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UNIQUE COVERING ON RADICAL CLASSES OF ℓ-GROUPS

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The unique covering question on radical classes of ℓ-groups was raised by J. Jakubík in 1977. The question is whether there is a radical class σ with $A(\sigma) \neq \emptyset$ such that $A(\sigma)$ is a class consisting of exactly one element. By some new concepts as supplementing radical class, polar radical class, certain valuable results are obtained in this paper. In Part 1, the above question is answered affirmatively (Theorem 1.2). In Part 2, a lot of necessary and sufficient conditions to the question are given for a large class of radical classes, namely, non-superatom radical classes (Theorem 2.5, 2.6, 2.8). Cardinality of a radical class is studied in the last part of this paper. For any cardinal α, the following theorem is proved in Part 3: there does exist a radical class σ such that the cardinality of $A(\sigma)$ is α. This results has filled the gaps in the study of ℓ-groups.

The notions and terminology in this paper are standard, cf. [1], [2], [3].

Let $\mathscr{G}$ be the class of all ℓ-groups, R the class of all radical classes.

For $G \in \mathscr{G}$, $c(G)$ is the lattice of all solid subgroups of G. $T(G)$ denotes the principal radical class generated by G. For $\sigma \in R$, $\sigma(G)$ is the largest solid subgroup of G which belongs to σ.

An atom τ over σ (or τ covers σ) means that $\sigma < \tau$ and there exists no $\varrho \in R$ such that $\sigma < \varrho < \tau$. Next, τ is an antiatom over σ if $\sigma < \tau$ and there is no $\varrho \leqslant \tau$ with ϱ an atom over σ. Atom or antiatom over 0 are called respectively atom or antiatom for short. $A(\sigma)$ stands for the class of all atoms over σ. If $T(G)$ is an atom, then G is called a homogeneous ℓ-group.

The interval $[\sigma, \varrho]$ has the usual meaning.

$$A_0 = \sup\{\sigma \in R \mid \sigma \text{ is an atom}\},$$
$$A_1 = \sup\{\sigma \in R \mid \sigma \text{ is an antiatom}\}.$$

Emphasis must be placed on the following concepts:

1. Regular atom: for $G \in \mathscr{G}$, a regular atom over $T(G)$ is the atom $T\big(G(\alpha)\big)$ generated by $G(\alpha) = F\big(I(\alpha), G\big)$, cf. [1], §3.

2. Regular antiatom: for $G \in \mathscr{G}$, a regular antiatom over $T(G)$ is the antiatom $T(G'(\alpha))$ generated by $G'(\alpha) = \Gamma Hm$, cf. [1], §5.

3. Big atom: Let $\sigma \in R$, $A(\sigma) \neq \emptyset$. The big atom over σ is the radical class $z(\sigma) = \bigvee \varepsilon(\sigma, \alpha)$, where α runs over all ordinals, cf. [1], §6.

For many results in this paper, the lemmas below are useful.

Lemma 1. *Let $T(G(\alpha))$ (or $T(G'(\alpha))$) be a regular atom (or antiatom) over $T(G)$. Then any non-zero solid subgroup of $G(\alpha)$ (or $G'(\alpha)$) has a non-zero solid subgroup isomorphic to a non-zero solid subgroup of G.*

This is from [1], §1, (A).

Lemma 2. *The big atom $z(\sigma)$ over σ has no atom, that is, $A(z(\sigma)) = \emptyset$.*

This is Proposition 6.4 of [1].

Lemma 3. *There are no radical classes $\tau < \varrho$ such that $\tau, \varrho \in [\sigma, z(\sigma)]$ and ϱ is an antiatom over τ.*

This is Proposition 6.8 of [1].

1. Polar radical classes and unique covering question

Let $\sigma \in R$. Put

$$S = \{\tau \in R \mid \tau \wedge \sigma = 0\}.$$

Since $0 \in S$ we have $S \neq \emptyset$. Let $\sigma' = \sup S = \bigvee \tau$, then $\sigma' \in S$ (for R is a complete Brouwerian lattice), that is to say, σ' is the largest radical class having trivial intersection with σ. The radical class σ' is called the supplementing radical of σ. The supplementing radical $(\sigma')'$ of σ' is called the polar radical of σ and is denoted by σ''. If σ is an atom, then σ' is called a supplementing atom. Obviously, $\sigma \leqslant \sigma''$. If $\sigma = \sigma''$, we call σ a polar. 0 and $\mathscr{G}$ are trivial polars.

Proposition 1.1. *Let $\sigma, \tau \in R$. Then*

(1) *$\sigma \geqslant \tau$ implies $\sigma' \leqslant \tau'$, $\sigma'' \geqslant \tau''$;*
(2) *$\sigma' = \sigma'''$, so every supplementing radical is a polar;*
(3) *$(\sigma \wedge \tau)' \geqslant \sigma' \wedge \tau'$, $(\sigma \vee \tau)' \leqslant \sigma' \vee \tau'$;*
(4) *$(\sigma \vee \tau)' = \sigma' \wedge \tau'$, $(\sigma \wedge \tau)' \geqslant \sigma' \vee \tau'$;*
(5) *$\sigma' = \{G \in \mathscr{G} \mid \sigma(G) = 0\} = \{G \in \mathscr{G} \mid$ if $H \in c(G) \cap \sigma$, then $H = 0\} = \{G \in \mathscr{G} \mid$ if $C \in c(G)$, then $\sigma(C) = 0\}$.*

The proof is straightforward. Note all inclusions in Proposition 1.1 may be proper.

Theorem 1.2. *Let $\sigma = T(G)$ be an atom. Then $A(\sigma')$ has exactly one element.*

Proof. Evidently, $G \notin \sigma'$. Then $T(G) \wedge \sigma' = 0$, hence $\sigma' \vee T(G)$ covers σ'. If $\tau \neq \varrho$, $\tau, \varrho \in A(\sigma')$, then $\tau \wedge \varrho = \sigma'$, thus $\tau \geqslant \sigma'$, $\varrho \geqslant \sigma'$. We obtain immediately $\tau \leqslant \sigma'$ or $\varrho \leqslant \sigma'$, for $G \notin \tau$ or $G \notin \varrho$. Therefore, $\tau = \sigma'$ or $\varrho = \sigma'$, a contradiction. This implies the result. □

Remark. Theorem 1.2 gives an affirmative answer to J. Jakubík's question. Later, we will see some radical classes with unique covering, which are not supplementing atoms.

Let $\sigma \in R$. A solid subgroup of G belonging to σ is called a σ-subgroup of G. $\{0\}$ is a trivial σ-subgroup.

Theorem 1.3. *Let $\sigma \in R$. Put*

$$P = \{G \in \mathcal{G} \mid \text{each non-zero solid subgroup of } G \text{ has a non-trivial } \sigma\text{-subgroup.}\}$$

Then $P = \sigma''$. Therefore σ is a polar if and only if for each $G \in \sigma$ and $0 < x \in G$ there is an element $y \in G$, $0 < y \leqslant nx$ (for some positive integer n), such that $G(y)$ has a non-zero σ-subgroup.

Proof. From Proposition 1.1, $G \in \sigma''$ iff $\sigma'(H) = 0$ for each $H \in c(G)$. Since $\sigma'(G)$ is the largest solid subgroup C of G having the property $\sigma(C) = 0$, $\sigma'(G) = 0$ iff each non-zero solid subgroup has a non-trivial sz-subgroup, this is equivalent to $G \in P$. Hence $P = \sigma''$. The last part is consequently obvious. □

Corollary 1.4. *σ is a polar if and only if for $G \in \mathcal{G}$, whenever each non-zero solid subgroup of G has a non-zero σ-subgroup, then $G \in \sigma$.*

2. The unique covering of a non-superatom radical class

Call a radical class which contains A_0 a superatom radical class (shortly, superatom). Otherwise, it will be called a non-superatom radical class (simply, non-superatom).

Proposition 2.1. *A_1 is a polar. A polar containing A_1 has no antiatom.*

Proof. Let $G \in \mathcal{G}$ and suppose that each non-zero solid subgroup of G has a non-trivial A_1-subgroup. If $T(G) \geqslant X$, X an atom, then $X(G)$ is a homogeneous ℓ-group, thus $X(G)$ has no non-trivial A_1-subgroup, a contradiction. Therefore $T(G)$ is an antiatom, which implies $G \in A_1$. From Corollary 1.4, A_1 is a polar.

Now assume that σ is a polar and $\sigma \geqslant A_1$. If τ is an antiatom over σ, then $\tau > \sigma \geqslant A_1$, hence τ contains an ℓ-group G such that $T(G)$ is not an antiatom. Put $H = \sigma'(G)$, then $T(H)$ is not an antiatom. From the projectivity of the intervals $[\sigma \wedge T(\sigma'(G)), T(\sigma'(G))]$ and $[\sigma, \sigma \vee T(\sigma'(G))]$ and from the relation $\sigma \wedge T(H) = T(\sigma(H)) = 0$ we infer that $\sigma \vee T(\sigma'(G))$ is not an antiatom over σ, thus $\sigma'(G) = 0$. Hence $G \in \sigma'' = \sigma$, a contradiction. This completes the proof. □

Theorem 2.2. *Let σ be a superatom and suppose that $\sigma'' = \sigma$. Then $A(\sigma) = \emptyset$.*

Proof. Let $\sigma'' = \sigma \geqslant A_0$. If $\sigma \vee T(G)$ covers σ, then either $\sigma \vee T(G) = \sigma \vee T(\sigma'(G))$ or $\sigma = \sigma \vee T(\sigma'(G))$, for $\sigma \leqslant \sigma \vee T(\sigma'(G)) \leqslant \sigma \vee T(G)$. The first case says that $G \in \sigma \vee T(\sigma'(G))$, thus $G = \sigma(G) \vee \sigma'(G)$. Since $T(G)$ covers $T(\sigma(G))$, we infer that $T(\sigma'(G))$ covers $T(\sigma'(G)) \wedge T(\sigma(G)) = 0$, hence $\sigma'(G)$ is a homogeneous ℓ-group. Therefore $0 \neq \sigma'(G) \in \sigma$, a contradiction. So it must be the case that $\sigma = \sigma \vee T(\sigma'(G))$, then $T(\sigma'(G)) = 0$. This implies $\sigma'(G) = 0$, hence $G \in \sigma'' = \sigma$, again a contradiction. Thus, there is no atom over σ. □

Corollary 2.3. *$\mathscr{G}$ is the unique polar which contains A_0 and A_1, so $(A_0 \vee A_1)'' = \mathscr{G}$.*

Corollary 2.4. *Let σ and τ be two distinct atoms. Then $(\sigma' \vee \tau')'' = \mathscr{G}$.*

Theorem 2.5. *Let σ be a non-superatom. Then $A(\sigma)$ is a one-element class if and only if there exists a homogeneous ℓ-group G, such that $\sigma = T(G)' \wedge z(\sigma)$, where $z(\sigma)$ is the big atom over σ.*

Proof. Assume $A(\sigma)$ is a one-element class. There is a homogeneous ℓ-group $G \notin \sigma$ for $\sigma \ngeqslant A_0$. Hence $\sigma \vee T(G)$ covers σ. This implies $\sigma \leqslant T(G)'$, thus $\sigma \leqslant T(G)' \wedge z(\sigma)$. For $H \notin \sigma$, either $H \notin T(G)'$ hence $H \notin T(G)' \wedge z(\sigma)$, or $H \in T(G)'$, hence $T(H) \wedge T(G) = 0$ and $\sigma \vee T(H)$ is an antiatom over σ, so that $H \notin z(\sigma)$ (Lemma 3). Therefore, $H \notin T(G)' \wedge z(\sigma)$, i.e. $\sigma = T(G)' \wedge z(\sigma)$.

Now suppose $\sigma = T(G)' \wedge z(\sigma)$, G is a homogeneous ℓ-group. Then $T(G)' \vee \sigma$ covers σ for $G \notin \sigma$. If $\sigma \vee T(H) \neq \sigma \vee T(G)$ and $\sigma \vee T(H)$ covers σ, then $T(H) \wedge T(G) = 0$, hence $T(H) \leqslant T(G)'$. We infer that $H \in z(\sigma)$ for $T(H) \vee \sigma \leqslant z(\sigma)$. So we have $T(H) \vee \sigma \leqslant T(G)' \wedge z(\sigma) = \sigma$, a contradiction. Therefore $A(\sigma)$ is a one-element class. □

The example below says that $A(\sigma)$ being a one-element class may be true when σ is not a polar.

Example 2.6. Let $\sigma_0 = 0$. Put $\sigma = z(\sigma_0) \wedge T(G)'$, where G is a homogeneous ℓ-group. Obviously, $\sigma \vee T(G)$ covers σ. If $\sigma \vee T(H) \neq \sigma \vee T(G)$ and $\sigma \vee T(H)$ covers σ, then $T(H) \leqslant T(G)'$. Thus $\sigma \wedge T(H) = z(\sigma_0) \wedge T(H)$, hence $T(H) \vee z(\sigma_0)$ covers

$z(\sigma_0)$, a contradiction (Lemma 2). Therefore, there is exactly one atom over σ.

Let H be an arbitrary homogeneous ℓ-group with $H \notin T(G)$. Let $T(H'(\alpha))$ be a regular antiatom over $T(H)$. Since there is no $\tau, \varrho \in [\sigma_0, z(\sigma_0)]$, such that ϱ is an antiatom over τ, we obtain $H'(\alpha) \notin z(\sigma_0)$, hence $H'(\alpha) \notin \sigma$. Note that each non-zero solid subgroup of $H'(\sigma)$ has a non-trivial $z(\sigma_0)$-subgroup (Lemma 1), we have $H'(\alpha) \in z(\sigma_0)''$. Moreover $T(H'(\alpha)) \wedge T(G) = 0$ implies $H'(\alpha) \in T(G)'$, hence $H'(\alpha) \in z(sz_0)'' \wedge T(G)' \leqslant \sigma''$. By Corollary 1.4, σ is not a polar.

In much the same way, we obtain

Theorem 2.7. *Let σ be a superatom. Then $A(\sigma)$ is a one-element class if and only if there is $\tau \in R$ and there is a homogeneous ℓ-group G, such that $\sigma = z(\tau) \wedge T(G)'$.*

Theorem 2.8. *Let σ be a polar. Then $A(\sigma)$ is a one-element class if and only if σ is an intersection of a supplementing atom with a superatom which is a polar.*

Proof. Let $T(G)$ be an atom. $\tau' \geqslant A_0$, $\sigma = T(G)' \wedge \tau'$. Trivially, $T(G) \vee \sigma$ covers σ. If $\sigma \vee T(H) \neq \sigma \vee T(G)$ and $\sigma \vee T(H)$ covers σ, then $T(H) \leqslant T(G)'$. Since $T(H)$ covers $\sigma \wedge T(H) = T(H) \wedge \tau'$, we infer that $T(H) \vee \tau'$ covers τ', which contradicts Theorem 2.2. Thus $A(\sigma)$ consists of exactly one element.

Now suppose that $A(\sigma)$ is a one-element class. Theorem 2.2 says $\sigma \ngeqslant A_0$. Hence there exists a homogeneous ℓ-group G such that $\sigma \vee T(G)$ covers σ, thus $\sigma \leqslant T(G)'$. Put

$$\varrho = \{H \in T(G)' \mid \sigma(H) = 0\} = T(G)' \wedge \sigma'.$$

We have $\varrho' \geqslant \sigma$ for $\varrho \wedge \sigma = 0$. Thus $\sigma \leqslant \varrho' \wedge T(G)'$. For $H \notin \sigma$, either $H \notin T(G)'$ then $H \notin T(G)' \wedge \varrho'$, or $H \in T(G)'$. In the latter case, $H > \sigma(H) \geqslant 0$. If $\sigma'(H) = 0$, then $H \in \sigma'' = \sigma$, a contradiction. So $\sigma'(H) \neq 0$. Since $\sigma'(H) \in T(G)'$ and $\sigma(\sigma'(H)) = 0$, we infer that $\sigma'(H) \in \varrho$, thus $\sigma'(H) \notin \varrho'$. Hence $H \notin \varrho'$, consequently $H \notin \varrho' \wedge T(G)'$. Therefore, $\sigma = \varrho' \wedge T(G)'$. For any homogeneous ℓ-group H, either $T(H) \wedge \sigma = T(H)$ or $T(H) = T(G)$. That is, either $H \in T(G)'$ and $\varrho(H) = 0$, thus $H \in \varrho'$, or $H \in T(G) \leqslant \varrho'$. In any case, we have $\varrho' \geqslant A_0$. This completes the proof. □

Remark. We have solved the question of a unique covering on non-superatoms and polars.

3. The cardinality of $A(\sigma)$

Throughout this section, $A(\sigma)$ is supposed to be a set (not a proper class), its cardinality will be denoted by $C(\sigma)$.

Let $\sigma \in R$. Call σ an atom-closed class if for $G \in \sigma$, $A(T(G)) \leqslant \sigma$ holds.

Theorem 3.1. *$C(\sigma) = 0$ if and only if σ is an atom-closed class.*

Proof. Let σ be an atom-closed class. If $A(\sigma)$ is not empty, then there is an ℓ-group G such that $\tau = \sigma \vee T(G)$ covers σ; clearly, $G \notin \sigma$. Since $\sigma(G) \in \sigma$ and $T(G)$ covers $T(G) \wedge \sigma = T(\sigma(G))$, we obtain a contradiction. Thus $A(\sigma) = \emptyset$, i.e. $C(\sigma) = 0$.

If $C(\sigma) = 0$, then $A(\sigma) = \emptyset$. Suppose that σ is not an atom-closed class, then there exists some $G \in \sigma$ such that $T(G)$ is covered by $T(H)$, for some $H \notin \sigma$. Obviously $\sigma \wedge T(H) \geqslant T(G)$. If $\sigma \wedge T(H) > T(G)$, then $T(H) > \sigma \wedge T(H) > T(G)$, a contradiction. Thus $\sigma \wedge T(H) = T(G)$. Since $T(H)$ covers $T(G)$, we obtain immediately that $\sigma \vee T(H)$ covers σ, again a contradiction. Therefore, σ is an atom-closed class. □

Theorem 3.2. *For any cardinal α, there exists a radical class σ such that $C(\sigma) = \alpha$.*

Proof. Let I be an indexed set whose cardinal is α. Let $T(G_i)$ be distinct atoms, $i \in I$. Put

$$\sigma = \bigvee T(G_i)' = \bigwedge T(G_i)'.$$

For $i \in I$, $T(G_i) \wedge \sigma = 0$, thus $T(G_i) \vee \sigma$ covers σ. For $i \neq j$, obviously $\sigma \vee T(G_i) \neq \sigma \vee T(G_j)$. Hence $C(\sigma) \geqslant \operatorname{Card} I = \alpha$.

On the other hand, assume that $\sigma \vee T(H)$ covers σ and $\sigma \vee T(H) \neq \sigma \vee T(G_i)$, $i \in I$. Then $T(H) \wedge T(G_i) = 0$ and $T(H) \leqslant T(G_i)'$. Hence $T(H) \leqslant \bigwedge T(G_i)' = \sigma$, a contradiction. Therefore $\{\sigma \vee T(G_i) \mid i \in I\} = A(\sigma)$, which means that $C(\sigma) = \alpha$. □

The above theorem gives a construction for a radical class having arbitrary many coverings. So far, the study on cardinality of $A(\sigma)$ is restricted to two cases, $A(\sigma) = \emptyset$ and $A(\sigma)$ is a proper class. Therefore, Theorem 3.2 is a surprising generalization to the unique covering question.

References

[1] *J. Jakubík*: Radical mappings and radical classes of lattice ordered groups. Symposia Math. *21* (1977), 451–477.
[2] *J. Jakubík*: Radical subgroups of lattice ordered groups. Czech. Math. J. *36(111)* (1986), 285–290.
[3] *P. Conrad*: Lattice ordered groups. Tulane University, 1970.

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