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AFFINE COMPLETENESS OF COMPLETE LATTICE ORDERED GROUPS

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Affine completeness of universal algebras and, in particular, of lattices, was investigated by several authors (cf. [2]–[9]).

A variety is called affine complete if each of its algebras is affine complete. An important example of an affine complete variety is the variety of Boolean algebras [3]; this result was extended in [5] and [6].

In [4] it was proved that a bounded distributive lattice is affine complete if and only if it does not contain an interval which is a Boolean lattice with more than one element. A generalization of this result was established in [7].

In the present paper we show that if G is an abelian lattice ordered group which can be expressed as a direct product $G = A \times B$ with $A \neq \{0\} \neq B$, then G is not affine complete.

By means of this result we prove the following theorem:

(A). *Let G be a complete lattice ordered group. Then the following conditions are equivalent:*

(i) *G is affine complete.*
(ii) $G = \{0\}$.

The question whether the conditions (i) and (ii) are equivalent for each lattice ordered group remains open.

We shall apply the following notation. For a universal algebra A we denote by $\operatorname{Con} A$ the set of all congruences of A. Let $P(A)$ be the set of all polynomials that can be constructed by using the symbols of basic operations of A, constants $a, b, c, \ldots$ which are elements of A and a finite number of variables $x, y, \ldots$.

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Let n be a positive integer and let $f\colon A^n \longrightarrow A$ be a mapping. f is called compatible with $\operatorname{Con} A$ if, whenever $\Theta \in \operatorname{Con} A$, $a_i, b_i \in A$, $a_i \mathrel{\Theta} b_i$ for $i = 1, 2, \dots, n$, then $f(a_1, a_2, \dots, a_n) \mathrel{\Theta} f(b_1, b_2, \dots, b_n)$.

The algebra A is said to be affine complete if each mapping $f\colon A^n \longrightarrow A$ which is compatible with $\operatorname{Con} A$ belongs to $P(A)$.

1. Auxiliary results

For lattice ordered groups we apply the standard terminology and notation (cf. e.g., Conrad [1]). The group operation in a lattice ordered group will be written additively.

Let G be a lattice ordered group. The underlying lattice will be denoted by $\overline{G}$. Then

(a) $\overline{G}$ is a distributive lattice;
(b) if $G \neq \{0\}$, then $\overline{G}$ has neither the greatest element nor the least element;
(c) for each $x, y, z \in G$ the relations
$$x + (y \wedge z) = (x + y) \wedge (x + z), \quad (y \wedge z) + x = (y + x) \wedge (z + x),$$
$$x + (y \vee z) = (x + y) \vee (x + z), \quad (y \vee z) + x = (y + x) \vee (z + x)$$
are valid.

From (a) and (b) we obtain by the obvious induction steps:

1.1. Lemma. *Let $p(x) \in P(G)$. Then there are nonempty finite sets I and $J(i)$ $(i \in I)$ such that $p(x)$ can be expressed in the form*

$$p(x) = \bigwedge_{i \in I} \bigvee_{j \in J(i)} (a_{ij}^1 + a_{ij}^2 + \dots + a_{ij}^{n(i,j)}),$$

where for each $i \in I$, $j \in J(i)$ and $k \in \{1, 2, \dots, n(i, j)\}$ we have either $a_{ij}^k \in G$ or $a_{ij}^k = x$.

1.2. Corollary. *Let $p(x) \in P(G)$ and assume that G is abelian. Then $p(x)$ can be expressed in the form*

$$p(x) = \bigwedge_{i \in I} \bigvee_{j \in J(i)} (a_{ij} + n_{ij} x),$$

where all n_{ij} are integers and $a_{ij} \in G$.

1.3. Lemma. *Let $p(x)$ and G be as in 1.2. Suppose that $p(x)$ fails to be a constant (i.e., there are $x_1, x_2 \in G$ such that $p(x_1) \neq p(x_2)$. Then there are $x_1 \in G^+$, $i(0) \in I$ and $j(0) \in J_{i(0)}$ such that*

$$p(x_1) = a_{i(0)j(0)} + n_{i(0)j(0)} x_1.$$

Proof. We have $G \neq \{0\}$. Hence according to (b) there is $x_1 \in G$ such that $x_1 > 0$ and

$$x_1 > \sum_{i \in I} \sum_{j \in J(i)} |a_{ij}|.$$

For $i \in I$ we denote

$$c_i(x) = \bigvee_{j \in J(i)} (a_{ij} + n_{ij}x).$$

Let $j(1), j(2) \in J(i)$, $j(1) \neq j(2)$. If $n_{ij(1)} = n_{ij(2)}$, then

$$(a_{ij(1)} + n_{ij(1)}x) \vee (a_{ij(2)} + n_{ij(2)}x) = (a_{ij(1)} \vee a_{ij(2)}) + n_{ij(1)}x.$$

Hence without loss of generality we can suppose that $n_{ij(1)} \neq n_{ij(2)}$ whenever $j(1), j(2) \in J(i)$, $j(1) \neq j(2)$.

Let $j(1)$ and $j(2)$ be distinct elements of $J(i)$. Suppose that $n_{ij(1)} < n_{ij(2)}$. Then

$$\begin{aligned} &(a_{ij(2)} + n_{ij(2)}x_1) - (a_{ij(1)} + n_{ij(1)}x_1) = \\ &= (n_{ij(2)} - n_{ij(1)})x_1 + (a_{ij(2)} - a_{ij(1)}) \geqslant x_1 + (a_{ij(2)} - a_{ij(1)}). \end{aligned}$$

We have

$$-|a_{ij(2)} - a_{ij(1)}| \leqslant a_{ij(2)} - a_{ij(1)} \leqslant |a_{ij(2)} - a_{ij(1)}|,$$

$$|a_{ij(2)} - a_{ij(1)}| \leqslant |a_{ij(2)}| + |a_{ij(1)}| < x_1.$$

Thus

$$x_1 + (a_{ij(2)} - a_{ij(1)}) > 0.$$

Hence

$$(a_{ij(2)} + n_{ij(2)}x_1) \vee (a_{ij(1)} + n_{ij(1)}x_1) = a_{ij(2)} + n_{ij(2)}x_1.$$

This yields that there is $j(i) \in J(i)$ such that

$$c_i(x_1) = a_{ij(i)} + n_{ij(i)}x_1.$$

Therefore

$$p(x_1) = \bigwedge_{i \in I} (a_{ij(i)} + n_{ij(i)}x_1).$$

Now, by an analogous method as we did above we obtain that there is $i(0) \in I$ such that

$$p(x_1) = a_{i(0),j(i(0))} + n_{i(0),j(i(0))}x_1.$$

□

1.3.1. Remark. From the consideration applied in the proof of 1.3 we infer that if x_1 is as in 1.3 and $x_1' \in G$, $x_1' > x_1$, then

$$p(x_1') = a_{i(0)j(0)} + n_{i(0)j(0)}x_1'$$

(i.e., the indices remain the same as in 1.3).

If $G = A \times B$ and $g \in G$, then the component of g in A will be denoted by $g(A)$. Thus $g(A) = g$ for each $g \in A$, and $g(A) = 0$ for each $g \in B$.

1.4. Lemma. *Let $G = A \times B$, $f(x) = x(A)$ for each $x \in G$. Then f is compatible with* $\operatorname{Con} G$.

Proof. Let $\Theta \in \operatorname{Con} G$. There exists an ℓ-ideal H of G such that for any $g_1, g_2 \in G$,

$$g_1 \mathrel{\Theta} g_2 \Leftrightarrow g_1 - g_2 \in H.$$

Let $u, v \in G$, $u \mathrel{\Theta} v$. Hence $u - v \in H$ and thus $|u - v| \in H$. We have

$$f(|u-v|) = |f(u) - f(v)|,$$

$$f(|u-v|) \leqslant |u-v|,$$

and so $f(|u-v|) \in H$, yielding $f(u) \mathrel{\Theta} f(v)$. □

1.5. Lemma. *Let G and f be as in* 1.4. *Suppose that G is abelian and that $A \neq \{0\} \neq B$. Then $f \notin P(G)$.*

Proof. By way of contradiction, suppose that $f \in P(G)$. It is obvious that $f(x)$ satisfies the assumption from 1.3 (we put $p = f$). Since $A \neq \{0\} \neq B$ there exist $0 < a \in A$ and $0 < b \in B$. In view of 1.3.1, the element x_1 in 1.3 can be replaced by $x_2' = x_1 \vee a \vee b$. Thus for $x_1 = x_1 \vee a \vee b$ we have

$$f(x_1') = a + nx_1',$$

where $a = a_{i(0)j(0)}$ and $n = n_{i(0)j(0)}$.

Put $x_1'(A) = x^A$ and $x_1'(B) = x^B$. Hence

$$f(x_1') = f(x^A + x^B) = f(x^A) + f(x^B) = x^A,$$
$$f(x_1') = a + n(x^A + x^B) = a + nx^A + nx^B.$$

At the same time, taking $2x_1'$ instead of x_1' we get (cf. 2.3.1)

$$f(2x_1') = 2x^A, \quad f(2x_1') = a + 2nx^A + 2nx^B.$$

Hence

$$x^A = nx^A + nx^B,$$

yielding that

$$(1-n)x^A = nx^B.$$

Since $(1-n)x^A \in A, nx^B \in B$ and $A \cap B = \{0\}$ we obtain that $(1-n)x^A = nx^B = 0$. Since

$$x^A \geqslant a > 0, \quad x^B \geqslant b > 0,$$

we have arrived at a contradiction. □

2. Proof of (A)

2.1. Proposition. *Let G be an abelian lattice ordered group, $G = A \times B$, $A \neq \{0\} \neq B$. Then G is not affine complete.*

Proof. This is a consequence of 1.4 and 1.5. □

For a subset X of a lattice ordered group G we put

$$X^\delta = \{y \in G \colon |y| \wedge |x| = 0 \quad \text{for each} \quad x \in X\}.$$

If $G = \{g\}^{\delta\delta} \times \{g\}^\delta$ for each $g \in G$, then G is said to be projectable.

2.2. Proposition. *Let G be a projectable lattice ordered group. Assume that G is abelian and that it is not linearly ordered. Then G is not affine complete.*

Proof. There exist incomparable elements a, b in G. Put

$$a_1 = a - (a \wedge b), \quad b_1 = b - (a \wedge b).$$

Then $0 < a_1$, $0 < b_1$ and $a_1 \wedge b_1 = 0$. Denote $A = \{a_1\}^{\delta\delta}$, $B = \{a_1\}^\delta$. We have $a_1 \in A$, $b_1 \in B$. Since G is projectable, $G = A \times B$. Now it suffices to apply 2.1. □

It is well-known that each complete lattice ordered group is abelian and projectable. Hence we have

2.3. Corollary. *Let G be a complete lattice ordered group which is not linearly ordered. Then G is not affine complete.*

We denote by $\mathbb{R}$ and $\mathbb{Z}$ the additive group of all reals or of all integers, respectively. Both $\mathbb{R}$ and $\mathbb{Z}$ are linearly ordered in the usual way.

We define a mapping $f_1 \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ as follows: for $z \in \mathbb{Z}$ we put $f_1(z) = 1$ if z is even and $f_1(z) = 2$ if z is odd. Next, we define $f_2 \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_2(t) = f_1(t)$ if $t \in \mathbb{Z}$ and $f_2(t) = 0$ otherwise. Since both the lattice ordered groups $\mathbb{Z}$ and $\mathbb{R}$ are simple (i.e., they have no non-trivial ℓ-ideal) we obtain

2.4. Lemma. *f_1 is compatible with* $\operatorname{Con} \mathbb{Z}$ *and f_2 is compatible with* $\operatorname{Con} \mathbb{R}$.

2.5. Lemma. *Let f_1 and f_2 be as above. Then $f_1 \notin P(\mathbb{Z})$ and $f_2 \notin P(\mathbb{R})$.*

Proof. By way of contradiction, suppose that $f_1 \in P(\mathbb{Z})$. Thus according to 1.3 and 1.3.1 there are $x_1, a, n \in \mathbb{Z}$ such that

$$f_1(x_1) = a + nx_1,$$
$$f_1(x_1 + 2) = a + n(x_1 + 2).$$

In wiew of the definition of f_1 we have $f_1(x_1) = f_1(x_1 + 2)$, whence $n = 0$ and thus $f_1(x_1) = a$. By applying 1.3.1 again we obtain

$$f_1(x_1 + 1) = a$$

and hence $f_1(x_1) = f_1(x_1 + 1)$, which is a contradiction. Therefore $f_1 \notin P(\mathbb{Z})$. This implies that $f_2 \notin P(\mathbb{R})$. □

Now, 2.4 and 2.5 yield

2.6. Corollary. *Neither $\mathbb{Z}$ nor $\mathbb{R}$ is affine complete.*

Proof of (A). Let G be a complete lattice ordered group. Let (i) and (ii) be as in (A). Clearly (ii) $\Rightarrow$ (i). Suppose that (i) is valid. In view of 2.3, G must be linearly ordered. Hence G is isomorphic to some of the lattice ordered groups $\{0\}$, $\mathbb{Z}$ or $\mathbb{R}$. Therefore according to 2.6 we obtain that (ii) holds. □

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