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Jozef Džurina

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ASYMPTOTIC PROPERTIES OF THIRD ORDER DELAY DIFFERENTIAL EQUATIONS

Jozef Džurina, Košice

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We consider the delay differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' - p(t)u(\tau(t)) = 0. \tag{1}$$

We always assume that

(i) $r_i(t)$, $0 \leqslant i \leqslant 2$, $\tau(t)$ and $p(t)$ are continuous on $[t_0, \infty)$, $r_i(t) > 0$, $p(t) > 0$, $\tau(t) < t$, $\tau(t) \to \infty$ as $t \to \infty$ and $\tau(t)$ is increasing;

(ii) $R_i(t) = \displaystyle\int_{t_0}^t \frac{\mathrm{d}s}{r_i(s)} \to \infty \quad \text{as} \quad t \to \infty \quad \text{for } i = 1 \text{ and } 2.$

For the sake of convenience we introduce the following functions:

$$\begin{aligned} L_0u(t) &= \frac{u(t)}{r_0(t)}, \\ L_iu(t) &= \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}u(t), \quad i = 1 \text{ and } 2, \\ L_3u(t) &= \frac{\mathrm{d}}{\mathrm{d}t}L_2u(t). \end{aligned}$$

By a solution of (1) we mean any function $u\colon [T_u, \infty) \to \mathbb{R}$ satisfying (1) on $[T_u, \infty)$ such that $L_iu(t)$, $0 \leqslant i \leqslant 3$, exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

The asymptotic behavior of the solutions of (1) is described in the following lemma which is a generalization of a lemma of Kiguradze [2, Lemma 3].

Lemma 1. *Let $u(t)$ be a nonoscillatory solution of (1). Then there exist an integer ℓ, $\ell \in \{1,3\}$ and $t_1 \geqslant t_0$ such that*

$$
\begin{aligned}
u(t)L_i u(t) > 0, &\quad 0 \leqslant i \leqslant \ell, \\
(-1)^{i-\ell} u(t) L_i u(t) > 0, &\quad \ell \leqslant i \leqslant 3
\end{aligned}
\tag{2}
$$

for all $t \geqslant t_1$.

A function $u(t)$ satisfying (2) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then by Lemma 1

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3.$$

Following Kiguradze we say that equation (1) has property (B) if $\mathcal{N} = \mathcal{N}_3$.

In a recent paper [3] Kusano and Naito have presented a useful comparison principle which under conditions (i) and (ii) enables us to deduce property (B) of a delay equation of the form (1) from that of the ordinary differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' - \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}u(t) = 0,$$

where $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. The objective of this paper is to show that this comparison theorem may fail and then it is a good idea to compare equation (1) with the first order delay equation

$$y'(t) + q(t)y(w(t)) = 0, \tag{E}$$

where $w(t) \not\equiv \tau(t)$. We present the relationship between property (B) of equation (1) and the oscillation of equation (E).

In the sequel we shall consider the functions $g(t)$ and $w(t)$ satisfying

$$g(t) \in C([t_0,\infty)), \quad g(t) > t, \quad w(t) = \tau(g(t)) < t. \tag{3}$$

For the sake of convenience and further references we make use of the following notation:

$$q(t) = r_2(t)\int_t^{g(t)} p(s) r_0(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s \tag{4}$$

for sufficiently large t with $\tau(t) > t_1$.

Theorem 1. *Let* (3) *hold. Assume that the linear differential inequality*

$$(\tilde{\mathrm{E}}) \qquad y'(t) + q(t)y(w(t)) \leqslant 0$$

has no eventually positive solutions. Then equation (1) *has property* (B).

P r o o f. By way of contradiction we assume that (1) has a nonoscillatory solution $u(t)$ which belongs to the class N_1. We may assume that $u(t)$ is positive. Then by Lemma 1, there exists a t_1 such that

$$L_0u(t) > 0, \quad L_1u(t) > 0, \quad L_2u(t) < 0 \quad \text{and} \quad L_3u(t) > 0$$

for $t \geqslant t_1$. Integration of the identity $L_1u(t) = L_1u(t)$ from t_1 to t leads to

$$(5) \qquad L_0u(t) \geqslant \int_{t_1}^{t} r_1(s)L_1u(s)\,\mathrm{d}s, \quad t \geqslant t_1.$$

On the other hand, integrating the identity $L_3u(t) = L_3u(t)$ from t ($\geqslant t_1$) to ∞ one gets

$$(6) \qquad -L_2u(t) \geqslant \int_{t}^{\infty} L_3u(s)\,\mathrm{d}s = \int_{t}^{\infty} p(s)u(\tau(s))\,\mathrm{d}s.$$

Combining (5) with (6) we get

$$\begin{aligned} -L_2u(t) &\geqslant \int_{t}^{\infty} p(s)r_0(\tau(s)) \int_{t_1}^{\tau(s)} r_1(x)L_1u(x)\,\mathrm{d}x\,\mathrm{d}s \\ &\geqslant \int_{t}^{g(t)} p(s)r_0(\tau(s)) \int_{t_1}^{\tau(s)} r_1(x)L_1u(x)\,\mathrm{d}x\,\mathrm{d}s, \quad t \geqslant t_2, \end{aligned}$$

where $t_2 \geqslant t_1$ is large enough. Since $L_1u(t)$ is decreasing we have in view of the last inequalities

$$-L_2u(t) \geqslant \int_{t}^{g(t)} p(s)r_0(\tau(s))L_1u(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s.$$

Hence, as $L_1u(t)$ is decreasing and $\tau(t)$ is increasing, the last inequalities imply

$$-L_2u(t) \geqslant L_1u(w(t)) \int_{t}^{g(t)} p(s)r_0(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s.$$

Put $z(t) = L_1u(t)$. Obviously, $z(t) > 0$ and $z(t)$ satisfies

$$-\frac{z'(t)}{r_2(t)} \geqslant z(w(t))\frac{q(t)}{r_2(t)}, \quad t \geqslant t_2,$$

which implies that $z(t)$ is a positive solution of the differential inequality

$$z'(t) + q(t)z(w(t)) \leqslant 0, \quad t \geqslant t_2,$$

which contradicts hypothesis. The proof is complete. □

Corollary 1. *Let* (3) *hold. Assume that either*

$$\liminf_{t\to\infty} \int_{w(t)}^{t} q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} q(s)\,\mathrm{d}s > 1.$$

Then equation (1) *has property* (B).

P r o o f. It is known (see [4]) that both conditions are sufficient for $(\widetilde{\mathrm{E}})$ to have no positive solutions. Our assertion follows from Theorem 1. □

Corollary 2. *Let* (3) *hold. Assume that the differential equation*

$$(E) \qquad y'(t) + q(t)y(w(t)) = 0$$

is oscillatory. Then equation (1) *has property* (B).

P r o o f. Corollary 2 follows from Theorem 1 and the fact that $(\widetilde{\mathrm{E}})$ has no positive solutions if and only if (E) is oscillatory (see [1]). □

Corollary 3. *Let* (3) *hold. For all large* t *define*

$$\tilde{q}(t) = r_2(t) \int_{t}^{g(t)} p(s) r_0(\tau(s)) R_1(\tau(s))\,\mathrm{d}s.$$

Assume that either

$$(7) \qquad \liminf_{t\to\infty} \int_{w(t)}^{t} \tilde{q}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}$$

or

$$(8) \qquad \limsup_{t\to\infty} \int_{w(t)}^{t} \tilde{q}(s)\,\mathrm{d}s > 1.$$

Then equation (1) *has property* (B).

In the following illustrative example we show how to choose the function $g(t)$ satisfying $g(t) > t$ and $\tau(g(t)) < t$.

Example 1. Let us consider the third order differential equation

$$y'''(t) - \frac{a}{t^2\sqrt{t}}y(\sqrt{t}) = 0, \quad a > 0, \quad \text{and} \quad t \geqslant 1. \tag{9}$$

We have $\tau(t) = \sqrt{t}$. Let us put $g(t) = 2t$. Then $w(t) = \sqrt{2t}$ satisfies (3) and

$$\tilde{q}(t) = \int_t^{2t} \frac{a}{s^2\sqrt{s}}\sqrt{s}\,\mathrm{d}s = \frac{a}{2t}.$$

By Corollary 3 equation (9) has property (B) if

$$\liminf_{t\to\infty} \int_{\sqrt{2t}}^{t} \tilde{q}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{10}$$

Simple computation shows that (10) holds for any $a > 0$. Note that we obtain the same result if we choose $g(t) = \alpha^2 t^2$, $0 < \alpha < 1$. On the other hand, by the above-mentioned result of Kusano and Naito one gets that (9) has property (B) if the ordinary equation without delay

$$y'''(t) - \frac{2a}{t^4}y(t) = 0 \tag{11}$$

has property (B). However, as (11) has not property (B), the criterion of Kusano and Naito fails for (9).

For a special choice of the function $g(t)$ we have the following result:

Theorem 2. *Suppose that*

$$\limsup_{t\to\infty} \frac{\tau(t)}{t} < c < 1. \tag{12}$$

Assume that either

$$\liminf_{t\to\infty} \int_{ct}^{t} r_2(s) \int_s^{\tau^{-1}(cs)} p(x)r_0(\tau(x))R_1(\tau(x))\,\mathrm{d}x\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{13}$$

or

$$\limsup_{t\to\infty} \int_{ct}^{t} r_2(s) \int_s^{\tau^{-1}(cs)} p(x)r_0(\tau(x))R_1(\tau(x))\,\mathrm{d}x\,\mathrm{d}s > 1. \tag{14}$$

Then equation (1) *has property* (B).

P r o o f. It is easy to verify that (12) is equivalent to

$$\limsup_{t\to\infty} \frac{t}{\tau^{-1}(t)} < c < 1,$$

which implies that for all large t

$$\tau^{-1}(t) > \frac{t}{c}. \tag{15}$$

Put $g(t) = \tau^{-1}(ct)$. Then $g(t)$ satisfies (3) as $w(t) = \tau(g(t)) = ct < t$ and in view of (15) we have $g(t) > t$. Noting that (7) and (8) are equivalent to (13) and (14), respectively, the assertion of this theorem follows from Corollary 3. The proof is complete. □

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Author's address: Department of Mathematical Analysis, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia.