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# HOPF BIFURCATION AND ORDINARY DIFFERENTIAL INEQUALITIES

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## 0. INTRODUCTION

Let $B_\lambda$ be a real matrix of the type $N \times N$ ($N \geqslant 3$) depending continuously on a real parameter $\lambda$, $G\colon \mathbb{R}^{N+1} \to \mathbb{R}^N$ a continuous mapping satisfying the conditions

$$\text{(G)} \qquad \lim_{|U|\to 0} \frac{|G(\lambda,U)|}{|U|} = 0 \ \text{ uniformly on compact } \lambda\text{-intervals,}$$

$$\text{(L)} \qquad \begin{cases} \text{for any } \Lambda > 0,\ R > 0 \text{ there exists } C > 0 \text{ such that} \\ |G(\lambda,U_1) - G(\lambda,U_2)| \leqslant C|U_1 - U_2| \text{ for all } |\lambda| \leqslant \Lambda,\ |U_1|,|U_2| \leqslant R. \end{cases}$$

Set $F(\lambda,U) = B_\lambda U + G(\lambda,U)$. Let $K$ be a closed convex cone in $\mathbb{R}^N$ with its vertex at the origin. We will consider a bifurcation problem for the inequality

$$\text{(I)} \qquad \begin{cases} U(t) \in K, \\ (\dot{U}(t) - F(\lambda,U(t)),\ Z - U(t)) \geqslant 0 \ \text{ for all } \ Z \in K, \ \text{ a.a. } \ t \in [0,T). \end{cases}$$

Our aim is to show that if a Hopf bifurcation of periodic solutions to the equation

$$\text{(E)} \qquad \dot{U}(t) = F(\lambda,U(t))$$

occurs at some $\lambda_0$ and certain additional assumptions are fulfilled then there exists a bifurcation point $\lambda_I$ of our inequality at which periodic solutions to (I) bifurcate from the branch of trivial solutions. The main results (Theorems 1.1, 1.2) either

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ensure the existence of such a bifurcation or explain in a certain sense why a bifurcation does not occur (see also Remark 1.5). A similar result illustrated by a simple example was proved in [6] for a special case when the eigenvectors of $B_\lambda$ are independent of $\lambda$. The basic idea is to join the inequality (I) with the corresponding equation (E) by a certain homotopy and to show that the bifurcation point $\lambda_0$ of the equation is transfered to a bifurcation point of the inequality by this homotopy. While the joining mentioned was given by a suitable deformation of the cone $K$ in [6], in the present paper we will join the inequality with the equation by a system of penalty equations (see also Remark 1.2 and Theorem 2.3). Note that this approach was used for a particular case of the linearized inequality in $\mathbb{R}^3$ in [3]. It represents a certain nontrivial modification of the method for the investigation of bifurcations of stationary solutions to inequalities given in [4] (see also [5]). A certain generalization of the well-known Rabinowitz global bifurcation theorem [13] (Theorem 3.1) forms a basis of the proof of existence of a branch of solutions to penalty equations representing the joining mentioned.

Of course, the corresponding linearized problems

$$\text{(LE)} \qquad \dot{U}(t) = B_\lambda U(t),$$

$$\text{(LI)} \qquad \begin{cases} U(t) \in K, \\ (\dot{U}(t) - B_\lambda U(t),\ Z-U(t)) \geqslant 0 \ \text{ for all } \ Z \in K, \quad \text{a.a. } t \geqslant 0 \end{cases}$$

play an essential role. (Note that the problem (LI) is strongly nonlinear again.)

Main results (Theorems 1.1, 1.2) are formulated and explained in Section 1. In Section 2, we describe basic properties of the penalty equation necessary for the proof of main results contained in Section 3.

Notice that an elementary approach to the investigation of bifurcations of periodic solutions to inequalities (I) in the special case $N = 3$ was given in [2] and developed for the study of stability of bifurcating solutions in [7].

## 1. Main results

**Remark 1.1.** By a solution of (I) on $[0, T)$ we mean an absolutely continuous function satisfying (I) for a.a. $t \in [0, T)$. It follows from general results [12] that such a solution is right differentiable and its right derivative is right continuous at any $t \in [0, T)$.

**Notation 1.1.** Let $W_1(\lambda),\dots,W_N(\lambda)$ be a basis of $\mathbb{C}^N$ composed of the elements of the chains corresponding to the eigenvalues of $B_\lambda$ (i.e. of the corresponding eigenvectors if $B_\lambda$ has $N$ eigenvalues for some $\lambda$, see e.g. [8]). Suppose that

$$(\mu) \quad \begin{cases} \text{there is a couple of simple eigenvalues } \mu_{1,2}(\lambda)=\alpha(\lambda)\pm \mathrm{i}\beta(\lambda) \\ \text{where } \alpha,\beta \text{ are continuous real functions, } \beta(\lambda)\geqslant\beta_0>0 \text{ for all } \lambda\in\mathbb{R}, \\ \alpha(\lambda)<0 \text{ for } \lambda<\lambda_0,\ \alpha(\lambda_0)=0,\ \alpha(\lambda)>0 \text{ for } \lambda>\lambda_0, \\ \text{the other eigenvalues of } B_\lambda \text{ have negative real parts for all } \lambda\in\mathbb{R}, \\ W_j(\lambda) \text{ depend continuously on } \lambda. \end{cases}$$

In particular, the chains corresponding to $\mu_1(\lambda)$ and $\mu_2(\lambda)$ contain only eigenvectors $W_1(\lambda)$ and $W_2(\lambda)$, respectively. We can write $W_j(\lambda)=U_j(\lambda)+\mathrm{i}U_{j+1}(\lambda)$, $W_{j+1}(\lambda)=U_j(\lambda)-\mathrm{i}U_{j+1}(\lambda)$ for $j$ such that $W_j(\lambda)$, $W_{j+1}(\lambda)$ is a pair of complex adjoint elements of some chain, $W_j(\lambda)=U_j(\lambda)$ for $j$ such that $W_j(\lambda)$ is real. Then $U_1(\lambda),\dots,U_N(\lambda)$ is a basis of $\mathbb{R}^N$ depending continuously on $\lambda$, see e.g. [8].

**Notation 1.2.** We will write

$(U,V)=\sum\limits_{i=1}^{N}u_iv_i$, $|U|^2=(U,U)$ for $U=[u_1,\dots,u_N]$, $V=[v_1,\dots,v_N]$,

$\boldsymbol{L}_\lambda=\operatorname{Lin}\{U_1(\lambda),U_2(\lambda)\}$,

$\boldsymbol{S}_\lambda=\operatorname{Lin}\{U_3(\lambda),\dots,U_N(\lambda)\}$,

$\boldsymbol{V}_\lambda=\{V\in\mathbb{R}^N\,;\ V=y_1U_1(\lambda)+\sum\limits_{j=3}^{N}y_jU_j(\lambda),\ y_j\in\mathbb{R}\}$,

$P_{\boldsymbol{L}_\lambda}V=y_1U_1(\lambda)+y_2U_2(\lambda)$ for $V=\sum\limits_{j=1}^{N}y_jU_j(\lambda)$ (projection onto $\boldsymbol{L}_\lambda$),

$P^*_{\boldsymbol{L}_\lambda}V=-y_2U_1(\lambda)+y_1U_2(\lambda)$ for $V=\sum\limits_{j=1}^{N}y_jU_j(\lambda)$,

$\mathcal{U}_r(z)$—the ball with the radius $r$ centered at $z$,

$U_\lambda(\cdot,V)$, $U_{0,\lambda}(\cdot,V)$, $U^\tau_\lambda(\cdot,V)$, $U^\tau_{0,\lambda}(\cdot,V)$, $U^\infty_\lambda(\cdot,V)$, $U^\infty_{0,\lambda}(\cdot,V)$—the solutions of (E), (LE), (PE), (LPE), (I), (LI), respectively, with the initial condition $V$ at $t=0$,

$\varrho^\tau_\lambda(t,V)$, $\varphi^\tau_\lambda(t,V)$ - polar coordinates of $P_{\boldsymbol{L}_\lambda}U^\tau_\lambda(t,V)$ with the angle $\varphi$ measured from $P_{\boldsymbol{L}_\lambda}V$, i.e. continuous functions defined by $\varphi^\tau_\lambda(0,V)=0$ and

$$P_{\boldsymbol{L}_\lambda}U^\tau_\lambda(t,V)=\varrho^\tau_\lambda(t,V)\ [\cos(\varphi^\tau_\lambda(t,V)+\varphi_V)\cdot U_1(\lambda)\ +\sin(\varphi^\tau_\lambda(t,V)+\varphi_V)\cdot U_2(\lambda)]$$

for $t\in[0,t_0)$ if $|P_{\boldsymbol{L}_\lambda}U^\tau_\lambda(t,V)|>0$ on $[0,t_0)$, where $\varphi_V$ satisfies

$$P_{\boldsymbol{L}_\lambda}V=\varrho^\tau_\lambda(0,V)\,[\cos\varphi_V\cdot U_1(\lambda)+\sin\varphi_V\cdot U_2(\lambda)],$$

$\varrho^\tau_{0,\lambda}(\cdot,V)$, $\varphi^\tau_{0,\lambda}(\cdot,V)$, $\varrho_\lambda(\cdot,V)$, $\varphi_\lambda(\cdot,V)$, $\varrho_{0,\lambda}(\cdot,V)$, $\varphi_{0,\lambda}(\cdot,V)$ are defined analogously but using $U^\tau_{0,\lambda}(\cdot,V)$, $U_\lambda(\cdot,V)$, $U_{0,\lambda}(\cdot,V)$, respectively,

$t_\lambda^\tau(V) = \inf\{t_0\,;\ \varrho_\lambda^\tau(t,V) > 0$ for $t \in [0,t_0]$, $\varphi_\lambda^\tau(t_0,V) = -2\pi\}$ if $V \notin \boldsymbol{S}_\lambda$—the time of one circuit of $P_{\boldsymbol{L}_\lambda} U_\lambda^\tau(\cdot,V)$ around the origin,

$t_{0,\lambda}^\tau(V)$, $t_\lambda(V)$, $t_{0,\lambda}(V)$ are defined analogously (clearly $t_{0,\lambda}(V) = \frac{2\pi}{\beta(\lambda)}$),

$t_\lambda = \frac{2\pi}{\beta(\lambda)}$ $(= t_{0,\lambda}(V)$ for all $V \notin \boldsymbol{S}_\lambda)$,

$n(U)$—the outer normal to $\partial K$ at $U$ if it exists.

The symbol for the derivative will be understood as the right derivative if the classical derivative does not exist (see Remark 1.1).

We will consider equations with penalty

$$\text{(PE)} \qquad \dot U(t) - F(\lambda, U(t)) + \tau\beta U(t) = 0$$

and

$$\text{(LPE)} \qquad \dot U(t) - B_\lambda U(t) + \tau\beta U(t) = 0.$$

Here $\lambda$ and $\tau$ are real parameters, $\beta = I - P_K$, where $P_K$ is the projection on $K$, i.e. $P_K U$ for $U \in \mathbb{R}^N$ is the unique point from $K$ satisfying

$$|P_K U - U| = \min_{V\in K} |V - U|.$$

**Remark 1.2.** We obtain (E) and (I) from (PE) for $\tau = 0$ and $\tau \to +\infty$, respectively (precisely see Theorem 2.3). Hence, the penalty equation (PE) can be understood in a certain sense as a homotopy joining our inequality with the corresponding equation.

**Remark 1.3.** The operators $P_K$, $\beta = I - P_K$ are lipschitzian and

$$\text{(P)} \qquad (\beta U, U) > 0 \text{ for all } U \notin K,\ \beta U = 0 \text{ if and only if } U \in K,$$

$$\text{(H)} \qquad \beta(tU) = t\beta U \text{ for all } t > 0,\ U \in \mathbb{R}^N \text{ (i.e. } \beta \text{ is positively homogeneous)},$$

$$\text{(M)} \qquad (\beta U - \beta V, U - V) \geqslant 0 \text{ for all } U, V \in \mathbb{R}^N \text{ (i.e. } \beta \text{ is monotone)},$$

$$\text{(Pt)} \qquad \beta W = \frac{1}{2}\operatorname{grad}|\beta W|^2 \text{ (i.e. } \beta \text{ is potential)}$$

(see [14]).

**Remark 1.4.** The assumption ($\mu$) implies that $\boldsymbol{L}_\lambda$, $\boldsymbol{S}_\lambda$ are invariant for the equation (LE) (for any given $\lambda$),

$$\begin{gathered}
\dot{\varphi}_{0,\lambda}(t,V) = -\beta(\lambda) \leqslant -\beta_0 \text{ for any } \lambda \in \mathbb{R}, V \in \mathbb{R}^N \setminus \boldsymbol{S}_\lambda,\ t \geqslant 0,\\
t_\lambda = t_{0,\lambda}(V) \leqslant \frac{2\pi}{\beta_0} \text{ for all } \lambda \in \mathbb{R}, V \in \mathbb{R}^N \setminus \boldsymbol{S}_\lambda,\\
\lim_{t\to+\infty} |U_{0,\lambda}(t,V)| = 0 \text{ for any } \lambda < \lambda_0, V \in \mathbb{R}^N,\\
\lim_{t\to+\infty} |U_{0,\lambda}(t,V)| = \infty \text{ for any } \lambda > \lambda_0, V \in \mathbb{R}^N \setminus \boldsymbol{S}_\lambda,\\
\lim_{t\to+\infty} |U_{0,\lambda}(t,V)| = 0 \text{ for any } \lambda \in \mathbb{R}, V \in \boldsymbol{S}_\lambda,\\
U_{0,\lambda}(\cdot,V) \text{ is periodic if and only if } \lambda = \lambda_0, V \in \boldsymbol{L}_\lambda.
\end{gathered}$$

According to the assumption (G), the behaviour of solutions to (E) is analogous to that of solutions to (LE) in a small neighbourhood of the origin. In particular, for any $\Lambda > 0$ and $t_M > 0$ there are $\varrho_0 > 0$ and $\eta > 0$ such that

$$\dot{\varphi}_\lambda(t,V) \leqslant -\eta \text{ for any } |\lambda| \leqslant \Lambda, V \in \mathbb{R}^N \setminus \boldsymbol{S}_\lambda,\ |V| \leqslant \varrho_0,\ t \in [0,t_M].$$

For any $V \in \boldsymbol{L}_\lambda$, $|V| \neq 0$, $\lambda \in \mathbb{R}$, the equation $\nu V = U_{0,\lambda}(t_\lambda, V)$ is fulfilled with only one $\nu = \nu(\lambda)$, where $\nu(\lambda) > 1$, $\nu(\lambda) = 1$ and $\nu(\lambda) < 1$ if $\lambda > \lambda_0$, $\lambda = \lambda_0$ and $\lambda < \lambda_0$, respectively. This equation can be fulfilled also for some $V \notin \boldsymbol{L}_\lambda$ but then always $\nu < 1$. Notice that $V \in \boldsymbol{V}_\lambda \cap \boldsymbol{L}_\lambda$ if and only if $V = cU_1(\lambda)$, $c \in \mathbb{R}$.

Further, the trivial solution of (E) is stable or unstable for $\lambda < \lambda_0$ or $\lambda > \lambda_0$, respectively. If, moreover, $\dot{\alpha}(\lambda_0) > 0$ then the Hopf bifurcation of periodic solutions to (E) occurs at $\lambda_0$ (see e.g. [10]).

We will suppose that

$$\text{(1.1)}\qquad \begin{cases} \text{for any } V \in \partial K \cap \boldsymbol{V}_\lambda \setminus \{0\},\ \lambda \in \mathbb{R},\ \text{there is } r > 0 \text{ such that}\\ \text{the normal } n(U) \text{ to } \partial K \text{ exists and is continuous on } \partial K \cap \mathcal{U}_r(V), \end{cases}$$

i.e. $\partial K$ is smooth near $\boldsymbol{V}_\lambda$ with the exception of the vertex of $K$. We could consider this condition with general

$$\boldsymbol{V}_\lambda = \{V \in \mathbb{R}^N ;\, P_{\boldsymbol{L}_\lambda} V = c\Big(a(\lambda)U_1(\lambda) + b(\lambda)U_2(\lambda)\Big), c \in \mathbb{R}\}$$

where $a(\lambda)$, $b(\lambda)$ are given continuous functions. For formal simplification, we will consider the special $\boldsymbol{V}_\lambda$ introduced in Notation 1.2.

For the proof of our bifurcation result, the following assumption (1.2) concerning the linearized penalty equation (LPE) and the linearized inequality (LI) is essential:

$$(1.2)\qquad \begin{cases} \text{if} & [\lambda, W, \tau] \in \mathbb{R} \times \boldsymbol{V}_\lambda \times [0, +\infty],\ W \neq 0,\ W = U^\tau_{0,\lambda}(t, W) \\ & \text{for some } t > 0 \\ \text{then} & |\lambda| < \Lambda,\ W \notin \boldsymbol{S}_\lambda,\ t^\tau_{0,\lambda}(W) < t_M,\ \dot{\varphi}^\tau_{0,\lambda}(t^\tau_{0,\lambda}(W), W) < 0 \end{cases}$$

(with some $\Lambda > 0$, $t_M > 0$ fixed). The Bifurcation Theorem 1.1 will be a consequence of Theorem 1.2 guaranteeing the existence of a branch of solutions of penalty equations satisfying a convenient norm condition. The assumption (1.2) will exclude certain unconvenient possibilities of the behaviour of this branch and will ensure that this branch must be unbounded in the parameter $\tau$. (See also Remark 1.5.) This will be essential for obtaining small periodic solutions to (I) by the limiting process $\tau \to \infty$ along this branch. We will study some concrete examples where the condition (1.2) can be verified, in a forthcoming paper. Let us mention here only that the assumption $P_{\boldsymbol{L}_\lambda} K = \boldsymbol{L}_\lambda$ for all $\lambda \in \mathbb{R}$ seems to be necessary (but not sufficient) for the validity of (1.2). Note that in the case $N = 3$, $P_{\boldsymbol{L}_\lambda} K = \boldsymbol{L}_\lambda$ is fulfilled if $U_3(\lambda) \in \operatorname{int} K$.

**Theorem 1.1.** *Let* ($\mu$), (G), (L), (1.1) *be fulfilled. Suppose that there exist* $\Lambda > 0$, $t_M > 0$ *such that* (1.2) *holds. Then there exists* $\lambda_I \in [-\Lambda, \Lambda]$ *at which periodic solutions of* (I) *bifurcate from the branch of trivial solutions. More precisely, for any* $\varrho \in (0, \varrho_0)$ *(with some* $\varrho_0 > 0$ *small enough) there exist* $\lambda_\varrho \in [-\Lambda, \Lambda]$, $V_\varrho \in \boldsymbol{V}_{\lambda_\varrho}$ *such that* $U^\infty_{\lambda_\varrho}(\cdot, V_\varrho)$ *is periodic,* $0 < |V_\varrho|^2 \leqslant \varrho$ *and there exists at least one accumulation point* $\lambda_I$ *of* $\lambda_\varrho$ *for* $\varrho \to 0_+$.

**Theorem 1.2.** *Let* ($\mu$), (G), (L), (1.1) *be fulfilled. Then there exist* $\varrho_0 > 0$, $\tau_0 > 0$ *such that for any* $\varrho \in (0, \varrho_0)$ *there is a closed connected set* $\mathcal{C}_\varrho$ *of triplets* $[\lambda, V, \tau] \in \mathbb{R} \times \boldsymbol{V}_\lambda \times [0, +\infty)$ *containing* $[\lambda_0, 0, 0]$ *and having the following properties:*

$$(1.3)\qquad \begin{cases} \textit{if } [\lambda, V, \tau] \in \mathcal{C}_\varrho,\ V = \sum\limits_{j=1}^{N} y_j U_j(\lambda),\ Y = [y_1, \dots, y_N] \textit{ then } |Y|^2 = \frac{\varrho\tau}{1+\tau} \\ \textit{and } U^\tau_\lambda(\cdot, V) \textit{ is periodic provided } \tau > 0, \end{cases}$$

$$(1.4)\qquad \textit{for any } \tau \in [0, \tau_0) \textit{ there are } \lambda, V \textit{ such that } [\lambda, V, \tau] \in \mathcal{C}_\varrho.$$

*Moreover, if* (1.2) *holds with some* $\Lambda > 0$, $t_M > 0$ *then* (1.4) *holds with* $\tau_0 = +\infty$ *and* $|\lambda| \leqslant \Lambda$, $t^\tau_\lambda(V) < t_M$ *for all* $[\lambda, V, \tau] \in \mathcal{C}_\varrho$, $\varrho \in (0, \varrho_0)$.

Proof of Theorems 1.1, 1.2 will be given in Section 3. □

**Remark 1.5.** It follows from Theorem 1.2 that the problem (PE) with $\tau$ small enough has a bifurcation point even if (1.2) is not fulfilled. However, in this case it can happen that the branches $\mathcal{C}_\varrho$ have no continuation for $\tau \geqslant \tau_0$ because either $|\lambda| \to +\infty$ along these branches or the circulation of $P_{\boldsymbol{L}_\lambda} U_\lambda^\tau(t, V)$ around the origin in $\boldsymbol{L}_\lambda$ is damped too strongly by the penalty term for $\tau \to \tau_0$. See the proof of Theorem 1.2 for details.

**Remark 1.6.** A solution $U(t)$ of (PE) is simultaneously a solution of (E) on any interval $(t_1, t_2)$ such that $U(t) \in K$ for $t \in (t_1, t_2)$ (see (P) from Remark 1.3). A solution $U(t)$ of (I) is simultaneously a solution of (E) on any interval $(t_1, t_2)$ such that $U(t) \in \operatorname{int} K$ for $t \in (t_1, t_2)$.

## 2. Properties of the penalty equations

In this section, we will collect some basic assertions necessary for the proof of main results. Lemma 2.1 and Theorems 2.1, 2.2 follow from the theory of ordinary differential equations (see e.g. [8]), Theorem 2.3 can by obtained by the penalty method technique (cf. e.g. [9]) and Theorem 2.4 follows by elementary considerations (cf. also [6]). Only the proof of Theorem 2.5 contains new ideas. For the completeness, all proofs are given in Appendix.

We will always suppose automatically that the conditions (G), (L), ($\mu$), (1.1) are fulfilled.

**Remark 2.1.** The solution $U_\lambda^\tau(\cdot, V)$ (for a fixed $\lambda \in \mathbb{R}$, $\tau \in [0, +\infty]$, $V \in \mathbb{R}^N$) is unique and exists at least on some interval $[0, T_0), T_0 > 0$. Further, if $T > 0$ and $U_\lambda^\tau(\cdot, V)$ is bounded on any subinterval of [0,T) on which it is defined then it exists on $[0, T)$. (For $\tau \in [0, +\infty)$ see e.g. [8], for $\tau = +\infty$ see [1].) In particular, $U_{0,\lambda}^\tau(\cdot, V)$ always exists on $[0, +\infty)$ for all $\lambda \in \mathbb{R}$, $\tau \in [0, +\infty]$, $V \in \mathbb{R}^N$. For $\tau$ finite, the boundedness on any finite interval follows from estimates analogous to those from the proof of Lemma 2.1 in Appendix which becomes simpler in the case $G = 0$. For $\tau = +\infty$ cf. [6], Lemma 2.1.

**Lemma 2.1.** *Let $\Lambda > 0$, $t_M > 0$. Then there exist $\varrho_0 > 0$, $r > 0$, $C > 0$ such that*

$$
(2.1) \qquad \left\{
\begin{array}{l}
U_\lambda^\tau(\cdot, V) \text{ exists on } [0, t_M + 1), \\
|U_\lambda^\tau(t, V)|^2 \leqslant |V|^2 \mathrm{e}^{rt}, \ |\dot{U}_\lambda^\tau(t, V)|^2 \leqslant (C + \tau)|V|^2 \mathrm{e}^{rt} \\
\text{for all } V \in \mathbb{R}^N, \ |V| \leqslant \varrho_0, \ |\lambda| \leqslant \Lambda, \ \tau \in [0, +\infty), \ t \in [0, t_M + 1).
\end{array}
\right.
$$

**Remark 2.2.** It follows from (Pt) (Remark 1.3) that if $U \in C^1([0,t_0])$, $U(0) = U(t_0)$ then

$$\int_0^{t_0} (\beta U, \dot{U})\,\mathrm{d}t = \frac{1}{2}\int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}t}|\beta U|^2\,\mathrm{d}t = \frac{1}{2}\Big(|\beta U(t_0)|^2 - |\beta U(0)|^2\Big) = 0.$$

**Remark 2.3.** Let $\Lambda > 0$, $t_M > 0$, let $\varrho_0$ be from Lemma 2.1. Then there exist $C_1, C_2 > 0$ such that

$$\text{(2.2)} \qquad \begin{cases} |U_\lambda^\tau(t,V)| \leqslant C_1, \ |F(\lambda, U_\lambda^\tau(t,V))| \leqslant C_2 \\ \text{for } |\lambda| \leqslant \Lambda, \ |V| \leqslant \varrho_0, \ \tau \geqslant 0, \ t \in [0,t_M]. \end{cases}$$

Suppose that a solution $U(t) = U_\lambda^\tau(t,V)$ is periodic with a period $t_0 \leqslant t_M$, $|\lambda| \leqslant \Lambda$, $|V| \leqslant \varrho_0$, $\tau \geqslant 0$. Multiply (PE) by $\dot{U}$ and integrate over $(0,t_0)$. We obtain by using Remark 2.2 that

$$\text{(2.3)} \qquad \int_0^{t_0} |\dot{U}|^2\,\mathrm{d}t = \int_0^{t_0} \Big(F(\lambda,U),\dot{U}\Big)\,\mathrm{d}t \leqslant C_2 \int_0^{t_0} |\dot{U}|\,\mathrm{d}t \leqslant C_2 t_0^{\frac{1}{2}} \bigg(\int_0^{t_0} |\dot{U}|^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}.$$

Setting $k_m = \max\{k \in \mathbb{N};\ kt_0 \leqslant t_M\}$ and using the periodicity we obtain

$$\text{(2.4)} \qquad \int_0^{t_M} |\dot{U}(t)|^2\,\mathrm{d}t \leqslant (k_m+1)t_0 C_2^2 \leqslant 2t_M C_2^2 \quad \text{for } |\lambda| \leqslant \Lambda, \ |V| \leqslant \varrho_0, \ \tau \geqslant 0.$$

**Theorem 2.1.** *Let $\Lambda > 0$, $t_M > 0$, let $\varrho_0$ be from Lemma 2.1. If $|\lambda_n| \leqslant \Lambda$, $V_n \in \mathbb{R}^N$, $|V_n| \leqslant \varrho_0$, $\tau_n \in [0,+\infty)$ and $[\lambda_n, V_n, \tau_n] \to [\lambda, V, \tau]$, $\tau \in [0,+\infty)$ then*

$$\text{(2.5)} \qquad U_{\lambda_n}^{\tau_n}(\cdot, V_n) \to U_\lambda^\tau(\cdot, V) \quad \text{in} \quad C^1([0,t_M]).$$

*If, moreover, $V = 0$, $\frac{V_n}{|V_n|} = W_n \to W$ then*

$$\text{(2.6)} \qquad \frac{U_{\lambda_n}^{\tau_n}(\cdot, V_n)}{|V_n|} \to U_{0,\lambda}^\tau(\cdot, W) \quad \text{in} \quad C^1([0,T]) \text{ for any } T > 0.$$

**Consequence 2.1.** *Let the assumptions of Theorem 2.1 be fulfilled. If*

$$\varrho_\lambda^\tau(t,V) \geqslant \eta \text{ for all } t \in [0,T] \text{ with some } \eta > 0, \ T \in [0,t_M]$$

*then*

$$\text{(2.7)} \qquad \varrho_{\lambda_n}^{\tau_n}(\cdot, V_n) \to \varrho_\lambda^\tau(\cdot, V), \ \varphi_{\lambda_n}^{\tau_n}(\cdot, V_n) \to \varphi_\lambda^\tau(\cdot, V) \text{ in } C^1([0,T]).$$

*If* $V = 0$, $\frac{V_n}{|V_n|} \to W$ *and*

$$\varrho^{\tau}_{0,\lambda}(t, W) \geqslant \eta \text{ for all } t \in [0, T] \text{ with some } \eta > 0,\ T > 0$$

*then*

$$\frac{\varrho^{\tau_n}_{\lambda_n}(\cdot, V_n)}{|V_n|} \to \varrho^{\tau}_{0,\lambda}(\cdot, W), \varphi^{\tau_n}_{\lambda_n}(\cdot, V_n) \to \varphi^{\tau}_{0,\lambda}(\cdot, W) \text{ in } C^1([0,T]). \tag{2.8}$$

**Theorem 2.2.** *Let* $\Lambda > 0$, $t_M > 0$, *let* $\varrho_0$ *be from Lemma* 2.1, $|\lambda_n| \leqslant \Lambda$, $V_n \in \mathbb{R}^N$, $|V_n| \leqslant \varrho_0$, $\tau_n \in [0, +\infty)$ *and* $[\lambda_n, V_n, \tau_n] \to [\lambda, V, \tau]$, $\tau \in [0, +\infty)$. *Let* $U^{\tau_n}_{\lambda_n}(\cdot, V_n)$ *be periodic solutions of* (PE) *with periods* $t_n \to 0$. *Then* $U^{\tau}_{\lambda}(\cdot, V)$ *is a stationary solution of* (PE). *Moreover, if* $V = 0$, $\frac{V_n}{|V_n|} = W_n \to W$ *then* $U^{\tau}_{0,\lambda}(\cdot, W)$ *is a stationary solution of* (LPE).

**Theorem 2.3.** *Let* $\Lambda > 0$, $t_M > 0$, *let* $\varrho_0$ *be from Lemma* 2.1, $|\lambda_n| \leqslant \Lambda$, $V_n \in \mathbb{R}^N$, $|V_n| \leqslant \varrho_0$, $\tau_n \in [0, +\infty)$. *Let* $U^{\tau_n}_{\lambda_n}(\cdot, V_n)$ *be periodic solutions of* (PE) *with periods* $t_n$. *Let* $\tau_n \to +\infty, \lambda_n \to \lambda$, $V_n \to V$, $t_n \leqslant t_M$, $t_n \to t_0$. *Then*

$$U^{\tau_n}_{\lambda_n}(\cdot, V_n) \to U^{\infty}_{\lambda}(\cdot, V) \text{ in } C([0, t_M]) \text{ and weakly in } W^1_2(0, t_M).$$

*If* $t_0 > 0$ *then* $U^{\infty}_{\lambda}(\cdot, V)$ *is a periodic solution of* (I) *with the period* $t_0$. *If* $t_0 = 0$ *then* $U^{\infty}_{\lambda}(\cdot, V)$ *is a stationary solution of* (I).

*If, moreover,* $V = 0$, $W_n = \frac{V_n}{|V_n|} \to W$, $t_n \leqslant t_M$ *then*

$$\frac{U^{\tau_n}_{\lambda_n}(\cdot, V_n)}{|V_n|} \to U^{\infty}_{0,\lambda}(\cdot, W) \text{ in } C([0, t_M]) \text{ and weakly in } W^1_2(0, t_M),\ t_n \to t_0 \in [0, t_M].$$

*If* $t_0 > 0$ *then* $U^{\infty}_{0,\lambda}(\cdot, W)$ *is a periodic solution of* (LI) *with the period* $t_0$, *if* $t_0 = 0$ *then* $U^{\infty}_{0,\lambda}(\cdot, W)$ *is a stationary solution of* (LI).

**Theorem 2.4.** *Let* $\Lambda > 0$, $t_M > 0$, *let* $\varrho_0$ *be from Lemma* 2.1, $|\lambda_n| \leqslant \Lambda$, $V_n \in \boldsymbol{V}_{\lambda_n}$, $|V_n| \leqslant \varrho_0$, $\tau_n \in [0, +\infty)$, $[\lambda_n, V_n, \tau_n] \to [\lambda, V, \tau]$, $\tau \in [0, +\infty]$. *If* $V \notin \boldsymbol{S}_{\lambda}$, $t^{\tau}_{\lambda}(V) < t_M$, $\dot{\varphi}^{\tau}_{\lambda}(t^{\tau}_{\lambda}(V), V) < 0$ *then* $t^{\tau_n}_{\lambda_n}(V_n) \to t^{\tau}_{\lambda}(V)$.

*If* $V = 0$, $W_n = \frac{V_n}{|V_n|} \to W \notin \boldsymbol{S}_{\lambda}$, $t^{\tau}_{0,\lambda}(W) < \infty$, $\dot{\varphi}^{\tau}_{0,\lambda}(t^{\tau}_{0,\lambda}(W), W) < 0$ *then* $t^{\tau_n}_{\lambda_n}(V_n) \to t^{\tau}_{0,\lambda}(W)$.

**Theorem 2.5.** *Let* $\Lambda > 0$, $t_M > 0$, $|\lambda_n| \leqslant \Lambda$, $V_n \in \boldsymbol{V}_{\lambda_n}$, $0 < t_n \leqslant t_M$, $\lambda_n \to \lambda$, $V_n \to 0$, $\frac{V_n}{|V_n|} \to W \notin \boldsymbol{S}_{\lambda}$, $\tau_n \to +\infty$, $U^{\tau_n}_{\lambda_n}(t_n, V_n) = V_n$, $\dot{\varphi}^{\infty}_{0,\lambda}(0, W) < 0$. *Then*

$$\limsup \dot{\varphi}^{\tau_n}_{\lambda_n}(0, V_n) < 0.$$

**Remark 3.1.** For a brief explanation of main ideas of the proof, let us suppose first that $U_j = U_j(\lambda)$ are independent of $\lambda$. Hence, also $\boldsymbol{V} = \boldsymbol{V}_\lambda$, $\boldsymbol{S} = \boldsymbol{S}_\lambda$ are independent of $\lambda$. Our aim is to define a mapping $R\colon D \to \boldsymbol{V}$, $D \subset \mathbb{R} \times \boldsymbol{V} \times \mathbb{R}$ such that its fixed points (in $V$ for given $\lambda$, $\tau$) are initial conditions of periodic solutions of (PE). We would like to obtain the branches $\mathcal{C}_\varrho$ in Theorem 1.2 as branches of nontrivial solutions of the equation $V = R(\lambda, V, \tau)$ supplemented by a suitable norm condition. For the proof of existence of such a global branch on the basis of the degree theory, we need $R$ to be continuous on a domain of definition $D$ which is open, contains $[\lambda_0, 0, 0]$ and is maximal in a certain sense. We intend to define $R$ as a Poincaré map on a part of $\boldsymbol{V} \setminus \boldsymbol{S}$ on which this is possible and prolong it continuously onto $\boldsymbol{S}$. Unfortunately, this can be done directly only if $N = 3$, $G \equiv 0$, $U_3 \in \operatorname{int} K$, $K \subset \{V = \sum\limits_{j=1}^{3} y_j U_j,\ y_3 > 0\}$. In this special case we can set

$$
\begin{aligned}
D = \{[\lambda, V, \tau] \in \mathbb{R} \times \boldsymbol{V} \times \mathbb{R};&\\
&\text{either } V \in \boldsymbol{S} \text{ or } V \notin \boldsymbol{S},\ t^\tau_\lambda(V) < +\infty,\ \dot\varphi^\tau_\lambda(t^\tau_\lambda(V), V) < 0\},\\
R(\lambda, V, \tau) = U^\tau_\lambda(t^\tau_\lambda(V), V) &\text{ for } [\lambda, V, \tau] \in D,\ V \notin \boldsymbol{S},\\
= U^\tau_\lambda\Big(\frac{2\pi}{\beta(\lambda)}, V\Big) &\text{ for } [\lambda, V, \tau] \in D,\ V \in \boldsymbol{S}.
\end{aligned}
$$

(See also Remark 3.2.) Notice that for the proof of continuity of $R$ at given $\lambda_1$, $V_1$, $\tau_1$ with $V_1 \notin \boldsymbol{S}$ it is necessary to know that $t^\tau_\lambda(V)$ continuously depends on all parameters at $\lambda_1$, $V_1$, $\tau_1$, and this is ensured only if $\dot\varphi^{\tau_1}_{\lambda_1}(t^{\tau_1}_{\lambda_1}(V_1), V_1) < 0$. See Theorem 2.4.

In the general case the following complications arise.

1. If $G \not\equiv 0$ then the existence of solutions is ensured on a given time interval and for $\lambda$ from a given compact only for sufficiently small initial conditions (see Lemma 2.1). Therefore we will consider fixed $t_M > 0$, $\Lambda > 0$ and study solutions with initial conditions $V \in \mathcal{U}_{\varrho_0}(0)$ (with the corresponding $\varrho_0$ small enough) and satisfying $t^\tau_\lambda(V) < t_M$, $|\lambda| < \Lambda$.

2. The second complication is that even if the Poincaré map is already defined on some $\mathcal{U}_r(V) \cap \boldsymbol{V} \setminus \boldsymbol{S}$, $V \in \boldsymbol{S}$, then it need not have a continuous prolongation to $\mathcal{U}_r(V) \cap \boldsymbol{V} \cap \boldsymbol{S}$. (The only exception is the case $N = 3$, see Remark 3.2.) This will be solved by an artificial definition of $R$ in a "sector $\boldsymbol{S}_\varepsilon$ around $\boldsymbol{S}$" introduced in Notation 3.2. The fixed points of $R = R_\varepsilon$ on $\boldsymbol{S}_\varepsilon$ will have nothing common with periodic solutions of (PE) but it will be shown that the branch $\mathcal{C}_\varrho$ will not touch $\boldsymbol{S}_\varepsilon$ (with the exception of $[\lambda_0, 0, 0]$) in the situation of our interest.

3. Another difficulty arises in the moment when $U_j(\lambda)$, i.e. also $\boldsymbol{V}_\lambda$, depend on $\lambda$. We need to work with a mapping $R$ on a domain of definition in a fixed space. Therefore for any $\lambda$, we will associate points $V = \sum_{j=1}^{N} y_j U_j(\lambda) \in \boldsymbol{V}_\lambda$ with the vectors of coordinates $Y = [y_1, \dots, y_n] \in \boldsymbol{V}$, $\boldsymbol{V} = \{X = [x_1, \dots, x_N] \in \mathbb{R}^N\,;\, x_2 = 0\}$. This is the reason for Notation 3.1 which enables us to study our problem in this new setting.

All these considerations together lead to a formally complicated definition of $D_\varepsilon$ and $R_\varepsilon$ (Definition 3.1 below).

**Remark 3.2.** Consider the case $N = 3$, $U_3 \in \operatorname{int} K$, $K \subset \{V = \sum_{j=1}^{3} y_j U_j,\ y_3 > 0\}$, $U_j(\lambda) = U_j$ independent of $\lambda$ again. Denote by $K^\perp = \{Z \in \mathbb{R}^N\,;\, (Z, V) \leqslant 0$ for all $V \in K\}$ the dual cone to $K$ (see e.g. [14]). Then $\operatorname{int} K^\perp \neq \emptyset$ and $\boldsymbol{S} \setminus \{0\} \subset \operatorname{int} K \cup \operatorname{int} K^\perp$. If $V \in \operatorname{int} K$ is sufficiently close to $\boldsymbol{S}$ then $U_{0,\lambda}(t, V) \in K$ for $t \in [0, t_\lambda]$. Hence, the condition (P) in Remark 1.3 implies that $U^\tau_{0,\lambda}(t, V) = U_{0,\lambda}(t, V)$ on $[0, t_\lambda]$, $t^\tau_{0,\lambda}(V) = t_\lambda$ for all $\tau \geqslant 0$. It follows that the mapping $R$ defined in Remark 3.1 for the case $N = 3$, $G \equiv 0$, $U_3 \in \operatorname{int} K$ is continuous on $\boldsymbol{S} \cap K$. Further, we have $P_K U = 0$ for all $U \in K^\perp$ and it follows that the penalty term $\tau\beta U^\tau_{0,\lambda}(t, V)$ in (PE) influences neither the tendency of the solution to leave $K^\perp$ nor its circulation around the axis $\boldsymbol{S}$ if $U^\tau_\lambda(t, V) \in K^\perp$. If $V \in \operatorname{int} K^\perp$ is sufficiently close to $\boldsymbol{S}$ then $U_{0,\lambda}(t, V) \in K^\perp$ for $t \in [0, t_\lambda]$ and therefore also $U^\tau_{0,\lambda}(t, V) \in K^\perp$ for $t \in [0, t_\lambda]$, $t^\tau_{0,\lambda}(V) = t_\lambda$ for all $\tau \geqslant 0$. It follows that the mapping $R$ mentioned in Remark 3.1 is continuous on $\boldsymbol{S} \cap K^\perp$. Of course, the continuity of $R$ in $\boldsymbol{V} \setminus \boldsymbol{S}$ follows from Theorems 2.1, 2.4. However, in the case $N > 3$ we have $(\boldsymbol{S} \setminus \{0\}) \cap \partial K \neq \emptyset$ in general and the situation is essentially more complicated.

**Notation 3.1.**

$y^V_j(\lambda)$ $(j = 1, \dots, N)$—the coordinates of $V \in \mathbb{R}^N$ with respect to $\{U_j(\lambda)\}$, i.e. $V = \sum_{j=1}^{N} y^V_j(\lambda) U_j(\lambda)$,

$V^Y_\lambda = \sum_{j=1}^{N} y_j U_j(\lambda)$ for $Y = [y_1, \dots, y_N]$,

$\boldsymbol{L} = \{X \in \mathbb{R}^N\,;\, x_j = 0,\ j = 3, \dots, N\}$,

$\boldsymbol{S} = \{X \in \mathbb{R}^N\,;\, x_1 = x_2 = 0\}$,

$\boldsymbol{V} = \{X \in \mathbb{R}^N\,;\, x_2 = 0\}$,

$P_{\boldsymbol{L}} X = [x_1, x_2, 0, \dots, 0]$ (projection onto $L$),

$P^*_{\boldsymbol{L}} X = [-x_2, x_1, 0, \dots, 0]$,

$\tilde{U}_\lambda(\cdot,Y)$, $\tilde{U}_{0,\lambda}(\cdot,Y)$, $\tilde{U}^{\mathcal{T}}_\lambda(\cdot,Y)$, $\tilde{U}^{\mathcal{T}}_{0,\lambda}(\cdot,Y)$, $\tilde{U}^{\infty}_\lambda(\cdot,Y)$, $\tilde{U}^{\infty}_{0,\lambda}(\cdot,Y)$—the vectors of coordinates of $U_\lambda(\cdot,V^Y_\lambda)$, $U_{0,\lambda}(\cdot,V^Y_\lambda)$, $U^{\mathcal{T}}_\lambda(\cdot,V^Y_\lambda)$, $U^{\mathcal{T}}_{0,\lambda}(\cdot,V^Y_\lambda)$, $U^{\infty}_\lambda(\cdot,V^Y_\lambda)$, $U^{\infty}_{0,\lambda}(\cdot,V^Y_\lambda)$, respectively,

$\tilde{\varrho}^{\mathcal{T}}_\lambda(t,Y)$, $\tilde{\varphi}^{\mathcal{T}}_\lambda(t,Y)$ – polar coordinates of $P_{\boldsymbol{L}}\tilde{U}^{\mathcal{T}}_\lambda(t,Y)$ with the angle $\tilde{\varphi}$ measured from $P_{\boldsymbol{L}}Y$, i.e. continuous functions defined by $\tilde{\varphi}^{\mathcal{T}}_\lambda(0,Y)=0$ and

$$P_{\boldsymbol{L}}\tilde{U}^{\mathcal{T}}_\lambda(t,Y)=$$
$$\tilde{\varrho}^{\mathcal{T}}_\lambda(t,Y)\ \{\cos(\tilde{\varphi}^{\mathcal{T}}_\lambda(t,Y)+\tilde{\varphi}_Y)\cdot[1,0,\dots,0]+\sin(\tilde{\varphi}^{\mathcal{T}}_\lambda(t,Y)+\tilde{\varphi}_Y)\cdot[0,1,0,\dots,0]\}$$

for $t\in[0,t_0)$ if $|P_{\boldsymbol{L}}\tilde{U}^{\mathcal{T}}_\lambda(t,Y)|>0$ on $[0,t_0)$, where $\tilde{\varphi}_Y$ satisfies

$$P_{\boldsymbol{L}_\lambda}Y=\tilde{\varrho}^{\mathcal{T}}_\lambda(0,Y)\ \{\cos\tilde{\varphi}_Y\cdot[1,0,\dots,0]+\sin\tilde{\varphi}_Y\cdot[0,1,0,\dots,0]\},$$

$\tilde{\varrho}^{\mathcal{T}}_{0,\lambda}(\cdot,Y)$, $\tilde{\varphi}^{\mathcal{T}}_{0,\lambda}(\cdot,Y)$, $\tilde{\varrho}_\lambda(\cdot,Y)$, $\tilde{\varphi}_\lambda(\cdot,Y)$, $\tilde{\varrho}_{0,\lambda}(\cdot,Y)$, $\tilde{\varphi}_{0,\lambda}(\cdot,Y)$ are defined analogously but by using $\tilde{U}^{\mathcal{T}}_{0,\lambda}(\cdot,Y)$, $\tilde{U}_\lambda(\cdot,Y)$, $\tilde{U}_{0,\lambda}(\cdot,Y)$, respectively,

$\tilde{t}^{\mathcal{T}}_\lambda(Y)=t^{\mathcal{T}}_\lambda(V^Y_\lambda)$,

$\tilde{t}^{\mathcal{T}}_{0,\lambda}(Y)$, $\tilde{t}_\lambda(Y)$, $\tilde{t}_{0,\lambda}(Y)$ are defined analogously (clearly $\tilde{t}_{0,\lambda}(Y)=\frac{2\pi}{\beta(\lambda)}$),

$\tilde{t}_\lambda=\frac{2\pi}{\beta(\lambda)}$ $(=\tilde{t}_{0,\lambda}(Y)=t_\lambda$ for all $Y\notin\boldsymbol{S}$, see also Notation 1.2).

Of course, all our former assertions could be reformulated in terms of this new notation. In the following we will have on mind such reformulations automatically if necessary.

Further, we will consider fixed $\Lambda>|\lambda_0|$, $t_M>\frac{2\pi}{\beta_0}$ and the corresponding $\varrho_0$ from Lemma 2.1.

**Notation 3.2.** For any $\varepsilon>0$ and $X=[x_1,\dots,x_n]\in\mathbb{R}^N$ we will denote

$P^0X=[0,0,x_3,\dots,x_N]$,

$\boldsymbol{S}_\varepsilon=\{X\in\boldsymbol{V}\,;\,|x_1|\leqslant\varepsilon|P^0X|\}$,

$\boldsymbol{S}^0_\varepsilon=\{X\in\boldsymbol{V}\,;\,|x_1|<\varepsilon|P^0X|\}$,

$P^\varepsilon X=[\varepsilon\ \mathrm{sign}\ x_1\cdot|P^0X|,0,x_3,\dots,x_N]$ for $X\in\boldsymbol{S}_\varepsilon\setminus\boldsymbol{S}$,

$P^\varepsilon X=X$ for $X\in\boldsymbol{V}\setminus\boldsymbol{S}_\varepsilon$,

$\tilde{\varrho}_0=\inf\{|Y|\,;\,|V^Y_\lambda|=\varrho_0,\ \lambda\in[-\Lambda,\Lambda]\}$ where $\varrho_0$ is from Lemma 2.1,

$\mathrm{d}(L)=\deg(I-L,0,\mathcal{U}_r(0))$—the Leray-Schauder degree of $I-L$ with respect to 0, $\mathcal{U}_r(0)$—for any linear completely continuous mapping $L$ in a Banach space satisfying $\mathrm{Ker}(I-L)=\{0\}$. (Note that for such $L$, $\deg(I-L,0,\mathcal{U}_r(0))$ exists and is independent of $r>0$, see e.g. [11].)

**Remark 3.3.** According to Remark 3.1 our mapping will be defined naturally as a Poincaré map at points $Y\in\boldsymbol{V}\setminus\boldsymbol{S}_\varepsilon$ such that $\tilde{t}^{\mathcal{T}}_\lambda(Y)<t_M$, $\dot{\tilde{\varphi}}^{\mathcal{T}}_\lambda(\tilde{t}^{\mathcal{T}}_\lambda(Y),Y)<0$. Then it will be prolonged to $Y\in\boldsymbol{V}$ such that $Y\in S_\varepsilon\setminus\boldsymbol{S}$, $\tilde{t}^{\mathcal{T}}_\lambda(P^\varepsilon Y)<t_M$,

$\dot{\tilde{\varphi}}^{\tau}_{\lambda}(\tilde{t}^{\tau}_{\lambda}(P^{\varepsilon}Y), P^{\varepsilon}Y) < 0$. (Recall that $P^{\varepsilon}Y \notin \boldsymbol{S}$ and therefore $\tilde{t}^{\tau}_{\lambda}(P^{\varepsilon}Y)$ is well defined.) It follows from Theorem 2.4 and Consequence 2.1 that for such $\lambda$, $Y$, $\tau$

$$\text{(3.1)} \qquad \begin{cases} \text{there is } r > 0 \text{ such that } \tilde{t}^{\xi}_{\mu}(P^{\varepsilon}Z) < t_M, \ \dot{\tilde{\varphi}}^{\xi}_{\mu}(\tilde{t}^{\xi}_{\mu}(P^{\varepsilon}Z), P^{\varepsilon}Z) < 0 \\ \text{for any } [\mu, Z, \xi] \in \mathcal{U}_r(\lambda, Y, \tau) \text{ such that } \ Z \in \boldsymbol{V}, \ Z \notin \boldsymbol{S}. \end{cases}$$

Of course, $Z \notin \boldsymbol{S}$ automatically for all $[\mu, Z, \xi] \in \mathcal{U}_r(\lambda, Y, \tau)$ if $Y \notin S$ and $r$ is small enough. If $Y \in \boldsymbol{S}$ then $\tilde{t}^{\tau}_{\lambda}(P^{\varepsilon}Y)$ is not determined but our mapping can be defined at such points provided (3.1) is fulfilled, as we will see below.

**Definition 3.1.** Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$, let $\tilde{\varrho}_0$ be from Notation 3.2. For any $\varepsilon > 0$ set

$$\boldsymbol{D}_{\varepsilon} = \{[\lambda, Y, \tau] \in (-\Lambda, \Lambda) \times \boldsymbol{V} \times \mathbb{R}; \ |Y| < \tilde{\varrho}_0, \ (3.1) \text{ holds}\}$$

and define a mapping $R_{\varepsilon} \colon \boldsymbol{D}_{\varepsilon} \to \boldsymbol{V}$ as follows:

$$\begin{aligned} R_{\varepsilon}(\lambda, Y, \tau) &= \tilde{U}^{\tau}_{\lambda}(\tilde{t}^{\tau}_{\lambda}(Y), Y) \text{ for } [\lambda, Y, \tau] \in \boldsymbol{D}_{\varepsilon}, \ Y \notin \boldsymbol{S}_{\varepsilon}, \ \tau \geqslant 0, \\ &= \frac{|y_1|}{\varepsilon|P^0Y|}\tilde{U}^{\tau}_{\lambda}(\tilde{t}^{\tau}_{\lambda}(P^{\varepsilon}Y), P^{\varepsilon}Y) + \Big(1 - \frac{|y_1|}{\varepsilon|P^0Y|}\Big)\tilde{U}_{0,\lambda}(\tilde{t}_{\lambda}, P^0Y) \\ &\qquad \text{for } [\lambda, Y, \tau] \in \boldsymbol{D}_{\varepsilon}, \ Y = [y_1, \ldots, y_N] \in \boldsymbol{S}_{\varepsilon} \setminus \boldsymbol{S}, \ \tau \geqslant 0, \\ &= \tilde{U}_{0,\lambda}(\tilde{t}_{\lambda}, Y) \text{ for } [\lambda, Y, \tau] \in \boldsymbol{D}_{\varepsilon}, \ Y \in \boldsymbol{S}, \ \tau \geqslant 0. \\ &= R_{\varepsilon}(\lambda, Y, |\tau|) \text{ for } [\lambda, Y, \tau] \in \boldsymbol{D}_{\varepsilon}, \ \tau < 0. \end{aligned}$$

**Lemma 3.1.** *Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$. Then for any $\varepsilon > 0$ there are $\varrho_{\varepsilon} > 0$, $\tau_{\varepsilon} > 0$ such that*

$$\tilde{t}^{\tau}_{\lambda}(Y) < t_M \text{ for any } |\lambda| \leqslant \Lambda, \ Y \in \boldsymbol{V} \setminus \boldsymbol{S}_{\varepsilon}, \ |Y| < \varrho_{\varepsilon}, \ |\tau| < \tau_{\varepsilon}.$$

Proof. Suppose the contrary. Then there are $[\lambda_n, Y_n, \tau_n]$ such that $|\lambda_n| < \Lambda$, $\lambda_n \to \lambda$, $|\lambda| \leqslant \Lambda$, $Y_n \in \boldsymbol{V} \setminus \boldsymbol{S}_{\varepsilon}$, $Y_n \to 0$, $Z_n = \frac{Y_n}{|Y_n|} \to Z$, $\tau_n \to 0$, $\tilde{t}^{\tau_n}_{\lambda_n}(Y_n) \geqslant t_M$. Clearly $Z \notin \boldsymbol{S}$. We have $\tilde{t}_{0,\lambda}(Z) = \tilde{t}_{\lambda} < t_M$, $\dot{\tilde{\varphi}}_{0,\lambda}(\tilde{t}_{\lambda}, Z) < 0$ by Remark 1.4 (and our agreement from the end of Notation 3.1). Hence, Theorem 2.4 (together with Notation 3.1) implies $\tilde{t}^{\tau_n}_{\lambda_n}(Y_n) \to \tilde{t}_{0,\lambda}(Z)$, which is the contradiction. $\square$

**Remark 3.4.** If $Y \in \boldsymbol{S}_{\varepsilon} \setminus \boldsymbol{S}$ then

$$|P^{\varepsilon}Y|^2 = \varepsilon^2|P^0Y|^2 + |P^0Y|^2 \leqslant (1 + \varepsilon^2)|Y|^2.$$

**Remark 3.5.** Let $\tilde{U}(t) = \tilde{U}_\lambda^\tau(t,Y)$, $\tilde{\varrho}(t) = \tilde{\varrho}_\lambda^\tau(t,Y)$, $\tilde{\varphi}(t) = \tilde{\varphi}_\lambda^\tau(t,Y)$ for some $\lambda \in \mathbb{R}$, $Y \in \boldsymbol{V}$, $\tau \in [0,+\infty]$ and $|P_{\boldsymbol{L}}\tilde{U}_\lambda^\tau(t,Y)| > 0$ for $t \in [0,T)$. Then

$$\begin{aligned} P_{\boldsymbol{L}}\dot{\tilde{U}}(t) &= \tilde{\varrho}(t)\{-\sin(\tilde{\varphi}(t)+\tilde{\varphi}_Y)\dot{\tilde{\varphi}}(t)[1,0,\ldots,0] + \cos(\tilde{\varphi}(t)+\tilde{\varphi}_Y)\dot{\tilde{\varphi}}(t)[0,1,\ldots,0]\} \\ &\quad + \dot{\tilde{\varrho}}(t)\{\cos(\tilde{\varphi}(t)+\tilde{\varphi}_Y)[1,0,\ldots,0] + \sin(\tilde{\varphi}(t)+\tilde{\varphi}_Y)[0,1,\ldots,0]\}, \\ P_{\boldsymbol{L}}^*\tilde{U}(t) &= \tilde{\varrho}(t)\{-\sin(\tilde{\varphi}(t)+\tilde{\varphi}_Y)[1,0,\ldots,0] + \cos(\tilde{\varphi}(t)+\tilde{\varphi}_Y)[0,1,\ldots,0]\} \\ &\qquad\qquad \text{for a.a. } t \in (0,T). \end{aligned}$$

Hence

$$\frac{(\dot{\tilde{U}}(t), P_{\boldsymbol{L}}^*\tilde{U}(t))}{|P_{\boldsymbol{L}}\tilde{U}(t)|^2} = \frac{(P_{\boldsymbol{L}}\dot{\tilde{U}}(t), P_{\boldsymbol{L}}^*\tilde{U}(t))}{|P_{\boldsymbol{L}}\tilde{U}(t)|^2} = \dot{\tilde{\varphi}}(t) \quad \text{for all} \quad t \in (0,T).$$

**Lemma 3.2.** *Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$, let $\tilde{\varrho}_0$ be from Notation 3.2. Then for any $\varepsilon > 0$, $\delta \in (0,\tilde{\varrho}_0)$ there exists $\gamma > 0$ such that if $|\lambda| \leqslant \Lambda$, $\delta \leqslant |Y| \leqslant \tilde{\varrho}_0$, $Y \in \boldsymbol{V} \setminus \boldsymbol{S}_\varepsilon^0$, $\tau \in [0,+\infty)$, $\tilde{U}_\lambda^\tau(\cdot,Y)$ is periodic then $\tilde{t}_\lambda^\tau(Y) \geqslant \gamma$.*

Proof. Suppose by contradiction that $[\lambda_n, Y_n, \tau_n] \to [\lambda, Y, \tau]$ with $|Y_n| \leqslant \tilde{\varrho}_0$, $Y \notin \boldsymbol{S}_\varepsilon^0$, $|Y| > 0$, $\tau \in [0,+\infty]$, $\tilde{t}_{\lambda_n}^{\tau_n}(Y_n) \to 0$. Set $\tilde{U}_n(t) = \tilde{U}_{\lambda_n}^{\tau_n}(t,Y_n)$, $\tilde{\varphi}_n(t) = \tilde{\varphi}_{\lambda_n}^{\tau_n}(t,Y_n)$, $\tilde{t}_n = \tilde{t}_{\lambda_n}^{\tau_n}(Y_n)$. It follows from Theorems 2.2, 2.3 that

$$\tilde{U}_\lambda^\tau(t,Y) = Y \quad \text{for all } t \geqslant 0. \tag{3.2}$$

Theorems 2.1, 2.3 ensure $|P_L\tilde{U}_n(t,Y)| > 0$ for all $t \geqslant 0$, $n$ large enough. We shall show that there exists $C > 0$ such that

$$|\dot{\tilde{\varphi}}_n(t)| \leqslant C|\dot{\tilde{U}}_n(t)| \quad \text{for all } n, \text{ a.a. } t \in [0,t_M]. \tag{3.3}$$

Suppose that (3.3) is not satisfied: for any $C > 0$ there exist $n_C > 0$ and $E_C \subset [0,t_M]$, $\operatorname{meas}(E_C) > 0$ such that

$$|\dot{\tilde{\varphi}}_{n_C}(t)| > C|\dot{\tilde{U}}_{n_C}(t)| \text{ for a.a. } t \in E_C. \tag{3.4}$$

Remark 3.5 implies that

$$|\dot{\tilde{\varphi}}_{n_C}(t)| \leqslant \frac{|\dot{\tilde{U}}_{n_C}(t)|}{|P_{\boldsymbol{L}}\tilde{U}_{n_C}(t)|} \text{ for a.a. } t \in [0,t_M].$$

This together with (3.4) implies that

$$|P_{\boldsymbol{L}}\tilde{U}_{n_C}(t)| < \frac{1}{C} \quad \text{for a.a. } t \in E_C.$$

It follows by using Theorem 2.1 that there is $t_0 \in [0, t_M]$ such that

$$P_L \tilde{U}_\lambda^\tau(t_0, Y) = 0.$$

This is a contradiction with the assumption $Y \in \boldsymbol{V} \setminus \boldsymbol{S}_\varepsilon^0$, $|Y| > 0$, and (3.2). The estimate (3.3) is proved.

It follows from (2.3) (or (2.4)) in Remark 2.3 and from (3.3) that there is $C_2 > 0$ such that

$$\int_0^{\tilde{t}_n} |\dot{\tilde{\varphi}}_n(t)|^2 \,\mathrm{d}t < C_2. \tag{3.5}$$

From the periodicity of $U_n$ we have

$$2\pi = \tilde{\varphi}_n(0) - \tilde{\varphi}_n(\tilde{t}_n) \leqslant \left| \int_0^{\tilde{t}_n} \dot{\tilde{\varphi}}_n(t)\,\mathrm{d}t \right| \leqslant \tilde{t}_n^{\frac{1}{2}} \cdot \left( \int_0^{\tilde{t}_n} |\dot{\tilde{\varphi}}_n(t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \leqslant \tilde{t}_n^{\frac{1}{2}} \cdot C_3 \to 0.$$

This is a contradiction and our assertion is proved. □

**Lemma 3.3.** *Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$, let $\tilde{\varrho}_0$ be from Notation 3.2. Then for any $\varepsilon > 0$, $\boldsymbol{D}_\varepsilon$ is open (in $\mathbb{R} \times \boldsymbol{V} \times \mathbb{R}$) and there is $\varrho_\varepsilon^0 > 0$ such that $[\lambda, Y, 0] \in \boldsymbol{D}_\varepsilon$ if $|\lambda| < \Lambda$, $|Y| < \varrho_\varepsilon^0$. The mapping $R_\varepsilon$ is continuous on $\boldsymbol{D}_\varepsilon$.*

P r o o f. It follows directly from Definition 3.1 that $\boldsymbol{D}_\varepsilon$ is open. Lemma 3.1 and Remark 3.4 imply that

$$t_\mu^\xi(P^\varepsilon Z) < t_M \text{ if } |\mu| < \Lambda,\ Z \notin \boldsymbol{S},\ |Z| < \varrho_\varepsilon^0 = \varrho_\varepsilon(1+\varepsilon^2)^{-\frac{1}{2}},\ |\xi| < \tau_\varepsilon,$$

where $\varrho_\varepsilon$ is the number from Lemma 3.1. Clearly, for any $|\lambda| < \Lambda$, $|Y| < \varrho_\varepsilon^0$ there is $r > 0$ such that $|\mu| < \Lambda$, $|\xi| < \tau_\varepsilon$, $|Z| < \varrho_\varepsilon^0$ for any $[\mu, Z, \xi] \in \mathcal{U}_r(\lambda, Y, 0)$. Hence, $[\lambda, Y, 0] \in \boldsymbol{D}_\varepsilon$ by Definition 3.1 and Consequence 2.1. The continuity of $R_\varepsilon$ follows from Definition 3.1, Theorems 2.1, 2.4 and the choice of $\tilde{\varrho}_0$. □

**Definition 3.2.** Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$, let $\tilde{\varrho}_0$ be from Notation 3.2. Set $\boldsymbol{E} = \{[Y, \tau] \in \boldsymbol{V} \times \mathbb{R}\}$. For any $\varrho \in (0, \tilde{\varrho}_0)$, $\varepsilon > 0$ introduce a mapping $T_\varepsilon^\varrho \colon \boldsymbol{D}_\varepsilon \to \boldsymbol{E}$ by

$$T_\varepsilon^\varrho(\lambda, \mathfrak{X}) = \left[ R_\varepsilon(\lambda, Y, \tau), |Y|^2 \frac{1+\tau}{\varrho} \right] \quad \text{for any } [\lambda, Y, \tau] = [\lambda, \mathfrak{X}] \in \boldsymbol{D}_\varepsilon. \tag{3.6}$$

Set

$$\|\mathfrak{X}\| = |Y| + |\tau| \quad \text{for any } \mathfrak{X} = [Y, \tau].$$

**Lemma 3.4.** *Let $[\lambda, Y, \tau] \in \boldsymbol{D}_\varepsilon$, $\mathfrak{X} - T_\varepsilon^\varrho(\lambda, \mathfrak{X}) = 0$, $\mathfrak{X} = [Y, \tau]$. Then $|Y|^2 = \frac{\varrho\tau}{1+\tau}$. If, moreover, $Y \notin \boldsymbol{S}_\varepsilon^0$ and $\tau > 0$ then $Y = \tilde{U}_\lambda^\tau(\tilde{t}_\lambda^\tau(Y), Y)$ and $\tilde{t}_\lambda^\tau(Y) < t_M$.*

P r o o f follows directly from Definitions 3.1, 3.2. □

**Definition 3.3.** For any $\lambda \in \mathbb{R}$ introduce a linear mapping $L(\lambda)\colon \boldsymbol{E} \to \boldsymbol{E}$ by

$$L(\lambda)\mathfrak{X} = [\tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Y), 0] \text{ for all } \mathfrak{X} = [Y, \tau] \in \boldsymbol{E}.$$

**Lemma 3.5.** *If* $\delta > 0$ *then*

$$\mathrm{d}(L(\lambda_0 + \delta)) \neq \mathrm{d}(L(\lambda_0 - \delta)). \tag{3.7}$$

Proof. Recall that

$$\mathrm{d}(L) = (-1)^{\sum n_i} \tag{3.8}$$

where the sum is taken over all eigenvalues $\nu_i > 1$ of the operator $L$, $n_i$ is the algebraic multiplicity of $\nu_i$. (This holds for any linear completely continuous operator $L$ in a real Banach space, see e.g. [11].) It follows from $(\mu)$ (see also Remark 1.4) and Notation 3.1 that for any $\lambda > \lambda_0$ there is precisely one $\nu = \nu(\lambda) > 1$ such that

$$\nu Y = \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Y)$$

has a nontrivial solution $Y \in \boldsymbol{V}$ and this solution is of the form $[c, 0, \dots, 0]$. If $\lambda < \lambda_0$ then the last equation can have a nontrivial solution only with $\nu < 1$. This means that if $\lambda > \lambda_0$ then $L(\lambda)$ has precisely one eigenvalue $\nu_1(\lambda) > 1$ with the only corresponding (normed) eigenvector $\mathfrak{X}_1 = [1, 0, \dots, 0]$, and if $\lambda < \lambda_0$ then $L(\lambda)$ has no eigenvalue greater than 1. Let us show that the eigenvalue mentioned is algebraically simple, i.e.

$$\dim \bigcup_{k=1}^{\infty} \mathrm{Ker}(\nu_1(\lambda)I - L(\lambda))^k = 1.$$

If $Y \in \boldsymbol{V} \setminus \mathrm{Lin}\{[1, 0, \dots, 0]\}$ then $P^0 Y \neq 0$ and $\nu P^0 Y - \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^0 Y) = \nu P^0 Y - P^0 \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Y) \neq 0$ for any $\nu > 1$, $\lambda \in \mathbb{R}$ by Remark 1.4. This means

$$\nu Y - \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Y) \notin \mathrm{Lin}\{[1, 0, \dots, 0]\} \text{ for all } Y \in \boldsymbol{V} \setminus \mathrm{Lin}\{[1, 0, \dots, 0]\},\ \nu > 1,\ \lambda \in \mathbb{R},$$

i.e.

$$\nu\mathfrak{X} - L(\lambda)\mathfrak{X} \notin \mathrm{Lin}\{\mathfrak{X}_1\} \quad \text{for all } \mathfrak{X} \notin \mathrm{Lin}\{\mathfrak{X}_1\},\ \nu > 1,\ \lambda \in \mathbb{R}.$$

If $(\nu_1(\lambda)I - L(\lambda))^k \mathfrak{X} = (\nu_1(\lambda)I - L(\lambda))(\nu_1(\lambda) - L(\lambda))^{k-1}\mathfrak{X} = 0$ then we obtain $(\nu_1(\lambda)I - L(\lambda))^j \mathfrak{X} \in \mathrm{Lin}\{\mathfrak{X}_1\}$ successively for $j = k-1, \dots, 1, 0$ because $\mathfrak{X}_1$ is the only eigenvector of $L(\lambda)$ corresponding to $\nu_1(\lambda)$. This means $\mathfrak{X} \in \mathrm{Lin}\{\mathfrak{X}_1\}$ and the simplicity of our eigenvalue is proved. Now the assertion of Lemma 3.5 follows from (3.8). □

**Remark 3.6.** If $Y_n \to 0$, $Y_n \in \boldsymbol{V} \setminus \boldsymbol{S}$, $Z_n = \frac{Y_n}{|Y_n|} \to Z \in \boldsymbol{V} \setminus \boldsymbol{S}$, $\lambda_n \to \lambda$, $\tau_n \to 0$, then

$$\frac{\tilde{U}_n(\tilde{t}_n(P^\varepsilon Y_n), P^\varepsilon Y_n)}{|Y_n|} \to \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^\varepsilon Z)$$

where we write $\tilde{U}_n = \tilde{U}^{\tau_n}_{\lambda_n}$, $\tilde{t}_n = \tilde{t}^{\tau_n}_{\lambda_n}$. Indeed, we have

$$\frac{P^\varepsilon Y_n}{|Y_n|} \to P^\varepsilon Z, \quad \frac{P^\varepsilon Y_n}{|Y_n|} = \frac{P^\varepsilon Y_n}{|P^\varepsilon Y_n|} \frac{|P^\varepsilon Y_n|}{|Y_n|}$$

and therefore

$$\frac{P^\varepsilon Y_n}{|P^\varepsilon Y_n|} \to W = \frac{P^\varepsilon Z}{|P^\varepsilon Z|} \notin \boldsymbol{S}.$$

Of course, $\tilde{t}^0_{0,\lambda}(W) = \tilde{t}_\lambda < t_M$, $\dot{\tilde{\varphi}}^0_{0,\lambda}(\tilde{t}_\lambda, W) < 0$. Hence, Theorems 2.1 and 2.4 imply $\tilde{t}_n(P^\varepsilon Y_n) \to \tilde{t}^0_{0,\lambda}(W) = \tilde{t}_\lambda$ and

$$\begin{aligned}\frac{\tilde{U}_n(\tilde{t}_n(P^\varepsilon Y_n), P^\varepsilon Y_n)}{|Y_n|} &= \frac{\tilde{U}_n(\tilde{t}_n(P^\varepsilon Y_n), P^\varepsilon Y_n)}{|P^\varepsilon Y_n|} \frac{|P^\varepsilon Y_n|}{|Y_n|} \\ &\to \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, W) \cdot |P^\varepsilon Z| = \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^\varepsilon Z).\end{aligned}$$

Similarly, it follows from Theorem 2.1 that

$$\frac{\tilde{U}_{\lambda_n}(\tilde{t}_{\lambda_n}, P^0 Y_n)}{|Y_n|} \to \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^0 Z).$$

**Lemma 3.6.** *For any fixed $\varrho \in (0, \tilde{\varrho}_0)$, $\varepsilon > 0$ we have*

$$\lim_{\|\mathfrak{X}\| \to 0} \frac{\|T^\varrho_\varepsilon(\lambda, \mathfrak{X}) - L(\lambda)\mathfrak{X}\|}{\|\mathfrak{X}\|} = 0 \text{ uniformly on compact } \lambda\text{-intervals.} \tag{3.9}$$

P r o o f. It is sufficient to show that if

$$\lambda_n \to \lambda, \ [y^n_1, \dots, y^n_N] = Y_n \to 0, \ \tau_n \to 0, \ \frac{Y_n}{|Y_n|} \to Z = [z_1, \dots, z_N], \ \mathfrak{X}_n = [Y_n, \tau_n]$$

then

$$\lim_{n \to \infty} \frac{\|T^\varrho_\varepsilon(\lambda_n, \mathfrak{X}_n) - L(\lambda_n)\mathfrak{X}_n\|}{\|\mathfrak{X}_n\|} = 0.$$

Using Theorems 2.1, 2.4 we obtain

$$\frac{\tilde{U}^{\tau_n}_{\lambda_n}(\tilde{t}^{\tau_n}_{\lambda_n}(Y_n), Y_n)}{|Y_n|} \to \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Z) \quad \text{if } Y_n \notin \boldsymbol{S}_\varepsilon,$$

$$\frac{\tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, Y_n)}{|Y_n|} \to \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Z) \quad \text{if } Y_n \in \boldsymbol{S}.$$

If $Y_n \in \boldsymbol{S}_\varepsilon \setminus \boldsymbol{S}$ then we obtain by using the definition of $P^\varepsilon$, Remark 3.6 and the linearity of (LE) that

$$\begin{aligned}\frac{1}{|Y_n|}\Big(\frac{|y_1^n|}{\varepsilon|P^0Y_n|}\tilde{U}^{\tau_n}_{\lambda_n}(\tilde{t}^{\tau_n}_{\lambda_n}(P^\varepsilon Y_n), P^\varepsilon Y_n) + \Big(1-\frac{|y_1^n|}{\varepsilon|P^0Y_n|}\Big)\tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, P^0Y_n)\Big)\\ \to \frac{|z_1|}{\varepsilon|P^0Z|}\tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^\varepsilon Z) + \Big(1-\frac{|z_1|}{\varepsilon|P^0Z|}\Big)\tilde{U}_{0,\lambda}(\tilde{t}_\lambda, P^0Z) = \tilde{U}_{0,\lambda}(\tilde{t}_\lambda, Z).\end{aligned}$$

Notice that $P^\varepsilon Z$ has no sense if $Z \in \boldsymbol{S}$ but in this case $\frac{|y_1^n|}{\varepsilon|P^0Y_n|} \to \frac{|z_1|}{\varepsilon|P^0Z|} = 0$, the terms containing $P^\varepsilon Z$ can be understood as zeros and we have $P^0Z = Z$. Hence, the resulting expression is obtained directly.

Now, Definition 3.1 implies

$$\frac{|R_\varepsilon(\lambda_n, Y_n, \tau_n) - \tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, Y_n)|}{|Y_n|+|\tau_n|} \leqslant \frac{|R_\varepsilon(\lambda_n, Y_n, \tau_n) - \tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, Y_n)|}{|Y_n|} \to 0.$$

Simultaneously

$$\frac{|Y_n|^2(1+\tau_n)}{\varrho(|Y_n|+|\tau_n|)} \to 0$$

and our assertion follows from Definitions 3.2, 3.3. □

**Lemma 3.7.** *Let $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$, $\varepsilon > 0$, let $\tilde{\varrho}_0$ be from Notation 3.2. If $[\lambda_n, Y_n, \tau_n] \in \boldsymbol{D}_\varepsilon$, $Y_n \neq 0$, $[\lambda_n, Y_n, \tau_n] \to [\lambda_0, 0, 0]$, $Y_n = R_\varepsilon(\lambda_n, Y_n, \tau_n)$, $Z_n = \frac{Y_n}{|Y_n|} \to Z$ then either $Z = [1, 0, \dots, 0]$ or $Z = [-1, 0, \dots, 0]$. There is $r > 0$ such that if $[\lambda, Y, \tau] \in \boldsymbol{D}_\varepsilon \cap \mathcal{U}_r(\lambda_0, 0, 0)$, $Y = R_\varepsilon(\lambda, Y, \tau)$, $|Y| > 0$ then $Y \notin \boldsymbol{S}_\varepsilon$.*

Proof. Write $\tilde{U}_n = \tilde{U}^{\tau_n}_{\lambda_n}$, $\tilde{t}_n = t^{\tau_n}_{\lambda_n}$. First, let us realize that Definition 3.1 gives

$$\tilde{t}_n(P^\varepsilon Y_n) < t_M \quad \text{for } n \text{ such that } Y_n \notin \boldsymbol{S}.$$

We can suppose without loss of generality that one of the following cases (i)–(iii) occurs:

(i) $Y_n \notin \boldsymbol{S}_\varepsilon$: Then $Y_n = R_\varepsilon(\lambda_n, Y_n, \tau_n)$ reads $Y_n - \tilde{U}_n(\tilde{t}_n(Y_n), Y_n) = 0$. Dividing it by $|Y_n|$ and letting $n \to \infty$ we obtain by using Theorems 2.1 and 2.4 (precisely see Remark 3.6)

$$Z = \tilde{U}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, Z).$$

It follows that $Z \in \boldsymbol{L}$ by the assumption $(\mu)$ (see Remark 1.4) and therefore $Z = [\pm 1, 0, \dots, 0]$ because $Z \in \boldsymbol{V}$.

(ii) $Y_n \in \boldsymbol{S}$: Then

$$Y_n - \tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, Y_n) = 0$$

and it follows that

$$Z - \tilde{U}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, Z) = 0, \ Z \notin \boldsymbol{L}$$

which is impossible by the assumption $(\mu)$ (see Remark 1.4).

(iii) $Y_n \in \boldsymbol{S}_\varepsilon \setminus \boldsymbol{S}$: Then

$$Y_n - \frac{|y_1^n|}{\varepsilon|P^0Y_n|}\tilde{U}_n(\tilde{t}_n(P^\varepsilon Y_n), P^\varepsilon Y_n) - \Big(1 - \frac{|y_1^n|}{\varepsilon|P^0Y_n|}\Big)\tilde{U}_{0,\lambda_n}(\tilde{t}_{\lambda_n}, P^0Y_n) = 0.$$

If $Z \notin \boldsymbol{S}$ then Theorems 2.1, 2.4 (see Remark 3.6) and the definition of $P^\varepsilon$ yield

$$Z - \frac{|z_1|}{\varepsilon|P^0Z|}\tilde{U}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, P^\varepsilon Z) - \Big(1 - \frac{|z_1|}{\varepsilon|P^0Z|}\Big)\tilde{U}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, P^0Z) = Z - \tilde{U}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, Z) = 0.$$

If $Z \in \boldsymbol{S}$ then $P^\varepsilon Z$ has no sense but in this case the terms containing $P^\varepsilon Y_n$ tend to zero, $P^0Z = Z$ and the resulting expression is obtained directly. We have $Z \notin \boldsymbol{L}$ and the last equality is excluded by Remark 1.4 as in (ii).

The last assertion of Lemma 3.7 is a consequence of the first because of $Z = [\pm 1, 0, \dots, 0] \notin \boldsymbol{S}_\varepsilon$. □

**Theorem 3.1.** *Let $\boldsymbol{E}$ be a real Banach space with the norm $\|\cdot\|$, $D$ an open set in $\mathbb{R}\times\boldsymbol{E}$, $T, L\colon D \to \boldsymbol{E}$ completely continuous mappings such that $L(\lambda)$ is linear for $\lambda$ fixed. Suppose that $[\lambda_0, 0] \in D$ and there is $\delta_0 > 0$ with the following property:*

(a) $\quad \operatorname{Ker}(I - L(\lambda)) = \{0\}$ *for all* $\lambda \in (\lambda_0 - \delta_0, \lambda_0 + \delta_0), \lambda \neq \lambda_0$,

(b) $\quad \mathrm{d}(L(\lambda_0 - \delta)) \neq \mathrm{d}(L(\lambda_0 + \delta))$ *for any* $\delta \in (0, \delta_0)$,

(c) $\quad \lim\limits_{\|\mathfrak{X}\|\to 0} \dfrac{\|T(\lambda,\mathfrak{X}) - L(\lambda)\mathfrak{X}\|}{\|\mathfrak{X}\|} = 0$ *uniformly on compact $\lambda$-intervals.*

*Set*

$$C = \overline{\{[\lambda,\mathfrak{X}] \in D;\, \mathfrak{X} - T(\lambda,\mathfrak{X}) = 0, \|\mathfrak{X}\| \neq 0\}}^{D} \ (\text{the closure in } D).$$

*Denote by $C_0$ the component of $C$ containing $[\lambda_0, 0]$. Then at least one of the following conditions is fulfilled:*

(i) *there is $\lambda_1 \neq \lambda_0$ such that* $\operatorname{Ker}(I - L(\lambda_1)) \neq \{0\}$, $[\lambda_1, 0] \in C_0$,

(ii) *$C_0$ is not compact.*

P r o o f can be done similarly as that of Theorem 1.3 and Corollary 1.12 in [13]. In [13], operators of the form $L(\lambda)\mathfrak{X} = \lambda L\mathfrak{X}$, $T(\lambda,\mathfrak{X}) = \lambda L\mathfrak{X} + N(\lambda,\mathfrak{X})$ are considered, where $L$ is linear completely continuous, $N$ is a small (at zero) compact perturbation. It is supposed that $\lambda_0$ is a characteristic value of $L$ of an odd multiplicity. Our conditions (a), (b), (c) are automatically fulfilled under these assumptions. In fact, only these conditions are used in the proof and the special form of the operators is unnecessary. □

Proof of Theorem 1.2. First, suppose that (1.2) is fulfilled and let $\Lambda$, $t_M$ and $\tilde{\varrho}_0$ be from this assumption and from Notation 3.2, respectively. For any $\varrho \in (0, \tilde{\varrho}_0)$, $\varepsilon > 0$ denote by $C^{\varepsilon}_{\varrho}$ the component of the set

$$\overline{\{[\lambda, \mathfrak{X}] \in \boldsymbol{D}_\varepsilon\, ;\, \mathfrak{X} = T^{\varepsilon}_{\varrho}(\lambda, \mathfrak{X}),\ \mathfrak{X} \neq 0\}}^{\boldsymbol{D}_\varepsilon} =$$
$$= \overline{\left\{[\lambda, Y, \tau] \in \boldsymbol{D}_\varepsilon\, ;\, Y = R_\varepsilon(\lambda, Y, \tau),\ |Y|^2 = \frac{\varrho\tau}{1+\tau}, \tau \neq 0\right\}}^{\boldsymbol{D}_\varepsilon}$$

(the closure in $\boldsymbol{D}_\varepsilon$) containing $[\lambda_0, 0]$, where $\boldsymbol{D}_\varepsilon$, $R_\varepsilon$, $T^{\varepsilon}_{\varrho}$ are from Definitions 3.1, 3.2. Lemma 3.4 implies that

$$\text{(3.10)}\qquad \begin{cases} \text{if } [\lambda, Y, \tau] \in C^{\varepsilon}_{\varrho} \text{ then } |Y|^2 = \frac{\varrho\tau}{1+\tau}\,; \text{ if moreover } Y \notin \boldsymbol{S}^0_\varepsilon,\ \tau \neq 0 \\ \text{then } Y = \tilde{U}^{\tau}_{\lambda}(\tilde{t}^{\tau}_{\lambda}(Y), Y),\ \tilde{t}^{\tau}_{\lambda}(Y) < t_M. \end{cases}$$

It follows from the definition of $C^{\varepsilon}_{\varrho}$ that $\tau \geqslant 0$ for any $[\lambda, Y, \tau] \in C^{\varepsilon}_{\varrho}$. First, let us show that $\tilde{\varrho}_0$ could be chosen such that there is $\varepsilon > 0$ satisfying

$$\text{(3.11)}\qquad Y \notin \boldsymbol{S}_\varepsilon \quad \text{for any } [\lambda, Y, \tau] \in C^{\varepsilon}_{\varrho},\ \tau \neq 0,\ \varrho \in (0, \tilde{\varrho}_0).$$

Suppose the contrary. Then it follows from the last assertion of Lemma 3.7 and the connectedness of any $C^{\varepsilon}_{\varrho}$ that there exist $\varrho_n > 0$, $\varepsilon_n > 0$, $\tau_n \in (0, +\infty)$, $[\lambda_n, Y_n, \tau_n] \in C^{\varepsilon_n}_{\varrho_n}$ such that $\varrho_n \to 0$, $\varepsilon_n \to 0$, $\lambda_n \to \lambda \in [-\Lambda, \Lambda]$, $\tau_n \to \tau \in [0, +\infty]$, $|Y_n| \to 0$, $Z_n = \frac{Y_n}{|Y_n|} \to Z$, $Y_n \in \boldsymbol{S}_{\varepsilon_n} \setminus \boldsymbol{S}^0_{\varepsilon_n}$. Consequently, $Z \in \boldsymbol{S}$. Writing $\tilde{U}_n = \tilde{U}^{\tau_n}_{\lambda_n}$, $\tilde{t}_n = \tilde{t}^{\tau_n}_{\lambda_n}$ we obtain

$$\text{(3.12)}\qquad Y_n = \tilde{U}_n(\tilde{t}_n(Y_n), Y_n), \quad 0 < \tilde{t}_n(Y_n) < t_M$$

by (3.10). We can suppose $\tilde{t}_n(Y_n) \to t_0 \in [0, t_M]$. Theorems 2.1, 2.4 give

$$\frac{\tilde{U}_n(\tilde{t}_n(Y_n), Y_n)}{|Y_n|} \to \tilde{U}^{\tau}_{0,\lambda}(t_0, Z),$$

i.e. we obtain

$$\text{(3.13)}\qquad Z = \tilde{U}^{\tau}_{0,\lambda}(t_0, Z)$$

with $Z \in \boldsymbol{S}$, which contradicts (1.2) if $t_0 > 0$. It follows from Theorems 2.2, 2.3 that for $t_0 = 0$, $\tilde{U}^{\tau}_{0,\lambda}(t, Z)$ is a stationary solution of (LPE) or of (LI) if $\tau < +\infty$ or $\tau = +\infty$, respectively. Hence $Z = \tilde{U}^{\tau}_{0,\lambda}(t, Z)$ for all $t \geqslant 0$ and this contradicts (1.2) again. The existence of $\tilde{\varrho}_0$ and $\varepsilon$ satisfying (3.11) is proved.

Further, we will consider this fixed $\varepsilon$ and an arbitrary $\varrho \in (0, \tilde{\varrho}_0)$ and write $C_\varrho$ instead of $C_\varrho^\varepsilon$.

We will use Theorem 3.1 for $T = T_\varrho^\varepsilon$ with an arbitrary fixed $\varrho \in (0, \tilde{\varrho}_0)$. In this case, $C_0$ in Theorem 3.1 coincides with our $C_\varrho$.

If $\operatorname{Ker}(I - L(\lambda)) \neq \{0\}$ then (3.13) holds with some $Z \neq 0$, $\tau = 0$, $t_0 = \tilde{t}_\lambda$ and this is excluded for $\lambda \neq \lambda_0$ by Remark 1.4. In particular, the assumption (a) is fulfilled with any $\delta_0 > 0$ and the case (i) in Theorem 3.1 is excluded. The other assumptions of Theorem 3.1 follow from Lemmas 3.3, 3.5, 3.6. Hence, the case (ii) in Theorem 3.1 occurs.

Set

$$\mathcal{C}_\varrho = \{[\lambda, V, \tau];\ V = \sum_{j=1}^N y_j U_j(\lambda),\ [\lambda, Y, \tau] \in \overline{C_\varrho}$$
$$\text{with } Y = [y_1, \ldots, y_N] \text{ and } \tau \in [0, +\infty)\}.$$

We shall show that $\mathcal{C}_\varrho$ has all the properties announced in Theorem 1.2 with $\tau_0 = +\infty$.

First, the condition (1.3) follows from (3.10), (3.11) and Notation 3.1. It follows from (ii) in Theorem 3.1 that there are $[\lambda_n, Y_n, \tau_n] \in C_\varrho$ such that

$$[\lambda_n, Y_n, \tau_n] \to [\lambda, Y, \tau] \notin C_\varrho, |\lambda| \leqslant \Lambda, Y \in \boldsymbol{V}, \tau \in [0, +\infty].$$

We will write $[\lambda, Y, \tau] \in \overline{C_\varrho}$ also in the case $\tau = +\infty$. It follows that $[\lambda, Y, \tau] \notin \boldsymbol{D}_\varepsilon$. We have $[\lambda, 0, 0] \in \boldsymbol{D}_\varepsilon$ by Lemma 3.3 and therefore $|Y| \neq 0$, $\tau > 0$. We obtain (3.12) by (3.10), (3.11) again. We can suppose $\tilde{t}_n(Y_n) \to t_0 > 0$ by Lemma 3.2. The limiting process in (3.12) (by using Theorem 2.1 and (3.11)) gives

$$Y = \tilde{U}_\lambda^\tau(t_0, Y),\ 0 < t_0 \leqslant t_M,\ Y \notin S_\varepsilon^0. \tag{3.14}$$

The fact that $[\lambda, Y, \tau] \notin \boldsymbol{D}_\varepsilon$ (together with (3.11)) implies that at least one of the following conditions is fulfilled:

$$\tau = +\infty, \tag{3.15}$$

$$|\lambda| = \Lambda, \tag{3.16}$$

$$\tilde{t}_\lambda^\tau(Y) \geqslant t_M,\ ^{*)} \tag{3.17}$$

$$\dot{\tilde{\varphi}}_\lambda^\tau(\tilde{t}_\lambda^\tau(Y), Y) \geqslant 0. \tag{3.18}$$

*) In fact, it is possible to show that the sign ">" is excluded in (3.17). (First, it would be necessary to know that $\tilde{U}_\lambda^\tau(t, Y) \notin \mathbf{S}$ for all $t$, i.e. that $\tilde{t}_\lambda^\tau(Y) < +\infty$.) But this is not important for our further considerations where (3.14) is essential.

Hence, for any $\varrho \in (0, \tilde{\varrho}_0)$ there is $[\lambda, Y, \tau] \in \overline{C_\varrho} \setminus D_\varepsilon$ satisfying (3.14) and at least one of the conditions (3.15)–(3.18). Our aim is to show that in fact (3.15) holds for $\varrho \in (0, \tilde{\varrho}_0)$ if $\tilde{\varrho}_0$ is small enough. Suppose by contradiction that there are $\varrho_n > 0$, $[\lambda_n, Y_n, \tau_n] \in \overline{C_{\varrho_n}} \setminus D_\varepsilon$ and $t_0^n > 0$ such that $\varrho_n \to 0$, $\tau_n \in [0, +\infty)$,

$$Y_n = \tilde{U}_n(t_0^n, Y_n),\ 0 < t_0^n \leqslant t_M,\ Y_n \notin S_\varepsilon^0 \tag{3.19}$$

and at least one of the following conditions is fulfilled:

$$|\lambda_n| = \Lambda, \quad n = 1, 2, \dots, \tag{3.20}$$

$$\tilde{t}_n(Y_n) \geqslant t_M, \quad n = 1, 2, \dots, \tag{3.21}$$

$$\dot{\tilde{\varphi}}_n(\tilde{t}_n(Y_n), Y_n) \geqslant 0, \quad n = 1, 2, \dots. \tag{3.22}$$

We can suppose $\lambda_n \to \lambda$, $Z_n = \frac{Y_n}{|Y_n|} \to Z$, $\tau_n \to \tau \in [0, +\infty]$, $t_0^n \to t_0$. Dividing (3.19) by $|Y_n|$ and using Theorem 2.1 or 2.3 we obtain (3.13). If $t_0 > 0$ then the assumption (1.2) implies that $|\lambda| < \Lambda$, $\tilde{t}_{0,\lambda}^\tau(Z) < t_M$, $\dot{\tilde{\varphi}}_{0,\lambda}^\tau(t_{0,\lambda}^\tau(Z), Z) < 0$. This together with Theorem 2.1, Consequence 2.1, Theorems 2.4, 2.5 leads to the contradiction with the fact that one of the conditions (3.20)–(3.22) holds. The case $t_0 = 0$ is excluded because $U_{0,\lambda}^\tau(\cdot, Z)$ would be stationary by Theorems 2.2, 2.3 and this is impossible by the assumption (1.2).

Hence, $\tilde{\varrho}_0$ could be chosen such that for any $\varrho \in (0, \tilde{\varrho}_0)$ there is a point $[\lambda, Y, \tau] \in \overline{C_\varrho}$ satisfying (3.15). The connectedness of $\overline{C_\varrho}$ implies that for any $\tau \in [0, +\infty]$ there are $\lambda$, $Y$ such that $[\lambda, Y, \tau] \in \overline{C_\varrho}$. The condition (1.4) with $\tau_0 = +\infty$ follows.

It remains to show that the first part of the assertion of Theorem 1.2 (about the existence of $\varrho_0$, $\tau_0$ with the properties required) holds even if the assumption (1.2) is omitted. In this case, choose arbitrary fixed $\Lambda > |\lambda_0|$, $t_M > \frac{2\pi}{\beta_0}$. Let $\tilde{\varrho}_0$ be the corresponding number from Notation 3.2. For any $\varrho \in (0, \tilde{\varrho}_0)$, $\varepsilon > 0$ we can introduce $C_\varrho^\varepsilon$ as above.

We will show by almost the same considerations as in the proof of (3.11) above that $\tilde{\varrho}_0$ could be chosen such that there are $\tau_1 > 0$, $\varepsilon > 0$ satisfying

$$Y \notin \boldsymbol{S}_\varepsilon \quad \text{for any } [\lambda, Y, \tau] \in C_\varrho^\varepsilon, \tau \in (0, \tau_1], \varrho \in (0, \tilde{\varrho}_0). \tag{3.23}$$

Suppose the contrary. Then there exist $\varrho_n > 0$, $\varepsilon_n > 0$, $[\lambda_n, Y_n, \tau_n] \in C_{\varrho_n}^{\varepsilon_n}$, $\tau_n > 0$ such that $\varrho_n \to 0, \varepsilon_n \to 0, \tau_n \to 0, \lambda_n \to \lambda, |Y_n| \to 0, Z_n = \frac{Y_n}{|Y_n|} \to Z, Y_n \in \boldsymbol{S}_{\varepsilon_n} \setminus \boldsymbol{S}_{\varepsilon_n}^0$. (Cf. the proof of (3.11).) We have (3.12) again and the limiting process (after dividing by $|Y_n|$ and using Theorem 2.1) gives (3.13) with $t_0 \in [0, t_M]$, $Z \in \boldsymbol{S}$, $|Z| \neq 0$ and $\tau = 0$. This contradicts the behaviour of the solutions of the equation (LE) (see Remark 1.4). (Recall that if $t_0 = 0$ then $\tilde{U}_{0,\lambda}(\cdot, Z)$ is stationary by Theorem 2.2.)

Further, we consider a fixed $\varepsilon$ satisfying (3.23) and denote by $C_\varrho$ the component of

$$\{[\lambda, Y, \tau] \in C_\varrho^\varepsilon;\ \tau \leqslant \tau_1\}$$

containing $[\lambda_0, 0, 0]$ (for any $\varrho \in (0, \tilde{\varrho}_0)$). We apply Theorem 3.1 to $T_\varrho^\varepsilon$ for an arbitrary fixed $\varrho \in (0, \tilde{\varrho}_0)$ again. We set

$$\mathcal{C}_\varrho = \Big\{[\lambda, V, \tau];\ V = \sum_{j=1}^{N} y_j U_j(\lambda),\ [\lambda, Y, \tau] \in \overline{C_\varrho} \text{ with } Y = [y_1, \ldots, y_N]\Big\}$$

and show that $\mathcal{C}_\varrho$ has all the properties announced in Theorem 1.2. The condition (1.3) follows from (3.10), (3.23). Suppose that $\tilde{\varrho}_0$, $\tau_0$ cannot be chosen such that (1.4) in Theorem 1.2 holds for all $\varrho \in (0, \tilde{\varrho}_0)$. Then there are $\varrho_n > 0$, $0 < \tau_n < \tau_1$ such that $\varrho_n \to 0$, $\tau_n \to 0$ and

$$\tau < \tau_n \quad \text{for all } [\lambda, Y, \tau] \in C_{\varrho_n}. \tag{3.24}$$

In particular, $C_{\varrho_n} = C^\varepsilon_{\varrho_n}$. Analogously as above we can show that the condition (i) (for $C_0$ replaced by $C_{\varrho_n}$) from Theorem 3.1 cannot be fulfilled. Thus, the case (ii) in Theorem 3.1 occurs, i.e. $C_{\varrho_n}$ are not compact. By using similar considerations as above together with (3.24) we obtain that there exist $[\lambda_n, Y_n, \tau_n] \in \overline{C_{\varrho_n}} \setminus C_{\varrho_n}$ satisfying (3.19) and at least one of the conditions (3.20), (3.21), (3.22). (The case $\tau_n = \tau_1$ is excluded by (3.24).) We can suppose $\lambda_n \to \lambda$, $Z_n = \frac{Y_n}{|Y_n|} \to Z$. Dividing (3.19) by $|Y_n|$ and letting $n \to \infty$ we obtain (by using Theorem 2.1) (3.13) with $\tau = 0$. The case $t_0 = 0$ is excluded because there is no nontrivial stationary solution of (LE). The case $t_0 > 0$ is possible only for $\lambda = \lambda_0$, $Z = [\pm 1, 0, \ldots, 0]$ by Remark 1.4. We have $|\lambda_0| < \Lambda$, $t^0_{0,\lambda_0}(Z) = \tilde{t}_{\lambda_0} < t_M$, $\dot{\varphi}^0_{0,\lambda_0}(\tilde{t}_{\lambda_0}, Z) = \dot{\tilde{\varphi}}_{0,\lambda_0}(\tilde{t}_{\lambda_0}, Z) < 0$. Consequence 2.1 and Theorem 2.4 give $\tilde{t}_n(Y_n) \to \tilde{t}_{\lambda_0}$, $\dot{\tilde{\varphi}}_n(\tilde{t}_n(Y_n), Y_n) < 0$ for $n$ large enough. Hence, (3.20), (3.21), (3.22) are excluded, which is a contradiction. $\square$

*Proof* of Theorem 1.1. For any given $\varrho \in (0, \varrho_0)$ we can find $\tilde{\varrho}$ small enough such that

$$|V| \leqslant \varrho \quad \text{for all } V = \sum_{j=1}^{N} y_j U_j(\lambda) \quad \text{with } |Y| \leqslant \tilde{\varrho},\ Y = [y_1, \ldots, y_N],\ |\lambda| \leqslant \Lambda.$$

We have $\tau_0 = +\infty$ in (1.4) under the assumptions of Theorem 1.1 by the last assertion of Theorem 1.2. Hence, there exists a sequence of triplets $[\lambda_n, V_n, \tau_n] \in \mathcal{C}_{\tilde{\varrho}}$ such that $\tau_n \to +\infty, \lambda_n \to \lambda_\varrho \in [-\Lambda, \Lambda]$, $V_n \to V_\varrho \in \boldsymbol{V}_{\lambda_\varrho}$, $V_n = \sum\limits_{j=1}^{N} y_j^n U_j(\lambda_n)$, $V_\varrho =$

$\sum_{j=1}^{N} y_j^\varrho U_j(\lambda_\varrho)$, $|Y_n|^2 = \frac{\tilde{\varrho}\tau_n}{1+\tau_n} \to |Y_\varrho|^2 = \tilde{\varrho}$, $|V_\varrho| \leqslant \varrho$ with $Y_n = [y_1^n, \dots, y_N^n]$, $Y_\varrho = [y_1^\varrho, \dots, y_N^\varrho]$, $U_{\lambda_n}^{\tau_n}(\cdot, V_n)$ are periodic with $t_{\lambda_n}^{\tau_n}(V_n) < t_M$, $t_{\lambda_n}^{\tau_n}(V_n) \to t_0$. Lemma 3.2 ensures $t_0 > 0$ and Theorem 2.3 implies that $U_{\lambda_n}^{\tau_n}(\cdot, V_n) \to U_{\lambda_\varrho}^{\infty}(\cdot, V_\varrho)$ in $C([0, t_M])$ and $U_{\lambda_\varrho}^{\infty}(\cdot, V_\varrho)$ is periodic. Notice that it follows from the properties of $C_\varrho$ that $U_{\lambda_\varrho}^{\infty}(\cdot, V_\varrho)$ is not stationary. Indeed, we have $P_{L_{\lambda_\varrho}} V_\varrho \neq 0$ because $V_\varrho \notin S_\varepsilon^0$ (see (3.11)). If it were $U_{\lambda_\varrho}^{\infty}(t, V_\varrho) = V_\varrho$ for all $t$ then we would have $P_{L_{\lambda_n}} U_{\lambda_n}^{\tau_n}(\cdot, V_n) \to P_{L_{\lambda_\varrho}} V_\varrho$ in $C([0, t_M])$ by Theorem 3.1 which would contradict the fact that $P_{L_{\lambda_n}} U_{\lambda_n}^{\tau_n}(\cdot, V_n)$ circulate around the origin.

Of course, there is at least one accumulation point $\lambda_I$ of $\lambda_\varrho$ for $\varrho \to 0_+$. Clearly, any such accumulation point is a bifurcation point of (I) at which periodic solutions bifurcate from the branch of trivial solutions. □

## 4. Appendix

Proof of Lemma 2.1. Choose $\varrho > 0$. It follows from (G), (L) that there exists $C > 0$ such that

$$\frac{|G(\lambda, U)|}{|U|} \leqslant C \quad \text{for all } |U| \leqslant \varrho,\ |\lambda| \leqslant \Lambda.$$

Set $U(t) = U_\lambda^\tau(t, V)$. Multiplying (PE) by $U(t)$ and using (M) (see Remark 1.3) we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|U(t)|^2) &= (\dot{U}(t), U(t)) = (F(\lambda, U(t)) - \tau\beta U(t), U(t)) \leqslant \\ &\leqslant (B_\lambda U(t) + G(\lambda, U(t)), U(t)) \leqslant C_1 |U(t)|^2 \\ &\text{for all } |\lambda| \leqslant \Lambda,\ \tau \geqslant 0 \text{ and a.a. } t \text{ such that } |U(t)| \leqslant \varrho \end{aligned}$$

with some $C_1 > 0$. The Gronwall lemma implies

$$|U(t)|^2 \leqslant |V|^2 \mathrm{e}^{rt} \quad \text{for all } t \text{ such that } |U(t)| \leqslant \varrho \text{ (with } r = 2C_1).$$

If $\varrho_0^2 = \varrho^2 \mathrm{e}^{-r(t_M+1)}$ then

$$|U(t)|^2 \leqslant \varrho^2 \quad \text{for } |V| \leqslant \varrho_0,\ t \in [0, t_M + 1)$$

and the first estimate in (2.1) follows. In particular, Remark 2.1 ensures the existence of our solution on $[0, t_M + 1)$.

Further, it follows from (PE) that there exists $C_2 > 0$ such that

$$\begin{aligned} (\dot{U}(t), \dot{U}(t)) &\leqslant |(B_\lambda U(t) + G(\lambda, U(t)) - \tau(\beta U(t), \dot{U}(t))| \\ &\leqslant C_2 |U(t)||\dot{U}(t)| + \tau |U(t)||\dot{U}(t)| \end{aligned}$$

and therefore the later estimate in (2.1) is a consequence of the former one. □

P r o o f of Theorem 2.1. The convergence (2.5) follows from the standard results on continuous dependence of solutions of ODE's on parameters (see e.g. [8]).

Set $U_n(t) = U^{\tau_n}_{\lambda_n}(t, V_n)$. Let $V = 0$, $\frac{V_n}{|V_n|} = W_n \to W$. Lemma 2.1 implies that for any $T > 0$ there is $n_0$ such that (2.1) with $t_M$ and $U^{\tau}_{\lambda}$ replaced by $T$ and $U_n$ holds for all $n \geqslant n_0$ (i.e. for $|V_n|$ small enough). It follows that $\frac{U_n}{|V_n|}$ is bounded in $C^1([0,T])$ and therefore there exists a subsequence convergent in $C([0,T])$. It is sufficient to show that any such subsequence converges in $C^1([0,T])$ to $U^{\tau}_{0,\lambda}(\cdot, W)$. We will suppose without loss of generality that $\frac{U_n}{|V_n|} \to U_0$ in $C([0,T])$ and prove $U_0 = U^{\tau}_{0,\lambda}(\cdot, W)$. It follows from (G) that

$$\frac{G(\lambda_n, U_n(t))}{|V_n|} = \frac{G(\lambda_n, U_n(t))}{|U_n(t)|}\frac{|U_n(t)|}{|V_n|} \to 0 \text{ in } C([0,T]).$$

Further,

$$\frac{\dot{U}_n(t)}{|V_n|} = \frac{B_{\lambda_n} U_n(t)}{|V_n|} + \frac{G(\lambda_n, U_n(t))}{|V_n|} - \frac{\tau_n \beta U_n(t)}{|V_n|}$$

and it follows by the limiting process that $\frac{\dot{U}_n(t)}{|V_n|}$ converges in $C([0,T])$,

$$\frac{\dot{U}_n(t)}{|V_n|} \to \dot{U}_0(t) \text{ in } C([0,T]),$$

$$\dot{U}_0(t) = B_\lambda U_0(t) - \tau\beta U_0(t) \text{ for all } t \in (0,T).$$

Hence, $U_0 = U^{\tau}_{0,\lambda}(\cdot, W)$. Our assertion follows. □

P r o o f of Theorem 2.2. Set $U_n(t) = U^{\tau_n}_{\lambda_n}(t, V_n)$. The periodicity of $U_n$ together with Theorem 2.1 (and the fact that $t = 0$ is the Lebesgue point of $B_\lambda U + G(\lambda, U) - \tau\beta U$) imply

$$0 = U_n(t_n) - V_n = \int_0^{t_n} \dot{U}_n(t)\,\mathrm{d}t = \int_0^{t_n} F(\lambda_n, U_n(t)) - \tau_n\beta U_n(t)\,\mathrm{d}t,$$

$$0 = \frac{1}{t_n}\int_0^{t_n} B_{\lambda_n} U_n(t) + G(\lambda_n, U_n(t)) - \tau_n\beta U_n(t)\,\mathrm{d}t \to B_\lambda V + G(\lambda, V) - \tau\beta V.$$

If $V = 0$ then (G) implies

$$0 = \frac{1}{t_n}\int_0^{t_n} \frac{B_{\lambda_n} U_n(t)}{|V_n|} + \frac{G(\lambda_n, U_n(t))}{|V_n|} - \tau_n \frac{\beta U_n(t)}{|V_n|}\,\mathrm{d}t \to B_\lambda W - \tau\beta W.$$

Our assertion is proved. □

Proof of Theorem 2.3. Set $U_n(t) = U_{\lambda_n}^{\tau_n}(t, V_n)$ again. Conditions (P), (M) from Remark 1.3 and the equation (PE) give

$$\int_0^{t_M} (\dot{U}_n - F(\lambda_n, U_n), V - U_n)\,\mathrm{d}t = \int_0^{t_M} (\tau_n \beta V - \tau_n \beta U_n, V - U_n)\,\mathrm{d}t \geqslant 0$$

$$\text{for all } V \in L^2(0, t_M) \text{ such that } V(t) \in K \text{ for a.a. } t \in [0, t_M].$$

It follows from (2.2), (2.4) in Remark 2.3 that $\{U_n\}$ is bounded in $W_2^1(0, t_M)$. Suppose that $U_n \to U$ weakly in $W_2^1(0, t_M)$. Then $U_n \to U$ in $C([0, t_M])$ according to the compactness of the imbedding, and the limiting process in the last inequality gives

$$\text{(4.1)} \qquad \begin{cases} \int_0^{t_M} (\dot{U} - F(\lambda, U), V - U)\,\mathrm{d}t \geqslant 0 \\ \text{for all } V \in L^2(0, t_M) \text{ such that } V(t) \in K \text{ for a.a. } t \in [0, t_M]. \end{cases}$$

We claim to show that

$$\text{(4.2)} \qquad \begin{cases} U(t) \in K \text{ for all } t \in [0, t_M], \\ (\dot{U}(t) - F(\lambda, U(t)), V - U(t)) \geqslant 0 \text{ for all } V \in K, \text{ a.a. } t \in [0, t_M]. \end{cases}$$

We have

$$\int_0^{t_M} (\dot{U}_n(t), U_n(t))\,\mathrm{d}t = \frac{1}{2}\int_0^{t_M} \frac{\mathrm{d}}{\mathrm{d}t}|U_n(t)|^2\,\mathrm{d}t = \frac{1}{2}(|U_n(t_M)|^2 - |U_n(0)|^2)$$

and it follows from (PE) (multiplied by $U_n$ and integrated) and the boundedness of $U_n$ that there exists $C_5 > 0$ such that

$$\tau_n \int_0^{t_M} (\beta U_n(t), U_n(t))\,\mathrm{d}t \leqslant C_5.$$

We have $\tau_n \to +\infty$ and it follows by using (M) from Remark 1.3 that

$$(\beta U(t), U(t)) = \lim(\beta U_n(t), U_n(t)) = 0 \quad \text{for } t \in [0, t_M].$$

Hence, (P) implies

$$\text{(4.3)} \qquad U(t) \in K \text{ for } t \in [0, t_M].$$

Suppose that the inequality in (4.2) does not hold. Let $E \subset [0, t_M]$ and $V_0 \in K$ be such that $\operatorname{meas}(E) > 0$ and $(\dot{U}(t) - F(\lambda, U(t)), V_0 - U(t)) < 0$ for all $t \in E$. Introduce a function

$$V(t) = V_0 \quad \text{for } t \in E,$$
$$V(t) = U(t) \quad \text{for } t \notin E.$$

It follows from (4.3) that $V(t) \in K$ for a.a. $t \in [0, t_M]$ and clearly $V \in L^2(0, t_M)$. Hence

$$\int_0^{t_M} (\dot{U} - F(\lambda, U), V - U)\,\mathrm{d}t = \int_E (\dot{U} - F(\lambda, U), V_0 - U)\,\mathrm{d}t < 0,$$

which contradicts (4.1) and (4.2) is proved.

All these considerations could be done for an arbitrary weakly convergent subsequence of $U_n$ (instead of $U_n$) which implies $U_n \to U = U_\lambda^\infty(\cdot, V)$ in $C([0, t_M])$ and weakly in $W_2^1([0, t_M])$. Further, it follows that $t_0$ is the period of $U_\lambda^\infty(\cdot, V)$ provided $t_0 > 0$.

Now let us show that if $t_n \to t_0 = 0$, then $U_\lambda^\infty(\cdot, V)$ is a stationary solution of (I), i.e.

$$(F(\lambda, V), Z - V) = (F(\lambda, U(0)), Z - U(0)) \leqslant 0 \text{ for all } Z \in K.$$

We have

$$0 = U_n(t_n) - V_n = \int_0^{t_n} \dot{U}_n(t)\,\mathrm{d}t = \int_0^{t_n} F(\lambda_n, U_n(t)) - \tau_n \beta U_n(t)\,\mathrm{d}t.$$

Multiply this equation by a fixed $Z \in K$. We obtain

$$\int_0^{t_n} (F(\lambda_n, U_n(t)), Z)\,\mathrm{d}t = \tau_n \int_0^{t_n} (\beta U_n(t), Z)\,\mathrm{d}t. \tag{4.4}$$

Further,

$$\begin{aligned} 0 = |U_n(t_n)|^2 - |V_n|^2 &= 2\int_0^{t_n} (\dot{U}_n(t), U_n(t))\,\mathrm{d}t \\ &= 2\int_0^{t_n} (F(\lambda_n, U_n(t)) - \tau_n \beta U_n(t), U_n(t))\,\mathrm{d}t \end{aligned}$$

and therefore

$$\int_0^{t_n} (F(\lambda_n, U_n(t)), U_n(t))\,\mathrm{d}t = \tau_n \int_0^{t_n} (\beta U_n(t), U_n(t))\,\mathrm{d}t. \tag{4.5}$$

It follows from (4.4), (4.5) by using (P) and (M) that for any $Z \in K$ we have

$$\frac{1}{t_n} \int_0^{t_n} (F(\lambda_n, U_n(t)), Z - U_n(t))\,\mathrm{d}t = \frac{\tau_n}{t_n} \int_0^{t_n} (\beta U_n(t) - \beta Z, Z - U_n(t))\,\mathrm{d}t \leqslant 0.$$

The limiting process (by using the fact that $t = 0$ is a Lebesgue point) gives

$$\frac{1}{t_n} \int_0^{t_n} (F(\lambda_n, U_n(t)), Z - U_n(t))\,\mathrm{d}t \to (F(\lambda, V), Z - V) \leqslant 0.$$

The case $V = 0$ can be treated similarly if we divide all expressions by $|V_n|$ and use the fact that

$$\frac{G(\lambda_n, U_n(\cdot, V_n))}{|V_n|} \to 0 \text{ in } C([0, t_M]).$$

□

P r o o f of Theorem 2.4. Set $\varphi_n(t, V_n) = \varphi^{\tau_n}_{\lambda_n}(t, V_n)$, $t_n(V) = t^{\tau_n}_{\lambda_n}(V)$. If $V \notin \boldsymbol{S}_\lambda$, $t^\tau_\lambda(V) < t_M$, $\dot{\varphi}^\tau_\lambda(t^\tau_\lambda(V), V) < 0$, then there is $\delta > 0$ such that

$$\varphi^\tau_\lambda(t, V) < -2\pi \text{ for all } t \in (t^\tau_\lambda(V), t^\tau_\lambda(V) + \delta). \tag{4.6}$$

Let $t_0 \in (t^\tau_\lambda(V), t^\tau_\lambda(V) + \delta)$ be fixed. Then $\varphi_n(t_0, V_n) < -2\pi$ for $n$ large by Consequence 2.1. Hence, there are $t'_n \in (0, t_0)$ such that $\varphi_n(t'_n, V_n) = -2\pi$ due to the continuity of $\varphi_n(\cdot, V_n)$. This means $t_n(V_n) < t_0$. But $t_0 > t^\tau_\lambda(V)$ was arbitrarily close to $t^\tau_\lambda(V)$ and therefore

$$\limsup t_n(V_n) \leqslant t^\tau_\lambda(V). \tag{4.7}$$

Let $t_{l_n}$ be an arbitrary subsequence of $t_n(V_n)$, $t_{l_n} \to t'$. We have $t_{l_n} < t_M + 1$ for $n$ large enough by (4.7) and Consequence 2.1 yields $\varphi_{l_n}(t_{l_n}, V_{l_n}) \to \varphi^\tau_\lambda(t', V)$, i.e. $\varphi(t', V) = -2\pi$. It follows that $t' > 0$, $t^\tau_\lambda(\cdot, V) \leqslant t'$. This holds for an arbitrary converging subsequence and therefore lim inf $t_n(V_n) \geqslant t^\tau_\lambda(V)$ which together with (4.7) gives $t_n(V_n) \to t^\tau_\lambda(V)$.

The case $V = 0, W_n = \frac{V_n}{|V_n|} \to W \notin \boldsymbol{S}_\lambda$ can be treated similarly using (2.8) instead of (2.7) from Consequence 2.1. □

**Remark 4.1.** Denote by $T_K(U)$ the contingent cone to $K$ at a point $U \in K$, i.e.

$$T_K(U) = \overline{\bigcup_{h>0} \bigcup_{V \in K} h(V - U)}.$$

For any $U \in K$, $W \in \mathbb{R}^N$, denote by $P_U W$ the projection of $W$ onto $T_K(U)$, i.e. the unique element from $T_K(U)$ satisfying

$$|P_U W - W| = \min_{Z \in T_K(U)} |Z - W|.$$

It is known (see [1]) that an absolutely continuous function $U\colon [0, T) \to K$ is a solution of (I) if and only if

$$\dot{U}(t) = P_{U(t)}\big(B_\lambda U(t) + G(\lambda, U(t))\big) \quad \text{for a.a. } t \in [0, T).$$

In this case the last equation holds for all $t \in [0, T)$ if $\dot{U}(t)$ is understood as the right derivative in accordance with our agreement from Notation 1.2 (cf. Remark 1.1).

Proof of Theorem 2.5. Set

$$\langle V, Z\rangle_\lambda = \sum_{j=1}^{N} y_j^V(\lambda) y_j^Z(\lambda), \quad \|V\|_\lambda = \langle V, V\rangle_\lambda^{\frac{1}{2}}$$

$$\text{for } V = \sum_{j=1}^{N} y_j^V(\lambda) U_j(\lambda), \quad Z = \sum_{j=1}^{N} y_j^Z(\lambda) U_j(\lambda).$$

We will write $U_n(t)$, $\varphi_n(t)$, $\boldsymbol{L}_n$, $U(t)$, $\varphi(t)$ instead of $U_{\lambda_n}^{\tau_n}(t, V_n)$, $\varphi_{\lambda_n}^{\tau_n}(t, V_n)$, $\boldsymbol{L}_{\lambda_n}$, $U_{0,\lambda}^{\infty}(t, W)$, $\varphi_{0,\lambda}^{\infty}(t, W)$, respectively.

If $W \in \operatorname{int} K$ then $\frac{\dot U_n(0)}{|V_n|} \to \dot U(0)$, $\dot\varphi_n(0) \to \dot\varphi(0)$ by Theorem 2.1, Consequence 2.1 and Remark 1.6. If $V_n \in K$ then $\dot\varphi_n(0) \leqslant -\eta$ for $n$ large enough with some $\eta > 0$ by Remarks 1.6 and 1.4. Hence, it remains to consider the case

$$W \in \partial K, \ V_n \notin K.$$

The assumption $W \notin \boldsymbol{S}_\lambda$ and Theorem 2.1 imply the existence of $T > 0$ such that $\|P_{\boldsymbol{L}_n} U_n(t)\|_{\lambda_n} > 0$, $\|P_{\boldsymbol{L}_\lambda} U(t)\|_\lambda > 0$ on $[0, T)$. It follows from Remarks 3.5, 4.1 that

$$(4.8) \quad \dot\varphi_n(t) = \frac{\left\langle \dot U_n(t), P_{\boldsymbol{L}_n}^* U_n(t)\right\rangle_{\lambda_n}}{\|P_{\boldsymbol{L}_n} U_n(t)\|_{\lambda_n}^2} = \frac{\langle F(\lambda_n, U_n(t)) - \tau_n \beta U_n(t), P_{\boldsymbol{L}_n}^* U_n(t)\rangle_{\lambda_n}}{\|P_{\boldsymbol{L}_n} U_n(t)\|_{\lambda_n}^2},$$

$$(4.9) \quad \dot\varphi(t) = \frac{\left\langle \dot U(t), P_{\boldsymbol{L}_\lambda}^* U(t)\right\rangle_\lambda}{\|P_{\boldsymbol{L}_\lambda} U(t)\|_\lambda^2} = \frac{\langle B_\lambda U(t) - (B_\lambda U(t) - P_{U(t)} B_\lambda U(t)), P_{\boldsymbol{L}_\lambda}^* U(t)\rangle_\lambda}{\|P_{\boldsymbol{L}_\lambda} U(t)\|_\lambda^2}$$

for $t \in [0, T)$. It follows from the assumption (1.1) that

$$(4.10) \qquad \lim_{U \to W, U \notin K} \frac{\beta U}{|\beta U|} = n(W) = \frac{B_\lambda W - P_W B_\lambda W}{|B_\lambda W - P_W B_\lambda W|}.$$

The assumption (G) implies

$$\lim_{n\to\infty} \frac{\langle F(\lambda_n, V_n), P_{\boldsymbol{L}_n}^* V_n\rangle_{\lambda_n}}{\|P_{\boldsymbol{L}_n} V_n\|_{\lambda_n}^2} = \frac{\langle B_\lambda W, P_{\boldsymbol{L}_\lambda}^* W\rangle_\lambda}{\|P_{\boldsymbol{L}_\lambda} W\|_\lambda^2}.$$

It follows from (4.8) that if $\langle \beta V_n, P_{\boldsymbol{L}_n}^* V_n\rangle_{\lambda_n} > 0$ then the circulation of $P_{L_n} U_{\lambda_n}^{\tau_n}(t, V)$ is accelerated by the penalty term and this circulation is stronger than in the case of the equation. In particular, there is $\eta > 0$ such that $\dot\varphi_n(0) \leqslant -\eta$ (see Remark 1.4). Hence, we can consider only the case $\langle \beta V_n, P_{L_n}^* V_n\rangle_\lambda \leqslant 0$. It follows that it is sufficient to show that

$$\limsup_{n\to\infty} \frac{-\tau_n \langle \beta V_n, P_{\boldsymbol{L}_n}^* V_n\rangle_{\lambda_n}}{\|P_{\boldsymbol{L}_n} V_n\|_{\lambda_n}^2} \leqslant -\frac{\langle B_\lambda W - P_W B_\lambda W, P_{\boldsymbol{L}_\lambda}^* W\rangle_\lambda}{\|P_{\boldsymbol{L}_\lambda} W\|_\lambda^2}.$$

This can be written as

$$\limsup_{n\to\infty} \frac{-\tau_n\|\beta V_n\|_{\lambda_n}}{\|V_n\|_{\lambda_n}} \frac{\|V_n\|_{\lambda_n}}{\|P_{L_n} V_n\|_{\lambda_n}} \left\langle \frac{\beta V_n}{\|\beta V_n\|_{\lambda_n}}, \frac{P^*_{L_n} V_n\|_{\lambda_n}}{\|P_{L_n} V_n\|_{\lambda_n}} \right\rangle_{\lambda_n}$$
$$\leqslant -\frac{\|B_\lambda W - P_W B_\lambda W\|_\lambda}{\|P_{L_\lambda} W\|_\lambda} \left\langle \frac{B_\lambda W - P_W B_\lambda W}{\|B_\lambda W - P_W B_\lambda W\|_\lambda}, \frac{P^*_{L_\lambda} W}{\|P^*_{L_\lambda} W\|_\lambda} \right\rangle_\lambda .$$

We have $\frac{\|P_{L_n} V_n\|_{\lambda_n}}{\|V_n\|_{\lambda_n}} \to \|P_{L_\lambda} W\|_\lambda$, $\frac{P^*_{L_n} V_n}{\|P_{L_n} V_n\|_{\lambda_n}} \to \frac{P^*_{L_\lambda} W}{\|P^*_{L_\lambda} W\|_\lambda}$ and it follows from (4.10) that the last inequality will be ensured if

$$\limsup_{n\to\infty} \frac{\tau_n\|\beta V_n\|_{\lambda_n}}{\|V_n\|_{\lambda_n}} \leqslant \|B_\lambda W - P_W B_\lambda W\|_\lambda.$$

According (4.10), this is equivalent to

$$\limsup_{n\to\infty} \frac{\tau_n|\beta V_n|}{|V_n|} \leqslant |B_\lambda W - P_W B_\lambda W|.$$

It is easy to see that

$$|B_\lambda W - P_W B_\lambda W| = (B_\lambda W, n(W)) \tag{4.11}$$

under the assumption (1.1). (Note that $(B_\lambda W, n(W)) \geqslant 0$ because otherwise the periodic solution $U(t)$ would be directed into the interior of $K$ in a neighbourhood of $W$ and $U(t)$ could not reach $W = U(0) \in \partial K$ for $t \to 0_-$.) Hence, it is sufficient to prove that

$$\limsup_{n\to\infty} \frac{\tau_n|\beta V_n|}{|V_n|} \leqslant (B_\lambda W, n(W)). \tag{4.12}$$

Suppose that (4.12) does not hold. We can suppose that there is $\delta > 0$ such that

$$(B_\lambda W, n(W)) - \frac{\tau_n|\beta V_n|}{|V_n|} \leqslant -\delta. \tag{4.13}$$

Theorem 2.3 together with (G) implies

$$\frac{U_n(t)}{|V_n|} \to U(t), \ \frac{F(\lambda_n, U_n(t))}{|V_n|} \to B_\lambda U(t) \ \text{ in } C([-t_M, t_M]). \tag{4.14}$$

(Notice that our functions are periodic and therefore we can consider an arbitrary time interval.) Hence, it follows from (4.13), (4.10), (4.14) and the continuity of $U$ that there exist $n_0$ and $t_0 > 0$ such that

$$\begin{cases} \left(\frac{F(\lambda_n, U_n(t))}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) - \frac{\tau_n|\beta V_n|}{|V_n|} \leqslant -\frac{\delta}{2} \\ \text{for all } n \geqslant n_0 \text{ and } t \in [-t_0, t_0] \text{ such that } U_n(t) \notin K. \end{cases} \tag{4.15}$$

We shall show that also

$$\left(\frac{F(\lambda_n, U_n(t)) - \tau_n \beta U_n(t)}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) \leqslant -\frac{\delta}{2} \tag{4.16}$$

holds for all $n \geqslant n_0$ and $t \in [-t_0, 0]$. For any $n \geqslant n_0$, consider the set

$$\mathcal{U}_n = \{\tilde{t} \in [-t_0, 0]\,;\, U_n(t) \notin K \text{ and (4.16) holds for all } t \in [\tilde{t}, 0]\}.$$

We have $0 \in \mathcal{U}_n$ for all $n \geqslant n_0$ because (4.16) reduces to (4.15) for $t = 0$. According to (PE) and (Pt), the formula (4.16) is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\beta U_n(t)|^2\right) \left(= (\dot{U}_n(t), \beta U_n(t))\ \right) \leqslant -\frac{\delta}{2}|\beta U_n(t)||V_n|. \tag{4.17}$$

In particular, if $\tilde{t} \in \mathcal{U}_n$ then $|\beta U_n(t)| = \operatorname{dist}(U_n(t), K)$ is strictly decreasing on $[\tilde{t}, 0]$ and $|\beta V_n| < |\beta U_n(t)|$ for all $t \in [\tilde{t} - \eta_n, 0)$ with some $\eta_n > 0$. Thus,

$$U_n(t) \notin K \text{ for all } t \in [\tilde{t} - \eta_n, 0]. \tag{4.18}$$

We obtain by (4.15) that

$$\begin{aligned}\left(\frac{F(\lambda_n, U_n(t)) - \tau_n \beta U_n(t)}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) &= \left(\frac{F(\lambda_n, U_n(t))}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) - \frac{\tau_n |\beta U_n(t)|}{|V_n|} \\ &\leqslant \left(\frac{F(\lambda_n, U_n(t))}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) - \frac{\tau_n |\beta V_n|}{|V_n|} \leqslant -\frac{\delta}{2} \text{ for all } n \geqslant n_0 \text{ and } t \in [\tilde{t} - \eta_n, 0].\end{aligned}$$

Hence $\mathcal{U}_n$ is open in $[-t_0, 0]$. Simultaneously, $\mathcal{U}_n$ is closed according to the continuity argument. (Note that $|\beta U_n(t)|$ remains nontrivial due to the monotonicity mentioned above.) Hence,

$$\mathcal{U}_n = [-t_0, 0] \text{ for all } n \geqslant n_0. \tag{4.19}$$

We have $U(t) \in K$ for all $t$ and it follows from (4.18) and (4.14) that

$$U(t) \in \partial K \text{ for all } t \in [-t_0, 0]. \tag{4.20}$$

Now, it is easy to see that

$$\frac{\beta U_n(t)}{|\beta U_n(t)|} \to n(U(t)) \text{ in } C([-t_0, 0]). \tag{4.21}$$

Further, (4.16) means by (PE)

$$\left(\frac{\dot{U}_n(t)}{|V_n|}, \frac{\beta U_n(t)}{|\beta U_n(t)|}\right) \leqslant -\frac{\delta}{2}. \tag{4.22}$$

Theorem 2.3 gives $\frac{U_n(t)}{|V_n|} \to U(t)$ weakly in $W_2^1(0, t_M)$. Integrating (4.22) and using (4.21), we obtain by the limiting process

$$\int_{-t_0}^{0} \left(\dot{U}(t), n(U(t))\right) \mathrm{d}t < 0.$$

However, simultaneously $\left(\dot{U}(t), n(U(t))\right) \geqslant 0$ for all $t \in [-t_0, 0]$ with respect to (4.20) because otherwise $U(t)$ would be directed to the interior of $K$ at some $t' \in [-t_0, 0]$, i.e. $U(t) \in \operatorname{int} K$ for $t \in (t', t' + \xi)$ with some $\xi > 0$. This contradiction completes the proof. $\square$

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