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Mária Tóthová; Oleg Palumbíny

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ON MONOTONE SOLUTIONS OF THE FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

MÁRIA TÓTHOVÁ and OLEG PALUMBÍNY, Trnava

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1. INTRODUCTION

The purpose of the paper is to study the existence of monotone solutions of the linear differential equation of the fourth order with quasi-derivatives

$$L(y) \equiv L_4 y + P(t) L_2 y + Q(t) y = 0, \tag{L}$$

where

$$\begin{aligned}
L_1 y(t) &= p_1(t) y'(t) = p_1(t)\,\mathrm{d}y(t)/\,\mathrm{d}t,\\
L_2 y(t) &= p_2(t)(p_1(t)y'(t))' = p_2(t)(L_1 y(t))',\\
L_3 y(t) &= p_3(t)(p_2(t)(p_1(t)y'(t))')' = p_3(t)(L_2 y(t))',\\
L_4 y(t) &= (p_3(t)(p_2(t)(p_1(t)y'(t))')')' = (L_3 y(t))',
\end{aligned}$$

$P(t)$, $Q(t)$, $p_i(t)$, $i = 1,2,3$, are real-valued continuous functions on an interval $I = [a, \infty)$, $-\infty < a < \infty$. It is assumed throughout that

(A) $P(t) \leqslant 0$, $Q(t) \leqslant 0$, $p_i(t) > 0$, $i = 1,2,3$, for all $t \in I$ and $Q(t)$ not identically zero in any subinterval of I.

Similar problems for the third order ordinary differential equations with quasi-derivatives were studied in several papers ([2], [3], [5], [6]). The equation (L), where $p_i(t) \equiv 1$, $i = 1,2,3$, was studied for example in ([1], [9], [10]). The equation of the fourth order with quasi-derivatives was also studied, for instance, in ([7], [8]). Therefore some results achieved in the papers mentioned above are special cases of ours.

Theorem 1 and Theorem 2 give sufficient conditions for the existence of monotone solutions of (L) and their quasi-derivatives as well. Theorem 3 deals with the uniqueness of such solutions (with the exception of constant multiples).

A nontrivial solution of a differential equation of the n-th order is called oscillatory if its set of zeros is not bounded from above. Otherwise, it is called nonoscillatory. A differential equation of the n-th order will be called nonoscillatory, when all its solutions are nonoscillatory; oscillatory, when at least one of its solutions (except the trivial one) is oscillatory. Let $C(I)$ denote the set of all real-valued functions which are continuous on I.

2. Preliminary results

We start by a generalization of Švec's result from [4].

Lemma 1. *Let $p(t) > 0$, $p(t)$, $q(t)$, $f(t)$ be functions of class $C([t_0, \infty))$, let the differential equation*

$$(p(t)w'(t))' + q(t)w(t) = 0 \tag{1}$$

be nonoscillatory. If $f(t)$ does not change the sign in $[t_0, \infty)$, then also the differential equation

$$(p(t)z'(t))' + q(t)z(t) = f(t) \tag{2}$$

is nonoscillatory in $[t_0, \infty)$.

Proof. If $y(t)$ and $z(t)$ are solutions of (1) and (2), respectively, then the function

$$W(z, y) = \begin{vmatrix} y(t) & z(t) \\ p(t)y'(t) & p(t)z'(t) \end{vmatrix}$$

fulfils the equation

$$W(z, y) = c + \int_{t_0}^{t} f(x)y(x)\,\mathrm{d}x,$$

where c is a constant. Let equation (1) be nonoscillatory. Then its solution $y(t)$ is a nonoscillatory function. Let $y(t) > 0$ eventually. Then the function $\int_{t_0}^{t} f(x)y(x)\,\mathrm{d}x$ as well as the function $W(z, y)$ do not change the sign for all $t > t_1 \geqslant t_0$. This fact implies the existence of such t_1 that W is a nonoscillatory function on (t_1, ∞). Now, the function

$$\left(\frac{z(t)}{y(t)}\right)' = \frac{1}{p(t)}\frac{W(z, y)}{y^2(t)}$$

as well as the function $W(z,y)$ have the same sign for all $t > t_1$. This fact implies that $z(t)/y(t)$ is either an increasing function or a decreasing one, i.e. there exists $t_2 \geqslant t_1$ such that either

a) the function z/y is still negative on $[t_2, \infty)$ or

b) the function z/y is still positive on $[t_2, \infty)$.

In both cases it is obvious that $z(t)$ is nonoscillatory, i.e. equation (2) is nonoscillatory. □

Lemma 2, [1]. *Let $A(t,s)$ be a nonnegative and continuous function for $t_0 \leqslant s \leqslant t$ (nonpositive for $a \leqslant t \leqslant s \leqslant t_0$). If $g(t)$, $\varphi(t)$ ($\psi(t)$) are continuous functions in the interval $[t_0, \infty)$ ($[a, t_0)$) and*

$$\varphi(t) \leqslant g(t) + \int_{t_0}^{t} A(t,s)\varphi(s)\,\mathrm{d}s \quad \text{for } t \in [t_0, \infty)$$

$$(\psi(t) \geqslant g(t) + \int_{t_0}^{t} A(t,s)\psi(s)\,\mathrm{d}s \text{ for } t \in [a, t_0]),$$

then every solution $y(t)$ of the integral equation

$$y(t) = g(t) + \int_{t_0}^{t} A(t,s)y(s)\,\mathrm{d}s \tag{3}$$

satisfies the inequality

$$y(t) \geqslant \varphi(t) \quad \text{in } [t_0, \infty)$$

$$(y(t) \leqslant \psi(t) \quad \text{in } [a, t_0]).$$

Proof. See [1]. □

Lemma 3. *Let (A) and $\int^{\infty} (1/p_1(t))\,\mathrm{d}t = \infty$ hold. Then for every nonoscillatory solution $y(t)$ of (L) there exists a number $t_0 \geqslant a$ such that either*

$$(y(t)L_1y(t) > 0,\ y(t)L_2y(t) > 0) \quad \text{or} \quad (y(t)L_1y(t) < 0,\ y(t)L_2y(t) > 0)$$

or

$$(y(t)L_1y(t) > 0,\ y(t)L_2y(t) < 0) \quad \text{for all } t \geqslant t_0.$$

Proof. Let $y(t)$ be a nonoscillatory solution of (L). Then there exists a number $t_1 \geqslant a$ such that $y(t) \neq 0$ in $[t_1, \infty)$. Without loss of generality we can assume that

$y(t) > 0$ on $[t_1, \infty)$. The substitution $z(t) = L_2y(t)$ into (L) leads to the differential equation

$$(5) \qquad (p_3(t)z'(t))' + P(t)z(t) = -Q(t)y(t).$$

Since $P(t) \leqslant 0$, the equation $(p_3z')' + Pz = 0$ is nonoscillatory on $[t_1, \infty)$. Then the fact that $Q(t)y(t)$ does not change the sign in $[t_1, \infty)$ implies that equation (5) is nonoscillatory by Lemma 1.

Hence, there exists a number $t_2 \geqslant t_1$ such that $z(t) \neq 0$, i.e. $L_2y(t) \neq 0$. This fact implies the existence of a number $t_0 \geqslant t_2$ such that $L_1y(t) \neq 0$ for all $t \geqslant t_0$. The following four cases may occur for $t \geqslant t_0$:

$$\begin{aligned}
&\text{a) } y(t)L_1y(t) > 0,\ y(t)L_2y(t) > 0,\\
&\text{b) } y(t)L_1y(t) < 0,\ y(t)L_2y(t) > 0,\\
&\text{c) } y(t)L_1y(t) > 0,\ y(t)L_2y(t) < 0,\\
&\text{d) } y(t)L_1y(t) < 0,\ y(t)L_2y(t) < 0.
\end{aligned}$$

We prove that the case d) is impossible. Without loss of generality we can assume that $y(t) > 0$, $L_1y(t) < 0$, $L_2y(t) < 0$. It follows that $L_1y(t) = p_1(t)y'(t)$ is a negative and decreasing function and hence there exists a constant $k \neq 0$ such that $p_1(t)y'(t) \leqslant -k^2$ for $t \geqslant t_0$. This implies that $y(t) \leqslant y(t_0) - \int_{t_0}^{t} (k^2/p_1(\tau))\,\mathrm{d}\tau$. According to the assumptions of the lemma we have $y(t) \to -\infty$, $t \to \infty$, which contradicts the fact that $y(t) > 0$. This completes the proof of the lemma. □

Lemma 4. *Suppose that* (A) *holds and let* $y(t)$ *be a nontrivial solution of* (L) *satisfying the initial conditions*

$$\begin{gathered}
y(t_0) = y_0 \geqslant 0,\ L_1y(t_0) = y_0' \geqslant 0,\\
L_2y(t_0) = y_0'' \geqslant 0,\ L_3y(t_0) = y_0''' \geqslant 0
\end{gathered}$$

($t \in I$ *arbitrary and* $y_0 + y_0' + y_0'' + y_0''' \neq 0$). *Then*

$$y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) > 0,\ L_3y(t) > 0 \text{ for all } t > t_0.$$

Proof. The initial-value problem $L_4y + P(t)L_2y + Q(t)y = 0$, $y(t_0) = y_0$, $L_1y(t_0) = y_0'$, $L_2y(t_0) = y_0''$, $L_3y(t_0) = y_0'''$ is equivalent to the following Volterra integral equation:

$$(6) \qquad L_3y(t) = g(t) + \int_{t_0}^{t} A(t,\tau)L_3y(\tau)\,\mathrm{d}\tau,$$

where

$$g(t) = y_0''' - y_0'' \int_{t_0}^{t} P(s)\,\mathrm{d}s - y_0'' \int_{t_0}^{t} Q(s)G(t_0,s)\,\mathrm{d}s - \int_{t_0}^{t} Q(s)(y_0' h(t_0,s) + y_0)\,\mathrm{d}s,$$

$$A(t,\tau) = \int_{\tau}^{t} ((-P(s) - Q(s)G(\tau,s))/p_3(\tau))\,\mathrm{d}s,$$

$$G(\tau,s) = \int_{\tau}^{s} (h(\xi,s)/p_2(\xi))\,\mathrm{d}\xi,$$

$$h(\xi,s) = \int_{\xi}^{s} (1/p_1(t))\,\mathrm{d}t.$$

It follows from (L) that $L_4 y = -P(t)L_2 y - Q(t)y$. Integrating the last equation we get

(7)

$$L_3 y(t) = y_0''' - y_0'' \int_{t_0}^{t} P(s)\,\mathrm{d}s - \int_{t_0}^{t} P(s)\left[\int_{t_0}^{s} (L_3 y(\tau)/p_3(\tau))\,\mathrm{d}\tau\right]\mathrm{d}s - \int_{t_0}^{t} Q(s)y(s)\,\mathrm{d}s.$$

If we express $y(s)$ by $L_1 y$ and $L_2 y$ we get

$$y(s) = \int_{t_0}^{s} \left[\left[\int_{t_0}^{\tau} (L_2 y(\xi)/p_2(\xi))\,\mathrm{d}\xi\right]/p_1(\tau)\right]\,\mathrm{d}\tau + y_0' \int_{t_0}^{s} (1/p_1(\tau))\,\mathrm{d}\tau + y_0.$$

Exchanging the limits of integration and denoting

$$h(t_0,s) = \int_{t_0}^{s} (1/p_1(\tau))\,\mathrm{d}\tau$$

we get

$$y(s) = \int_{t_0}^{s} (L_2 y(\xi) h(\xi,s)/p_2(\xi))\,\mathrm{d}\xi + y_0' h(t_0,s) + y_0.$$

If we express $L_2 y$ by $L_3 y$, we obtain

$$y(s) = \int_{t_0}^{s} \left[\int_{t_0}^{\xi} (L_3 y(\tau)/p_3(\tau))\,\mathrm{d}\tau\right] h(\xi,s)/p_2(\xi)\,\mathrm{d}\xi$$
$$+ y_0'' \int_{t_0}^{s} (h(\xi,s)/p_2(\xi))\,\mathrm{d}\xi + y_0' h(t_0,s) + y_0.$$

Exchanging the limits of integration and denoting

$$G(t_0,s) = \int_{t_0}^{s} (h(\xi,s)/p_2(\xi))\,\mathrm{d}\xi$$

we get

$$y(s) = \int_{t_0}^{s} (G(\tau, s) L_3 y(\tau)/p_3(\tau))\, \mathrm{d}\tau + y_0'' G(t_0, s) + y_0' h(t_0, s) + y_0.$$

We substitute this expression for $y(s)$ into (7) obtaining

$$L_3 y(t) = y_0''' - y_0'' \int_{t_0}^{t} P(s)\, \mathrm{d}s - \int_{t_0}^{t} P(s) \Big[\int_{t_0}^{s} (L_3 y(\tau)/p_3(\tau))\, \mathrm{d}\tau \Big]\, \mathrm{d}s$$
$$- \int_{t_0}^{t} Q(s) \Big[\int_{t_0}^{s} (G(\tau, s) L_3 y(\tau)/p_3(\tau))\, \mathrm{d}\tau + y_0'' G(t_0, s) + y_0' h(t_0, s) + y_0 \Big]\, \mathrm{d}s.$$

After little arrangements we get

$$L_3 y(t) = y_0''' - y_0'' \int_{t_0}^{t} P(s)\, \mathrm{d}s - y_0'' \int_{t_0}^{t} Q(s) G(t_0, s)\, \mathrm{d}s - \int_{t_0}^{t} Q(s)(y_0' h(t_0, s) + y_0)\, \mathrm{d}s$$
$$+ \int_{t_0}^{t} \Big[- \int_{t_0}^{s} ((P(s) + Q(s) G(\tau, s)) L_3 y(\tau)/p_3(\tau))\, \mathrm{d}\tau \Big]\, \mathrm{d}s.$$

Exchanging the limits of integration and rearranging the equation we obtain the Volterra integral equation (6). The hypotheses of the lemma imply that $A(t, \tau) \geqslant 0$ and $g(t) > 0$ for all $t \in (t_0, \infty)$. According to Lemma 2 we get $L_3 y(t) \geqslant \varphi(t) = g(t) > 0$ for all $t \in (t_0, \infty)$. Integrating this inequality over $[t_0, \infty)$ we obtain (owing to the initial conditions) the assertion of Lemma 4. □

Lemma 5. *Suppose that* (A) *holds and let* $y(t)$ *be a nontrivial solution of* (L) *satisfying the initial conditions*

$$y(t_0) = y_0 \geqslant 0,\ L_1 y(t_0) = y_0' \leqslant 0,\ L_2 y(t_0) = y_0'' \geqslant 0,\ L_3 y(t_0) = y_0''' \leqslant 0,$$

($t_0 \in I$ *arbitrary*, $y_0^2 + y_0'^2 + y_0''^2 + y_0'''^2 > 0$). *Then*

$$y(t) > 0,\ L_1 y(t) < 0,\ L_2 y(t) > 0,\ L_3 y(t) < 0 \text{ for all } t \in [a, t_0).$$

Proof. The initial-value problem is equivalent to the Volterra integral equation (6), where

$$g(t) = y_0''' + y_0'' \int_{t_0}^{t} P(s)\, \mathrm{d}s + y_0'' \int_{t}^{t_0} Q(s)[G(s, t_0) - y_0' h(s, t_0) + y_0]\, \mathrm{d}s,$$
$$G(b, a) = \int_{b}^{a} (h(b, \xi)/p_2(\xi))\, \mathrm{d}\xi,$$
$$A(t, \tau) = \int_{t}^{\tau} [(P(s) + G(s, \tau)\ Q(s))/p_3(\tau)]\, \mathrm{d}s,$$
$$h(s, \xi) = \int_{s}^{\xi} (1/p_1(\tau))\, \mathrm{d}\tau.$$

The hypotheses of the lemma imply that $g(t) < 0$, $A(t,\tau) \leqslant 0$ for $a \leqslant t \leqslant \tau \leqslant t_0$. Then by Lemma 2 we have $L_3y(t) < 0$ for all $t \in [a, t_0)$. Hence the assertion of Lemma 5 follows from the initial conditions. □

3. The existence of monotone solutions

Let z_0, z_1, z_2, z_3 be solutions of (L) on $[a,\infty)$ which fulfil the initial conditions

$$z_i(a) = \begin{cases} 1, & i = 0, \\ 0, & i = 1,2,3, \end{cases} \qquad L_1z_i(a) = \begin{cases} 1, & i = 1, \\ 0, & i = 0,2,3, \end{cases}$$

$$L_2z_i(a) = \begin{cases} 1, & i = 2, \\ 0, & i = 0,1,3, \end{cases} \qquad L_3z_i(a) = \begin{cases} 1, & i = 3, \\ 0, & i = 0,1,2. \end{cases}$$

We want to show the existence of solutions $y(t)$ and $z(t)$ such that $y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) > 0$, $L_3y(t) > 0$ for $t \in I$ and $z(t) > 0$, $L_1z(t) < 0$, $L_2z(t) > 0$, $L_3z(t) < 0$ for $t \in I$.

Theorem 1. *Suppose that* (A) *holds. Then there exists a solution $y(t)$ of* (L) *such that*

$$y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) > 0,\ L_3y(t) > 0 \text{ for all } t \in I_0 = (a,\infty).$$

Proof. The assertion of the theorem follows from Lemma 4 for $t_0 = a$. □

Theorem 2. *Suppose that* (A) *holds. Then there exists a solution $y(t)$ of* (L) *such that*

$$y(t) > 0,\ L_1y(t) < 0,\ L_2y(t) > 0,\ L_3y(t) < 0 \quad \text{for all } t \in I = [a,\infty).$$

Proof. Let $(c_{0n}, c_{1n}, c_{2n}, c_{3n})$ be a solution of the system (S_n) which consists of the relationships (8), (9), (10), (11) and (12):

$$c_{0n}z_0^{(0)}(n) + c_{1n}z_1^{(0)}(n) + c_{2n}z_2^{(0)}(n) + c_{3n}z_3^{(0)}(n) = 0, \tag{8}$$

$$c_{0n}z_0^{(1)}(n) + c_{1n}z_1^{(1)}(n) + c_{2n}z_2^{(1)}(n) + c_{3n}z_3^{(1)}(n) = 0, \tag{9}$$

$$c_{0n}z_0^{(2)}(n) + c_{1n}z_1^{(2)}(n) + c_{2n}z_2^{(2)}(n) + c_{3n}z_3^{(2)}(n) = 0, \tag{10}$$

$$c_{0n}z_0^{(3)}(n) + c_{1n}z_1^{(3)}(n) + c_{2n}z_2^{(3)}(n) + c_{3n}z_3^{(3)}(n) < 0, \tag{11}$$

$$c_{0n}^2 + c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1, \tag{12}$$

where n is an arbitrary integer, $n > \max\{0, a\}$, $z_i^{(j)}(n) = L_j z_i(n)$, $z_i(t)$ form the fundamental system of solutions of (L) such that $z_i^{(j)}(a) = 0$ for $i \neq j$, $z_i^{(j)}(a) = 1$ for $i = j$, $i, j = 0, 1, 2, 3$. We will show that (S_n) admits a solution $(c_{0n}, c_{1n}, c_{2n}, c_{3n})$ for all $n > \max\{0, a\}$. Let $W(z_0(t), z_1(t), z_2(t), z_3(t))$ denote Wronski's determinant of z_i at the point t. Then at least one of all the four subdeterminants of the system of equations (8), (9), (10) is not equal to zero. Let it be, for instance, the determinant

$$W_3 = \begin{vmatrix} z_0^{(0)}(n), & z_1^{(0)}(n), & z_2^{(0)}(n) \\ z_0^{(1)}(n), & z_1^{(1)}(n), & z_2^{(1)}(n) \\ z_0^{(2)}(n), & z_1^{(2)}(n), & z_2^{(2)}(n) \end{vmatrix}.$$

According to the Frobenius theorem, the system of equations (8), (9), (10) with the unknowns c_{0n}, c_{1n}, c_{2n} and the right hand side $(-c_{3n}z_3^{(0)}(n), -c_{3n}z_3^{(1)}(n), -c_{3n}z_3^{(2)}(n))$ admits the only solution $(c_{0n}, c_{1n}, c_{2n}) = (A_n c_{3n}, B_n c_{3n}, C_n c_{3n})$. Then (12) has the form $c_{0n}^2 + c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = (A_n^2 + B_n^2 + C_n^2 + 1)c_{3n}^2 = 1$. Therefore $|c_{3n}| = 1/(A_n^2 + B_n^2 + C_n^2 + 1)^{1/2} \neq 0$. The left hand side of (11) has the form $(A_n z_0^{(3)}(n) + B_n z_1^{(3)}(n) + C_n z_2^{(3)}(n) + z_3^{(3)}(n))c_{3n}$. The expression in the last parentheses is not equal to zero. If it were equal to zero, then the system consisting of (8), (9), (10) and (11′), where

$$c_{0n}z_0^{(3)}(n) + c_{1n}z_1^{(3)}(n) + c_{2n}z_2^{(3)}(n) + c_{3n}z_3^{(3)}(n) = 0, \tag{11'}$$

would admit a nontrivial solution, which is impossible because $W(z_0(n), z_1(n), z_2(n), z_3(n)) \neq 0$. Now it suffices to choose the sign of c_{3n} for (11) to be valid. Therefore (S_n) admits a solution for all $n > \max\{0, a\}$. Let us put $y_n(t) = \sum_{i=0}^{3} c_{in} z_i(t)$. Because of $(c_{0n}, c_{1n}, c_{2n}, c_{3n}) \neq (0, 0, 0, 0)$, $y_n(t)$ is not identically zero. According to Lemma 5, we have $(-1)^k L_k y_n(t) > 0$ on $[a, n)$ for $k = 0, 1, 2, 3$. It is obvious that c_{in}, $i = 0, 1, 2, 3$ are bounded. For this reason, there exist subsequences c_{ir_n} of c_{in} which are convergent. Let $c_{ir_n} \to c_i$ for $n \to \infty$, $i = 0, 1, 2, 3$. Let us put $y(t) = \sum_{i=0}^{3} c_i z_i(t) = \lim_{n\to\infty} y_n(t)$ for all $t \in [a, \infty)$. Let $n_0 > \max\{0, a\}$. Then $(-1)^k L_k y_n(t) > 0$ on $[a, n_0)$ for $n \geqslant n_0$ and so $(-1)^k L_k y(t) \geqslant 0$ on $[a, n_0)$ for all $n_0 > \max\{0, a\}$. Therefore $(-1)^k L_k y(t) \geqslant 0$ on $[a, \infty)$. Since $y(t)$ is a nontrivial solution of (L) on $[a, \infty)$ (because $\sum_{i=0}^{3} c_i^2 > 0$), $Q(t) \leqslant 0$ and $Q(t)$ is not identically zero in any subinterval of I, we have $L_4 y(t) \geqslant 0$ with $L_4 y(t) = 0$ at most at isolated points of $[a, \infty)$. This implies that $L_3 y(t)$ is increasing on I, so $L_3 y(t) < 0$ on $[a, \infty)$. Similarly, it can be proved that $L_2 y(t) > 0$, $L_1 y(t) < 0$, $L_0 y(t) = y(t) > 0$ on $[a, \infty)$. □

The next theorem deals with the uniqueness of such a solution.

Theorem 3. *Suppose that* (A) *holds,* $\int^{\infty}(1/p_1(t))\,\mathrm{d}t = \int^{\infty}(1/p_2(t))\,\mathrm{d}t = \infty$, *and* (L) *is nonoscillatory. Then there exists at most one solution (with the exception of constant multiples) of* (L) *such that*

$$(13) \qquad (\operatorname{sign} y \neq \operatorname{sign} L_1 y \neq \operatorname{sign} L_2 y \neq \operatorname{sign} L_3 y \quad \text{on } I = [a,\infty), \ \lim_{t\to\infty} y(t) = 0).$$

Proof. Suppose that there exists another solution $z(t)$ linearly independent of $y(t)$, which fulfils (13). Let $\tau \in [a,\infty)$. Then there exists $c \in (-\infty,\infty)$ such that $z(\tau) + cy(\tau) = 0$. The number τ has been taken such that $y(\tau) \neq 0$. We prove that such τ exists. Suppose on the contrary that the required τ does not exist. This implies that $y(t) \equiv 0$ for all $t > t^*$ and that is why $y'(t) \equiv 0 \equiv L_1 y(t)$, which contradicts (13). Let $Y(t) = z(t) + cy(t)$. It is obvious that $Y(\tau) = 0$, $\lim Y(t) = \lim z(t) + c \lim y(t) = 0$ for $t \to \infty$. According to Lemma 3 there exists $t_0 \geqslant a$ such that either

$$\text{(i)} \qquad [(YL_1Y > 0,\ YL_2Y > 0) \ \text{ or } \ (YL_1Y > 0,\ YL_2Y < 0)]$$

or

$$\text{(ii)} \qquad [YL_1Y < 0,\ YL_2Y > 0]$$

for all $t \geqslant t_0$. Let t_0 be taken such that $t_0 > \tau$. Without loss of generality we can assume $Y > 0$ for all $t \geqslant t_0$. Suppose that (ii) holds, i.e.

$$Y > 0,\ L_1Y < 0,\ L_2Y > 0.$$

Since Y is a solution of (L) we have

$$L_4Y = -PL_2Y - QY \geqslant 0.$$

This fact implies that the function L_3Y is increasing ($\mathrm{d}L_3Y/\mathrm{d}t = L_4Y$) because $L_4Y = 0$ at isolated points of the interval $[a,\infty)$ only. Two cases may occur now. Either

$$\text{(a)} \qquad \text{there exists } t_1 \geqslant t_0 \text{ such that } L_3Y(t_1) = 0$$

or

$$\text{(b)} \qquad L_3Y(t) < 0 \ \text{ for all } t \in [t_0,\infty).$$

If (a) is fulfilled then $L_3Y > 0$ for all $t > t_1$. Take $t_2 > t_1$. This implies that $L_3Y(t_2) = b > 0$ and $L_3Y(t) \geqslant b$ for all $t \geqslant t_2$, i.e. $\mathrm{d}L_2Y(t)/\mathrm{d}t \geqslant b/p_3(t)$. Let $t > t_2$. Integrating the last inequality over $[t_2,t]$ we obtain

$$L_2Y(t) - L_2Y(t_2) \geqslant \int_{t_2}^{t} (b/p_3(s))\,\mathrm{d}s > 0,$$

i.e. $L_2Y(t) > L_2Y(t_2) > 0$ because of $L_2Y(t) > 0$ for all $t \geqslant t_0$ and $t_2 > t_0$. Hence $\mathrm{d}L_1Y(t)/\,\mathrm{d}t > L_2Y(t_2)/p_2(t)$. Integration over $[t_2, t]$ yields

$$L_1Y(t) \geqslant L_1Y(t_2) + L_2Y(t_2)\int_{t_2}^{t}(1/p_2(s))\,\mathrm{d}s.$$

It is obvious that t_3 can be taken such that $t_3 > t_2$ and the right hand side of the last inequality is positive for all $t \geqslant t_3$. This fact follows from the assumption

$$\int^{\infty}(1/p_2(t))\,\mathrm{d}t = \infty.$$

This implies that $L_1Y(t) = p_1(t)Y'(t) > 0$ for all $t \geqslant t_3$, which is a contradiction. Therefore the case (a) is impossible, i.e. the case (b) occurs, i.e. $Y > 0$, $L_1Y < 0$, $L_2Y > 0$, $L_3Y < 0$ for all $t \geqslant t_0$. According to Lemma 5 we have $Y(t) > 0$ for all $t \in [a, t_0)$. But $\tau \in [a, t_0)$. This implies that $Y(\tau) > 0$, which contradicts our assumptions. This contradiction implies impossibility of (ii). For this reason the condition (i) holds. It implies that $Y(t) > 0$, $L_1Y(t) > 0$, i.e. $Y'(t) > 0$ for all $t \geqslant t_0$ and so $\lim Y(t) \neq 0$ for $t \to \infty$. This contradiction proves our theorem. □

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Authors' address: Department of Mathematics, Faculty of Material Engineering, Slovak Technical University, Paulínska 16, 917 24 Trnava, Slovakia.