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CERTAIN CUBIC MULTIGRAPHS AND THEIR UPPER EMBEDDABILITY

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Let G be a connected cubic multigraph such that each edge of G belongs to a cycle of length $\leqslant 5$. We shall find a global property of G (Theorem 1). Then we shall show that G is upper embeddable (Theorem 2).

1. Let G be a multigraph (in the sense of [1], for example) with a vertex set $V(G)$ and an edge set $E(G)$. (Note that G is a graph if and only if it has no multiple edges; and G is a path if and only if it is a tree with no vertex of degree $\geqslant 3$.) If $u, v \in V(G)$, $e \in E(G)$, $u \neq v$ and e is the only edge of G incident with u and v, then we shall write $e = uv$. Let U be a nonempty subset of $V(G)$; we denote by $\langle U\rangle$ the multigraph F defined as follows: $V(F) = U$,

$$E(F) = \{e \in E(G);\ e \text{ is incident with no vertex in } V(G) - U\},$$

and e and u are incident in F if and only if they are incident in G, for any $e \in E(F)$ and $u \in V(F)$.

Let G be a multigraph, and let $\mathscr{P}$ be a partition of $V(G)$. If $\mathscr{R} \subseteq \mathscr{P}$, then we denote by $E_{\mathscr{R}}$ the set of all edges e of G with the property that the vertices incident with e belong to distinct elements of $\mathscr{R}$. We shall say that $\mathscr{P}$ is a *D-partition* of G if $\langle \mathscr{P}\rangle$ is a connected multigraph different from a path for each $P \in \mathscr{P}$.

One of the two main results of the present paper is given by the next theorem:

Theorem 1. *Let G be a connected cubic multigraph. Assume that each edge of G belongs to a cycle of length $\leqslant 5$. Then*

$$|E_{\mathscr{P}}| \geqslant 2\big(|\mathscr{P}| - 1\big) \tag{1}$$

for every D-partition $\mathscr{P}$ of G.

The proof of Theorem 1 depends on the following lemma:

Lemma 1. *Let G be a connected cubic multigraph, and let $\mathscr{P}$ be a D-partition of G such that $|\mathscr{P}| \geqslant 2$. Assume that each edge of G belongs to a cycle of length $\leqslant 5$. Then there exists $\mathscr{R} \subseteq \mathscr{P}$ such that*

$$(2) \qquad |\mathscr{R}| \geqslant 2, \quad \Big\langle \bigcup_{R \in \mathscr{R}} R \Big\rangle \text{ is connected and } |E_{\mathscr{R}}| \geqslant 2\big(|\mathscr{R}| - 1\big).$$

Proof. If there exist distinct $P^*, P^{**} \in \mathscr{P}$ such that $|E_{\{P^*,P^{**}\}}| \geqslant 2$, then we put $\mathscr{R} = \{P^*, P^{**}\}$ and (2) holds.

Thus, we will assume that

$$(3) \qquad |E_{\{P',P''\}}| \leqslant 1 \quad \text{for any distinct } P', P'' \in \mathscr{P}.$$

We denote by $\mathscr{C}$ the set of all cycles C in G such that $E(C) \cap E_{\mathscr{P}} \neq \emptyset$ and $|E(C)| \leqslant 5$. If $C \in \mathscr{C}$, then, as follows from (3), $3 \leqslant |E(C) \cap E_{\mathscr{P}}|$. Let $A \subseteq E_{\mathscr{P}}$. We denote by $\mathscr{X}(A)$ and $\mathscr{Z}(A)$ the sets of all $P \in \mathscr{P}$ with the property that at least one vertex in P is incident with an edge in A and with the property that exactly one vertex in P is incident with an edge in A, respectively. If $P \in \mathscr{Z}(A)$, then the vertex in P incident with an edge in A will be denoted by $w(P, A)$. Finally, we denote by $\mathscr{Y}(A)$ the set of all $P \in \mathscr{Z}(A)$ such that $w(P, A)$ is incident with exactly two edges in A.

We shall construct infinite sequences $E_0, E_1, \ldots$ and $f_0, f_1, \ldots$ such that $E_i \subseteq E_{\mathscr{P}}$ and $f_i \in \{0, 1\}$ for every $i = 0, 1, \ldots$.

We put $f_0 = 0$. Consider an arbitrary $C^0 \in \mathscr{C}$ and put $E_0 = E(C^0) \cap E_{\mathscr{P}}$. Certainly, $\mathscr{Y}(E_0) = \mathscr{Z}(E_0)$. It follows from (3) that $\mathscr{Y}(E_0) \neq \emptyset$.

Let $i \geqslant 1$ and suppose that we have already constructed E_{i-1} and f_{i-1}. If $\mathscr{Y}(E_{i-1}) = \emptyset$ and $f_{i-1} = 0$, then we put $E_i = E_{i-1}$ and $f_i = 0$.

We shall assume that either (a) $\mathscr{Y}(E_{i-1}) \neq \emptyset$ and $f_{i-1} = 0$ or (b) $f_{i-1} = 1$. We first discuss (a). Consider an arbitrary $S_i \in \mathscr{Y}(E_{i-1})$. Put $F_i = \langle S_i \rangle$. Since F_i is connected and different from a path, we see that there exist an integer $g_i \geqslant 1$ and mutually distinct vertices $u_{i0}, \ldots, u_{ig_i} \in V(F_i)$ such that

$$u_{i0} = w(S_i, E_{i-1}), \ \deg_{F_i} u_{ig_i} = 3, \ \deg_{F_i} u_{ij} = 2 \quad \text{for each } j,$$
$$1 \leqslant j < g_i, \text{ and } u_{i0}u_{i1}, \ldots, u_{ig_i-1}u_{ig_i} \in E(F_i)$$

(note that $\deg_{F_i} u$ denotes the degree of u in F_i). Since G is cubic, we see that $|V(C) \cap V(F_i)| \geqslant 3$ for every cycle C in G such that $u_{ig_i-1}u_{ig_i} \in E(C)$. We put $h_i = 1$.

We now discuss (b), i.e. the case when $f_{i-1} = 1$. We put $S_i = S_{i-1}$, $g_i = g_{i-1}$, and $u_{ij} = u_{(i-1)j}$ for each j, $0 \leqslant j \leqslant g_i$. Moreover, we put $h_i = h_{i-1} + 1$.

In the sequel, we will not distinguish between (a) and (b). Consider an arbitrary $C^i \in \mathscr{C}$ such that $u_{ih_i-1}u_{ih_i} \in E(C^i)$. Put $E_i = E_{i-1} \cup \big(E(C^i) \cap E_{\mathscr{P}}\big)$. Clearly, $\big|V(C^i) \cap V(F_i)\big| \geqslant 2$. Moreover, if $h_i = g_i$, then $\big|V(C^i) \cap V(F_i)\big| \geqslant 3$. If $\big|V(C^i) \cap V(F_i)\big| \geqslant 3$, then we put $f_i = 0$. If $\big|V(C^i) \cap V(F_i)\big| = 2$, then we put $f_i = 1$.

Obviously, $E_0 \subseteq E_1 \subseteq \dots$. Since $E_{\mathscr{P}}$ is finite, it is easy to see that there exists $n > 0$ such that $E_n \neq E_{n-1}$ and $E_n = E_{n+j}$ for every $j \geqslant 0$. Consider an arbitrary $k \in \{1, \dots, n\}$. Since $E_k = E_{k-1} \cup \big(E(C^k) \cap E_{\mathscr{P}}\big)$, we see that $\mathscr{Y}(E_k) = \mathscr{X}(E_k)$. Moreover, it is easy to see that

(4) u is incident with an edge in E_{k-1} if and only if $u \in \{u_{k0}, \dots, u_{kh_k-1}\}$ for each $u \in S_k$.

We define

$$\begin{aligned} E^k &= E_k - E_{k-1}, \\ \mathscr{T}^k &= \big\{P \in \mathscr{X}(E_k) - \mathscr{Y}(E_k);\ P \notin \mathscr{X}(E_{k-1})\big\}, \\ \mathscr{U}^k &= \big\{P \in \mathscr{X}(E_k) - \mathscr{Y}(E_k);\ P \in \mathscr{Y}(E_{k-1}),\ P \neq S_k\big\}, \text{ and} \\ \mathscr{Y}^k &= \big\{P \in \mathscr{Y}(E_k);\ P \notin \mathscr{X}(E_{k-1})\big\}. \end{aligned}$$

We shall show that

$$|E^k| \geqslant 2|\mathscr{T}^k| + |\mathscr{U}^k| + |\mathscr{Y}^k| + 1 - f_k. \tag{5}$$

Clearly, $E^k \subseteq E(C^k)$. Combining (3) and (4) with the fact that G is cubic, we see that exactly one edge in E^k is incident with a vertex in S_k. Thus, $1 \leqslant |E^k| \leqslant 3$. Moreover, if $f_k = 0$, then $1 \leqslant |E^k| \leqslant 2$. Consider an arbitrary $P \in \mathscr{X}\big(E(C^k) \cap E_{\mathscr{P}}\big)$ such that $P \neq S_k$. Clearly, exactly two edges in $E(C^k) \cap E_{\mathscr{P}}$, say edges e' and e'', are incident with vertices in P. Without loss of generality we assume that if $e' \in E^k$, then $e'' \in E^k$. Let v' and v'' denote the vertices in P incident with e' and with e'', respectively. It is easy to show that

if $e', e'' \notin E^k$, then $P \notin \mathscr{T}^k \cup \mathscr{U}^k \cup \mathscr{Y}^k$,

if $e' \notin E^k$, $e'' \in E^k$ and $v' = v''$, then $P \notin \mathscr{T}^k \cup \mathscr{U}^k \cup \mathscr{Y}^k$,

if $e' \notin E^k$, $e'' \in E^k$ and $v' \neq v''$, then $P \notin \mathscr{T}^k \cup \mathscr{Y}^k$,

if $e' \in E^k$ and $v' = v''$, then $P \notin T^k$, and

if $v' \neq v''$, then $f_k = 1$.

It is now to see that (5) holds.

Obviously,

$$|E_0| \geqslant 2\big(|\mathscr{X}(E_0)| - 1\big) - |\mathscr{Y}(E_0)| - f_0. \tag{6}$$

Combining (5) and (6) and using induction on m, we get

$$|E_m| \geqslant 2\big(|\mathscr{X}(E_m)| - 1\big) - |\mathscr{Y}(E_m)| - f_m$$

for each $m \in \{0, \dots, n\}$.

It is clear that $\mathscr{Y}(E_n) = \emptyset$ and $f_n = 0$. We have

$$|E_n| \geqslant 2\big(|\mathscr{X}(E_n)| - 1\big).$$

Put $\mathscr{R} = \mathscr{X}(E_n)$. Obviously, $E_n \subseteq E_{\mathscr{R}}$ and $\mathscr{X}(E_{\mathscr{R}}) = \mathscr{R}$. Hence, (2) holds. The proof of the lemma is complete. □

Proof of Theorem 1. Let $\mathscr{P}$ be a D-partition of G. We proceed by induction on $|\mathscr{P}|$. If $|\mathscr{P}| = 1$, then $\mathscr{P} = \{V(G)\}$ and thus (1) holds. Let $|\mathscr{P}| \geqslant 2$. According to Lemma 1, there exists $\mathscr{R} \subseteq \mathscr{P}$ such that (2). Put

$$P_0 = \bigcup_{R \in \mathscr{R}} R \quad \text{and} \quad \mathscr{P}_0 = (\mathscr{P} - \mathscr{R}) \cup \{P_0\}.$$

Clearly, $\mathscr{P}_0$ is a D-partition of G and $|\mathscr{P}_0| < |\mathscr{P}|$. The induction hypothesis implies that $|E_{\mathscr{P}_0}| \geqslant 2\big(|\mathscr{P}_0| - 1\big)$. By virtue of (2), $|E_{\mathscr{R}}| \geqslant 2\big(|\mathscr{R}| - 1\big)$. Combining these facts we get (1), which completes the proof of the theorem. □

Remark 1. Let G be a connected multigraph, and let $\mathscr{P}$ be a partition of $V(G)$; we say that $\mathscr{P}$ is a C-partition of G if $\langle \mathscr{P} \rangle$ is a connected multigraph with at least two vertices, for each $P \in \mathscr{P}$. The concept of a C-partition was introduced in [7] for graphs and in [8] for multigraphs. In [7] a class of graphs with a certain local property was studied; it was proved that if G is a graph in that class, then (1) holds for every C-partition $\mathscr{P}$ of G. The same result was obtained for a larger class of multigraphs in [8]. On the other hand, the concept of a D-partition cannot be changed to that of a C-partition in Theorem 1. Fig. 1 shows a connected cubic graph G_1 such that each edge of G_1 belongs to a cycle of length $\leqslant 5$. Fig. 1 also shows a C-partition $\mathscr{P}$ of G_1: the edges in $E_{\mathscr{P}}$ are drawn by thick lines. We can see that $|E_{\mathscr{P}}| = 9$ and $2\big(|\mathscr{P}| - 1\big) = 10$.

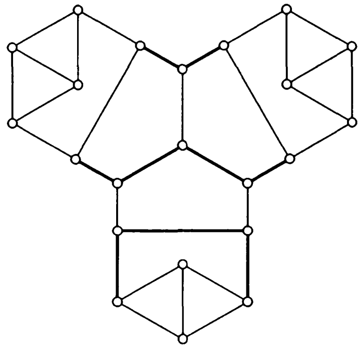

Figure 1

2. A connected pseudograph G is said to be upper embeddable if there exists a 2-cell embedding of G into the closed orientable surface of genus $\left[\frac{1}{2}\beta(G)\right]$, where

$$\beta(G) = |E(G)| - |V(G)| + 1.$$

The upper embeddability plays an important role in studying the maximum genus of a pseudograph (cf. [10] of Chapter 5 in [1]).

If F is a pseudograph, then we denote by $c(F)$ and $b(F)$ the number of all components of F and the number of the components H of F such that $\beta(H)$ is odd, respectively.

The following theorem will be useful for us:

Theorem A. *Let G be a connected pseudograph. Then the statements* (7), (8) *and* (9) *are equivalent:*

(7) *G is upper embeddable,*

(8) *there exists a spanning tree T of G with the property that at most one component of $G - E(T)$ has an odd number of edges,*

(9) *$c(G - A) + b(G - A) - 2 \leqslant |A|$ for each $A \subseteq E(G)$.*

The equivalence (7) $\Leftrightarrow$ (8) was proved in [5] and [11]; a similar result was proved in [4]. The equivalence (8) $\Leftrightarrow$ (9) was proved in [6]; a similar result was proved in [3].

Let G be a pseudograph, and let $\mathscr{P}$ be a partition of $V(G)$. We shall say that $\mathscr{P}$ is a B-partition of G if $\langle P\rangle$ is connected and $\beta(\langle P\rangle)$ is odd for every $P \in \mathscr{P}$.

Lemma 2. *Let G be a connected pseudograph. Then G is upper embeddable if and only if (1) holds for every B-partition $\mathscr{P}$ of G.*

P r o o f. Let G be upper embeddable. It follows from (9) that (1) holds for every B-partition $\mathscr{P}$ of G.

Conversely, let G be not upper embeddable. We denote $\tilde{y}(A) = c(G-A) + b(G-A) - 2 - |A|$ for each $A \subseteq E(G)$. There exists $A^* \subseteq E(G)$ such that $\tilde{y}(A^*) \geqslant \tilde{y}(A')$ for each $A' \subseteq E(G)$ and $\tilde{y}(A^*) > \tilde{y}(A'')$ for each proper subset A'' of A^*.

Denote

$$\mathscr{P} = \{P;\ \text{there exists a component } F \text{ of } G - A^* \text{ such that } P = V(F)\}.$$

Obviously, $\mathscr{P}$ is a partition of $V(G)$ and $E_{\mathscr{P}} \subseteq A^*$. The definition of A^* implies that $E_{\mathscr{P}} = A^*$.

Since G is not upper embeddable, it follows from (9) that $\tilde{y}(A^*) \geqslant 1$. Hence $b(G-A^*) \geqslant 2$. Assume that $b(G-A^*) < c(G-A^*)$. Since $b(G-A^*) > 0$, there exist $P_1, P_2 \in \mathscr{P}$ such that $\beta(\langle P_1 \rangle)$ is odd, $\beta(\langle P_2 \rangle)$ is even and $E_{\{P_1,P_2\}} \neq \emptyset$. Then

$$\tilde{y}(A^* - E_{\{P_1,P_2\}}) \geqslant \tilde{y}(A^*),$$

which is a contradiction. Thus $b(G-A^*) = c(G-A^*)$. This implies that $\mathscr{P}$ is a B-partition of G. We have $|A^*| < 2(|\mathscr{P}| - 1)$, which completes the proof. □

Note that a pseudograph is a multigraph if and only if it contains no loop. Obviously, if G is a cubic pseudograph such that each edge of G belongs to a cycle, then G is a multigraph.

Certainly, every B-partition of a multigraph is a D-partition. Thus, combining Theorem 1 and Lemma 2 we get the second main result of the present paper:

Theorem 2. *Let G be a connected cubic multigraph. If each edge of G belongs to a cycle of length $\leqslant 5$, then G is upper embeddable.*

Remark 2. Let G be a connected multigraph. We can see that if (1) holds for every C-partition of G, then G is upper embeddable. This fact was used in [7] and [8].

Remark 3. Fig. 2 shows a connected cubic graph G_2 such that each edge of G_2 belongs to a cycle of length $\leqslant 6$. Fig. 3 shows a connected graph G_3 with the maximum degree four and such that each edge of G_3 belongs to a cycle of length $\leqslant 5$. We see that neither G_2 nor G_3 are upper embeddable.

Remark 4. Glukhov [2] proved that if G is a 2-connected multigraph such that each edge of G belongs to a cycle of length $\leqslant 3$, then G is upper embeddable. It was shown in [7] that there exists a 2-connected graph G with the properties that G is not upper embeddable and each edge of G belongs to a cycle of length $\leqslant 4$.

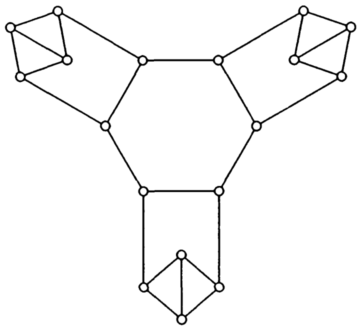

Figure 2

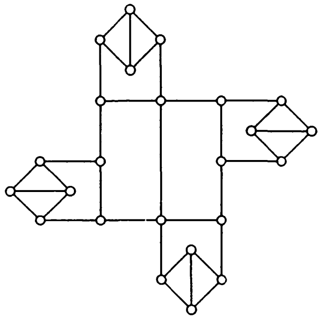

Figure 3

Remark 5. If G is a connected multigraph and k is a positive integer, then a 2-cell embedding ε of G into a closed orientable or nonorientable surface such that the length of the boundary of no region of ε is exceeding k will be called a k-embedding. Nedela and Škoviera [9] proved that if a connected multigraph has a 4-embedding, then it is upper embeddable. Moreover, Nedela and Škoviera [9] conjectured that if a connected multigraph has a 5-embedding, then it is upper embeddable, too. Let G be a connected cubic multigraph; it is not difficult to show that if G has a 5-embedding, then each edge of G belongs to a cycle of length $\leqslant 5$. Thus, as follows from Theorem 2, the above conjecture is correct for connected cubic multigraphs.

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