrm{d}x + \int_{B_1} \sum_{i=1}^N |D_i u(x)|^p |x|^{p-N} \, \mathrm{d}x \right)^{1/p}$$

which was introduced by Kufner, Kadlec in [6]. Similarly, by $L^p(P_1, d^{-N+1}(\ln \frac{2}{d})^{-p})$ we understand the set of all functions u defined on P_1 with a finite norm

$$\left\|u\Big|L^p\left(P_1, d^{-N+1}\left(\ln \frac{2}{d}\right)^{-p}\right)\right\| = \left(\int_{P_1} |u(x', 0)|^p |x'|^{-N+1} \left(\ln \frac{2}{|x'|} \right)^{-p} \mathrm{d}S_{N-1}(x) \right)^{1/p}.$$

The following two assertions will be used in the proof of Lemma 5.11 below.

Proposition 4.2 (see [1]). *The spaces* $W^{1,p}(B_1, d^{p-N})$ *and* $H^{1,p}_{(1)}(B_1, d^{p-N})$ *coincide and the norms are equivalent.*

Proposition 4.3 (see [1]). *There exists a unique bounded trace operator*

$$T\colon H^{1,p}_{(1)}(B_1, d^{p-N}) \to L^p\left(P_1, d^{-N+1}\left(\ln \frac{2}{d}\right)^{-p}\right).$$

Lemma 4.4. *There exists a unique bounded trace operator*

$$T\colon W^{1,p}(B_1, d^{p-N}) \to L^p(P_1, d^{p-N}).$$

Proof. This follows immediately from Propositions 4.2 and 4.3 and from the obvious fact that $|x|^{p-N} \leqslant c|x|^{-N}\left(\ln \frac{2}{|x|}\right)^{-p}$ on B_1. □

Now, we will prove that the trace operator T is bounded as a mapping from $W^{1,p}(B_1, d^{p-N})$ in $\widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$. We will proceed as follows: We decompose the space $W^{1,p}(B_1, d^{p-N})$ in accordance with Theorem 3.4. In Theorem 4.7 we establish the boundedness of $T\colon W^{1,p}(B_1, d^{p-N}) \to \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$. Proposition 1.3 implies that T is a bounded operator from $H^{1,p}(B_1, d^{p-N})$ into $H^{1-1/p,p}(P_1, d^{p-N})$. This and the result of Theorem 4.9 yield the boundedness of $T\colon H^{1,p}(B_1, d^{p-N}) \to \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$.

Lemma 4.5. *There exists such a positive constant c that for all $u \in C^\infty([0,1])$,*

$$\int_0^1\int_0^1 \frac{|u(x)-u(y)|^p}{|x-y|^p}\,dy\, x^{p-2}\,dx \leqslant c\int_0^1 |u'(x)|^p x^{p-1}\,dx.$$

Proof. The left hand side of the inequality can be expressed as the sum of two integrals,

$$I_1 = \int_0^1\int_0^x \frac{|u(x)-u(y)|^p}{|x-y|^p}\,dy\, x^{p-2}\,dx$$

and

$$I_2 = \int_0^1\int_x^1 \frac{|u(x)-u(y)|^p}{|x-y|^p}\,dy x^{p-2}\,dx.$$

Let us estimate I_1. Obviously,

$$I_1 \leqslant \int_0^1\int_0^x \left(\frac{1}{x-y}\int_y^x |u'(t)|\,dt\right)^p dy x^{p-2}\,dx.$$

Assume first that $p \geqslant 2$. Then $x^{p-2} \leqslant \max(1, 2^{p-3})[(x-y)^{p-2} + y^{p-2}]$, and so

$$I_1 \leqslant \max(1,2^{p-3})\Bigg[\int_0^1\int_0^x \left(\frac{1}{x-y}\int_y^x |u'(t)|\,dt\right)^p (x-y)^{p-2}\,dy\,dx$$
$$+ \int_0^1\int_0^x \left(\frac{1}{x-y}\int_y^x |u'(t)|\,dt\right)^p y^{p-2}\,dy\,dx\Bigg] = \max(1,2^{p-3})(I_{11}+I_{12}).$$

Using Example 6.8 in [7] with $\varepsilon = p-2$ and Fubini's theorem we obtain

$$I_{11} = \int_0^1 \int_y^1 \left(\frac{1}{x-y} \int_y^x |u'(t)| \,\mathrm{d}t \right)^p (x-y)^{p-2} \,\mathrm{d}y \,\mathrm{d}x$$
$$\leqslant c \int_0^1 \int_y^1 |u'(x)|^p (x-y)^{p-2} \,\mathrm{d}x \,\mathrm{d}y = \frac{c}{p-1} \int_0^1 |u'(x)|^p x^{p-1} \,\mathrm{d}x.$$

To estimate I_{12} we use Fubini's theorem and Example 6.8 in [7] with $\varepsilon = 0$:

$$I_{12} \leqslant c \int_0^1 \int_y^1 |u'(x)|^p y^{p-2} \,\mathrm{d}x \,\mathrm{d}y = \frac{c}{p-1} \int_0^1 |u'(x)|^p x^{p-1} \,\mathrm{d}x.$$

Thus,

$$I_1 \leqslant c_1 \int_0^1 |u'(x)|^p x^{p-1} \,\mathrm{d}x$$

for $p \geqslant 2$. Now, suppose $1 < p < 2$. Fubini's theorem and Example 6.8 in [7] with $\varepsilon = 0$ yield

$$(4.1) \qquad I_1 \leqslant \int_0^1 \int_y^1 \left(\frac{1}{x-y} \int_y^x |u'(t)| \,\mathrm{d}t \right)^p \mathrm{d}x y^{p-2} \,\mathrm{d}y$$
$$\leqslant c \int_0^1 \int_0^x |u'(x)|^p y^{p-2} \,\mathrm{d}x \,\mathrm{d}y = \frac{c}{p-1} \int_0^1 |u'(x)|^p x^{p-1} \,\mathrm{d}x.$$

To estimate I_2 we use Example 6.8 in [7] with $\varepsilon = 0$ and Fubini's theorem; we obtain

$$I_2 \leqslant \int_0^1 \int_x^1 |u'(y)|^p \,\mathrm{d}y \, x^{p-2} \,\mathrm{d}x \leqslant c \int_0^1 \int_x^1 |u'(x)|^p \,\mathrm{d}y \, x^{p-2} \,\mathrm{d}x = \frac{c}{p-1} \int_0^1 |u'(y)|^p y^{p-1} \,\mathrm{d}y.$$

The last estimate and (4.1) give the desired inequality. □

Lemma 4.6. *Let $N \geqslant 2$. Then there exists a positive constant c such that the inequality*

$$\int_{|y|=r} \frac{1}{|x-y|^{N+p-1}} \,\mathrm{d}S_{N-1}(y) \leqslant \frac{c}{\big||x|-r\big|^p}$$

holds for all $x \in \mathbb{R}^N$ and $r > 0$.

P r o o f. Denote the integral on the left hand side by $I_N(x)$. For $|x| = r$ the inequality is obvious. Therefore assume $|x| \neq r$. In view of the spherical symmetry of $I_N(x)$ we can suppose that $x = (0, 0, \ldots, |x|) \in \mathbb{R}^N$. Recall that for $y \in \mathbb{R}^N$ we write $y = (y', y_N)$, $y' \in \mathbb{R}^{N-1}$. The substitution $y_N = r\cos\varphi$ gives

$$\begin{aligned} I_N(x) &= \int_0^{\pi} \int_{|y'|=r\cos\varphi} \frac{\mathrm{d}S_{N-2}(y')\,\mathrm{d}\varphi}{\big((|x| - r\cos\varphi)^2 + (r\sin\varphi)^2\big)^{\frac{N+p-1}{2}}} \\ &= \sigma_{N-1} \int_0^{\pi} \frac{r^{N-1}\sin^{N-2}\varphi}{(|x|^2 + r^2 - 2|x|r\cos\varphi)^{\frac{N+p-1}{2}}}\,\mathrm{d}\varphi. \end{aligned}$$

Let $0 \leqslant \varphi \leqslant \pi$. We have

$$(|x| - r)^2 \leqslant (|x| - r)^2 + 4|x|r\sin^2\frac{\varphi}{2} = |x|^2 + r^2 - 2|x|r\cos\varphi. \tag{4.2}$$

If $|x| \leqslant \frac{1}{2}r$, then $r < 2(r - |x|)$, and so

$$r^2\varphi^2 \leqslant 4\pi^2(|x| - r)^2 \leqslant 4\pi^2(|x|^2 + r^2 - 2|x|r\cos\varphi).$$

If $|x| \geqslant \frac{1}{2}r$, then

$$r^2\varphi^2 \leqslant 2|x|r\varphi^2 \leqslant \frac{\pi^2}{2}\Big[\frac{4|x|r}{\pi^2}\varphi^2 + (|x| - r)^2\Big] \leqslant \frac{\pi^2}{2}\Big[(|x| - r)^2 + 4|x|r\sin^2\frac{\varphi}{2}\Big].$$

In both cases we have

$$r\varphi \leqslant 2\pi(|x|^2 + r^2 - 2|x|r\cos\varphi)^{1/2},$$

which, together with (4.2), yields

$$\big||x| - r\big| + r\varphi \leqslant (1 + 2\pi)(|x|^2 + r^2 - 2|x|r\cos\varphi)^{1/2}.$$

Consequently,

$$I_N \leqslant (1 + 2\pi)^{N+p-1} \int_0^{\pi} \frac{r^{N-1}\sin^{N-2}\varphi\,\mathrm{d}\varphi}{\big(\big||x| - r\big| + r\varphi\big)^{N+p-1}}.$$

For $N = 2$ we have

$$I_2(x) = (1 + 2\pi)^{p+1} \int_0^{\pi} \frac{r}{\big(\big||x| - r\big| + r\varphi\big)^{p+1}}\,\mathrm{d}\varphi \leqslant \frac{(1 + 2\pi)^{p+1}}{p}\frac{1}{\big||x| - r\big|^p}.$$

If $N \geqslant 3$, integration by parts gives

$$I_N(x) \leqslant (1 + 2\pi)^{N+p-1}\frac{N-2}{N+p-2}I_{N-1}(x) \leqslant \ldots \leqslant \frac{c}{\big||x| - r\big|^p},$$

which completes the proof. □

Theorem 4.7. *Let $N \geqslant 2$. Then the trace operator*

$$T\colon V^{1,p}(B_1, d^{p-N}) \to \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$$

is bounded.

Proof. In accordance with Definition 3.3 assume that $u \in C^\infty(\overline{B}_1)$. Define a function v of one real variable by $v(r) = u(x)$ for $r = |x|$. Using Lemmas 4.6 and then 4.5 we obtain

$$\begin{aligned}
&\int\limits_{|x'|<1}\int\limits_{|y'|<1} \frac{|u(x',0)-u(y',0)|^p}{|x'-y'|^{N+p-2}}\,\mathrm{d}S_{N-1}(y')|x'|^{p-N}\,\mathrm{d}x' \\
&= \int\limits_0^1\int\limits_0^1 |v(r)-v(\varrho)|^p \int\limits_{|x'|=r}\int\limits_{|y'|=r} \frac{1}{|x'-y'|^{N+p-2}}\,\mathrm{d}S_{N-2}(y')\,\mathrm{d}S_{N-2}(x')r^{p-N}\,\mathrm{d}\varrho\,\mathrm{d}r \\
&\leqslant c_1 \int\limits_0^1\int\limits_0^1 |v(r)-v(\varrho)|^p \int\limits_{|x'|=r} \frac{1}{\big||x'|-\varrho\big|^p}\,\mathrm{d}S_{N-2}(x')r^{p-N}\,\mathrm{d}\varrho\,\mathrm{d}r \\
&= \sigma_{N-1}c_1 \int\limits_0^1\int\limits_0^1 \frac{|v(r)-v(\varrho)|^p}{|r-\varrho|^p}\,\mathrm{d}\varrho\; r^{p-2}\,\mathrm{d}r \leqslant c_2 \int\limits_0^1 |v'(r)|^p r^{p-1}\,\mathrm{d}r \\
&= \frac{2c_2}{\sigma_N}\int\limits_0^1 |v'(r)|^p \int\limits_{S_r} |x|^{p-N}\,\mathrm{d}S_{N-1}(x)\,\mathrm{d}r \\
&\leqslant c_3 \int\limits_0^1\int\limits_{S_r} \sum_{i=1}^N |D_i u(x)|^p |x|^{p-N}\,\mathrm{d}S_{N-1}(x)\,\mathrm{d}r \leqslant c_3\big\|u\big|W^{1,p}(B_1, d^{p-N})\big\|^p.
\end{aligned}$$

To complete the proof we observe that, by Lemma 4.4,

$$\big\|(u\big|_{P_1})|L^p(P_1, d^{p-N})\big\| \leqslant c\big\|u\big|W^{1,p}(B_1, d^{p-N})\big\|^p.$$

□

Lemma 4.8. *Let $N \geqslant 2$. Then there exists a positive constant c such that*

$$\int\limits_{\mathbb{R}^{N-1}} \frac{\big||x'|^{(p-N)/p} - |y'|^{(p-N)/p}\big|^p}{|x'-y'|^{N+p-2}}\,\mathrm{d}y' \leqslant c|x'|^{1-N}$$

for all $x' \in \mathbb{R}^{N-1}$.

Proof. Remark that the inequality is trivial for $x' = 0$. Therefore we can assume $x' \neq 0$. Denote the integral on the left hand side by $I(x')$. Substituting $y' = |x'|t$ we get

$$\begin{aligned} I(x') &= |x'|^{1-N} \int\limits_{\mathbb{R}^{N-1}} \frac{\left|1-|t|^{(p-N)/p}\right|^p}{|\lambda - t|^{N+p-2}} \,\mathrm{d}t \\ &= |x'|^{1-N} \Bigg(\int\limits_{|t|\leqslant \frac{1}{2}} \frac{\left|1-|t|^{(p-N)/p}\right|^p}{|\lambda - t|^{N+p-2}} \,\mathrm{d}t + \int\limits_{|t|>\frac{1}{2}} \frac{\left|1-|t|^{(p-N)/p}\right|^p}{|\lambda - t|^{N+p-2}} \,\mathrm{d}t \Bigg) \\ &= |x'|^{1-N} \big(J_1(\lambda) + J_2(\lambda)\big), \end{aligned}$$

where $\lambda = \frac{x'}{|x'|}$. Let us first estimate $J_2(\lambda)$. According to Lemma 4.6 we obtain

$$J_2(\lambda) = \int\limits_{1/2}^{\infty} |1 - r^{(p-N)/p}|^p \int\limits_{|t|=r} \frac{\mathrm{d}S_{N-1}(t)}{|\lambda - t|^{N+p-2}} \,\mathrm{d}r \leqslant c \int\limits_{1/2}^{\infty} \frac{|1 - r^{(p-N)/p}|^p}{|1-r|^p} \,\mathrm{d}r.$$

Since the integrand is $O(1)$ as $r \to 1$ and $O(r^{-N})$ as $r \to \infty$ we have $J_2(\lambda) < \infty$. Now, let us estimate $J_1(\lambda)$. If $p \geqslant N$, then the integrand is a continuous function on $\left[0, \frac{1}{2}\right]$ and, consequently, integrable. If $p < N$, the integrand is $O(|t|^{p-N})$ as $|t| \to 0$ and so, using the spherical coordinates, we obtain again $J_1(\lambda) < \infty$ which completes the proof. □

Theorem 4.9. *Let $N \geqslant 2$. Then $H^{1-1/p,p}(P_1, d^{p-N}) \hookrightarrow \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$.*

Proof. Let $u \in H^{1-1/p,p}(P_1, d^{p-N})$. Since $|x|^{p-N} \leqslant c|x|^{1-N}$ on P_1, we have

$$\int\limits_{P_1} |u(x)|^p |x|^{p-N} \,\mathrm{d}S_{N-1}(x) \leqslant c \big\| u \big| H^{1-1/p,p}(P_1, d^{p-N}) \big\|^p.$$

To estimate the corresponding seminorm we use Lemma 4.8:

$$\begin{aligned} &\int\limits_{|x'|<1} \int\limits_{|y'|<1} \frac{|u(x',0) - u(y',0)|^p}{|x' - y'|^{N+p-2}} \,\mathrm{d}y' |x'|^{p-N} \,\mathrm{d}x' \\ &\leqslant 2^{p-1} \Bigg(\int\limits_{|x'|<1} \int\limits_{|y'|<1} \frac{\left|u(x',0)\,|x'|^{(p-N)/p} - u(y',0)\,|y'|^{(p-N)/p}\right|^p}{|x' - y'|^{N+p-2}} \,\mathrm{d}y' \,\mathrm{d}x' \\ &\quad + \int\limits_{|x'|<1} \int\limits_{|y'|<1} |u(y',0)|^p \frac{\left||x'|^{(p-N)/p} - |y'|^{(p-N)/p}\right|^p}{|x' - y'|^{N+p-2}} \,\mathrm{d}y' \,\mathrm{d}x' \Bigg) \\ &\leqslant 2^{p-1}(1+c) \big\| u \big| H^{1-1/p,p}(P_1, d^{p-N}) \big\|^p. \end{aligned}$$

□

As an easy consequence of Proposition 1.3, Lemma 3.4, 4.7 and 4.9 we have the following Lemma.

Lemma 4.10. *Let $N \geqslant 2$. Then the trace operator*

$$T\colon W^{1,p}(B_1, d^{p-N}) \to \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$$

is bounded.

Using the local covering of the boundary from Definition 1.1 and standard techniques it is not difficult to extend Lemma 4.10 in the following way.

Theorem 4.11. *Let $N \geqslant 2$. Then the trace operator*

$$T\colon W^{1,p}(\Omega, d^{p-N}) \to \; W^{1-1/p,p}(\partial\Omega, d^{p-N})$$

is bounded.

5. Extension operator

Now we will construct an extension operator R corresponding to the operator T. First we will deal with the particular case of the cylindrical domain $C = \{x \in \mathbb{R}^N : x = (x', x_N), |x'| < 1, 0 < x_N < 1\}$. Recall that the Hardy–Littlewood maximal operator is defined for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ by $(Mu)(x) = \sup \frac{1}{|B|} \int_B |u(y)|\,\mathrm{d}y$, where the supremum is taken over all balls B in $\mathbb{R}^N$ such that $x \in B$. Let $\varphi_{N-1} \in C^\infty(R^{N-1})$ be a function satisfying $\int\limits_{R^{N-1}} \varphi_{N-1}(x)\,\mathrm{d}x = 1$, $\varphi_{N-1}(x) \geqslant 0$ and $\varphi_{N-1}(x) = 0$ for $|x| \geqslant 1$.

Lemma 5.1 (see [8]). *Let α be a real number. Then the inequality*

$$\big\|Mu\big|L^p(\mathbb{R}^N, |x|^\alpha)\big\| \leqslant c\big\|u\big|L^p(\mathbb{R}^N, |x|^\alpha)\big\|$$

holds for all $u \in L^p(\mathbb{R}^N, |x|^\alpha)$ if and only if $-N < \alpha < N(p-1)$.

Lemma 5.2. *Let $N \geqslant 2$. Then the operator R defined by*

$$(Ru)(x', x_N) = \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} \varphi_{N-1}\Big(\frac{x'-y'}{x_N}\Big) u(y')\,\mathrm{d}y'$$

is a linear and bounded mapping from $\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})$ into $W^{1,p}(C, d^{p-N})$.

Proof. Let $u \in \widetilde{W}^{1-1/p,p}(P_2, d^{p-N})$. Extend the function u by zero for $(x', 0)$, $|x'| \geqslant 2$. We will denote this extension again by u. We can write

$$(D_i Ru)(x', x_N) = \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} D_i\varphi_{N-1}\Big(\frac{x'-y'}{x_N}\Big)\frac{u(y',0)-u(x',0)}{x_N}\,\mathrm{d}y'$$

for $i = 1, 2, \ldots, N-1$ and

$$(D_N Ru)(x', x_N) = \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} \Big((1-N)\varphi_{N-1}\Big(\frac{x'-y'}{x_N}\Big) - \sum_{i=1}^{N-1} D_i\varphi_{N-1}\Big(\frac{x'-y'}{x_N}\Big)\frac{x_i-y_i}{x_N}\Big)\frac{u(y',0)-u(x',0)}{x_N}\,\mathrm{d}y'.$$

Let us estimate $I_0 = \big\|Ru\big|L^p(C, d^{p-N})\big\|^p$. We have

$$I_0 \leqslant c\Big(\int\limits_{|x'|<1}\int\limits_0^{|x'|}\Big(\frac{1}{x_N^{N-1}}\int\limits_{|x'-y'|<x_N}|u(y',0)|^p\,\mathrm{d}y'\Big)^p |x'|^{p-N}\,\mathrm{d}x_N\,\mathrm{d}x' + \int\limits_{|x'|<1}\int\limits_{|x'|}^{1}\Big(\frac{1}{x_N^{N-1}}\int\limits_{|x'-y'|<x_N}|u(y',0)|^p\,\mathrm{d}y'\Big)^p x_N^{p-N}\,\mathrm{d}x_N\,\mathrm{d}x'\Big) = c(I_{01}+I_{02}).$$

By Fubini's theorem we obtain

$$I_{01} \leqslant \int\limits_0^1\Big(\int\limits_{x_N<|x'|<1}|Mu(x',0)|^p|x'|^{p-N}\,\mathrm{d}x'\Big)\,\mathrm{d}x_N.$$

According to Lemma 5.1 we have

$$\text{(5.1)}\qquad I_{01} \leqslant \int\limits_{|x'|<1}|Mu(x')|^p|x'|^{p-N}\,\mathrm{d}x' \leqslant c_1\int\limits_{|x'|<2}|u(x')|^p|x'|^{p-N}\,\mathrm{d}x' \leqslant c_1\big\|u\big|\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})\big\|^p.$$

Let us estimate I_{02}. If $p \leqslant N$, then the inequality $|x'| \leqslant x_N$ yields $x_N^{p-N} \leqslant |x'|^{p-N}$. Analogously as in the estimate of I_{01} we get

$$\text{(5.2)}\qquad I_{02} \leqslant \int\limits_0^1\Big(\int\limits_{|x'|<x_N}|Mu(x',0)|^p|x'|^{p-N}\,\mathrm{d}x'\Big)\,\mathrm{d}x_N \leqslant c_2\big\|u\big|\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})\big\|^p.$$

Let $p > N$. Using Hölder's inequality we obtain

$$I_{02} \leqslant \int\limits_{|x'|<1} \int\limits_{|x'|}^{1} \frac{x_N^{p-N}}{x_N^{p(N-1)}} \Bigg(\int\limits_{|x'-y'|<x_N} |u(y',0)|^p |y'|^{p-N} \, dy' \Bigg) \times \Bigg(\int\limits_{|x'-y'|<x_N} |y'|^{\frac{N-p}{p-1}} \, dy' \Bigg)^{p-1} dx_N \, dx'.$$

Since $\frac{N-p}{p-1} < 0$, we can use the obvious estimate

$$\int\limits_{|x'-y'|<x_N} |y'|^{\frac{N-p}{p-1}} \, dy' \leqslant \int\limits_{|y'|<x_N} |y'|^{\frac{N-p}{p-1}} \, dy' = \sigma_{N-1} \int\limits_0^{x_N} r^{\frac{N-p}{p-1}+N-2} \, dr = c_3 x_N^{\frac{N-p}{p-1}+N-1}$$

It implies that

$$\begin{aligned} I_{02} &\leqslant c_3 \int\limits_{|x'|<1} \int\limits_{|x'|}^{1} \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} |u(y',0)|^p |y'|^{p-N} \, dy' \, dx_N \, dx' \\ &\leqslant c_3 \int\limits_0^1 \int\limits_{|x'|<1} \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} |u(y',0)|^p |y'|^{p-N} \, dy' \, dx' \, dx_N. \end{aligned}$$

Using the substitution $\frac{x'-y'}{x_N} = t'$ and Fubini's theorem we obtain

$$I_{02} \leqslant c_3 \int\limits_0^1 \int\limits_{|t'|<1} \int\limits_{|x'|<1} |u(x' - t'x_N, 0)|^p |x' - t'x_N|^{p-N} \, dx' \, dt' \, dx_N.$$

The substitution $z' = x' - t'x_N$ gives $|z'| = |x' - t'x_N| \leqslant |x'| + |t'|x_N \leqslant 2$, which immediately yields

$$I_{02} \leqslant c_3 \int\limits_{|t'|<1} \int\limits_{|z'|<2} |u(z')|^p |z'|^{p-N} \, dz' \, dt' \leqslant c_3 \big\| u | \widetilde{W}^{1-1/p,p}(P_2, d^{p-N}) \big\|^p.$$

The last estimate, (5.1) and (5.2) imply

$$I_0 \leqslant c_4 \big\| u | \widetilde{W}^{1-1/p,p}(P_2, d^{p-N}) \big\|^p. \tag{5.3}$$

Now, let us estimate $I_i = \|D_i Ru|W^{1,p}(C, d^{p-N})\|^p$. Using Fubini's theorem we obtain

$$
\begin{aligned}
I_i &\leqslant c_5\Bigg(\int\limits_{|x'|<1} \int\limits_0^{|x'|} \Bigg(\frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} \frac{|u(y',0)-u(x',0)|}{x_N} \, \mathrm{d}y' \Bigg)^p |x'|^{p-N} \, \mathrm{d}x_N \, \mathrm{d}x' \\
&\quad + \int\limits_{|x'|<1} \int\limits_{|x'|}^{1} \Bigg(\frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} \frac{|u(y',0)-u(x',0)|}{x_N} \, \mathrm{d}y' \Bigg)^p x_N^{p-N} \, \mathrm{d}x_N \, \mathrm{d}x' \Bigg) \\
&= c_5(I_{i1} + I_{i2}).
\end{aligned}
$$

By Hölder's inequality and Fubini's theorem we have

$$
\begin{aligned}
I_{i1} &\leqslant \int\limits_{|x'|<1} \int\limits_{|x'-y'|<|x'|} \frac{|u(y',0)-u(x',0)|^p}{|y'-x'|^{N+p-2}} \int\limits_{|x'-y'|}^{|x'|} \frac{|y'-x'|^{N+p-2}}{x_N^{N+p-1}} \, \mathrm{d}x_N |x'|^{p-N} \, \mathrm{d}y' \, \mathrm{d}x' \\
&\leqslant \frac{1}{N+p-2} \int\limits_{|x'|<1} \int\limits_{|x'-y'|<|x'|} \frac{|u(y',0)-u(x',0)|^p}{|y'-x'|^{N+p-2}} |x'|^{p-N} \, \mathrm{d}y' \, \mathrm{d}x'.
\end{aligned}
$$

Since $|y'| - |x'| \leqslant |y' - x'| \leqslant |x'|$, we have $|y'| \leqslant 2|x'| \leqslant 2$. Thus, extending the integration domain we obtain

$$
I_{i1} \leqslant c_6 \|u|\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})\|^p. \tag{5.4}
$$

To estimate I_{i2} we use analogous techniques as in the estimate of I_{02}. If $p \leqslant N$, then Hölder's inequality and Fubini's theorem yield

$$
\begin{aligned}
I_{i2} &\leqslant \int\limits_{|x'|<1} \int\limits_{|x'-y'|<1} \frac{|u(y')-u(x')|^p}{|y'-x'|^{N+p-2}} \int\limits_{|x'-y'|}^{1} \frac{|y'-x'|^{N+p-2}}{x_N^{N+p-1}} \, \mathrm{d}x_N |x'|^{p-N} \, \mathrm{d}y' \, \mathrm{d}x' \\
&\leqslant c_7 \|u|\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})\|^p.
\end{aligned} \tag{5.5}
$$

In the case $p > N$ we get

$$
\begin{aligned}
I_{i2} &\leqslant c_8 \int\limits_{|x'|<1} \int\limits_{|x'|}^{1} \frac{1}{x_N^{N-1}} \int\limits_{|x'-y'|<x_N} \frac{|u(y',0)-u(x',0)|^p}{x_N^p} |y'|^{p-N} \, \mathrm{d}y' \, \mathrm{d}x_N \, \mathrm{d}x, \\
&\leqslant c_9 \int\limits_{|x'|<1} \int\limits_{|x'-y'|<1} \frac{|u(y',0)-u(x',0)|^p}{|y'-x'|^{N+p-2}} |y'|^{p-N} \, \mathrm{d}y' \, \mathrm{d}x' \\
&\leqslant c_9 \|u|\widetilde{W}^{1-1/p,p}(P_2, d^{p-N})\|^p.
\end{aligned}
$$

□

The assertion of the lemma follows from the last estimate, (5.3), (5.4) and (5.5).

Lemma 5.3. *Let $N \geqslant 2$. Then there exists a linear bounded operator*

$$R\colon \widetilde{W}^{1-1/p,p}(P_1, d^{p-N}) \to W^{1,p}(C, d^{p-N})$$

such that $RTu = u$ for all $u \in \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$.

Proof. Lemma 3.2 in [9] yields the existence of a linear bounded operator $S\colon \widetilde{W}^{1-1/p,p}(P_1, d^{p-N}) \to \widetilde{W}^{1-1/p,p}(P_2, d^{p-N})$ such that $Su = u$ on P_1. The operator R is now the superposition of S and of the extension operator from Lemma 5.2. It is easily seen that $TRu = u$ if $u \in \widetilde{W}^{1-1/p,p}(P_1, d^{p-N})$, which completes the proof. □

As an immediate consequence we have the following theorem.

Theorem 5.4. *Let $N \geqslant 2$. Then there exists a linear bounded operator*

$$R\colon \widetilde{W}^{1-1/p}(\partial\Omega, d^{p-N}) \to W^{1,p}(\Omega, d^{p-N})$$

such that $TRu = u$ for all $u \in \widetilde{W}^{1-1/p,p}(\partial\Omega, d^{p-N})$.

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Authors' addresses: J. Lang, Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic; A. Nekvinda, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Praha 6, Czech Republic.