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ON SOME OPERATIONAL REPRESENTATIONS OF q-POLYNOMIALS

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1. INTRODUCTION

In an earlier paper [16] the present author defined the $T_{k,q,x}$-operator by the relation

$$T_{k,q,x} \equiv x(1-q)\{[k] + q^k x D_{q,x}\}, \tag{1}$$

where k is a constant, $|q| < 1$, $[k]$ is a q-number and $D_{q,x}$ is the q-derivative with respect to x.

The present paper gives applications of the $T_{k,q,x}$-operator in finding operational representations for certain q-polynomials. In a separate communication it has been demonstrated how successfully this operator can be used to obtain generating functions and recurrence relations for q-Laguerre and other polynomials.

Some of the results obtained in this paper are q-analogues of those obtained by Al-Salam [5], Mittal [19] and Rainville [20] while the rest are believed to be new.

2. DEFINITIONS AND NOTATION

For most of the definitions and the notation needed in this paper, the reader is referred to the papers by Agarwal and Verma [2], Hahn [9], Khan [13–18] and to the books by Exton [8] and Slater [21]. However, definitions of some q-polynomials are given below:

The q-Jacobi polynomials are defined by

$$J_n(q,\gamma,\beta;x) = \frac{(-1)^n (q^\gamma)_n q^{n\gamma+n(n-1)/2}}{(q^{\beta+n-1})_n} {}_2\varphi_1[q^{-n}, q^{\beta+n-1}; q^\gamma; q^{1-\gamma}x] \tag{1}$$

and

$$(2) \qquad P_{n,q}^{(\alpha,\beta)}(x) = \frac{(q^{1+\alpha})_n}{(q)_n} {}_2\varphi_1[q^{-n}, q^{1+\alpha+\beta+n}; q^{1+\alpha}; x].$$

Here the q-polynomial (2.1) is due to Hahn [10].

The q-Rice and generalized q-Rice polynomials are given by the relations

$$(3) \qquad H_{n,q}(\xi, p, x) = {}_3\varphi_2[q^{-n}, q^{1+n}, q^{\xi}; q, q^p; x]$$

and

$$(4) \qquad H_{n,q}^{(\alpha,\beta)}(\xi, p, x) = \frac{(q^{1+\alpha})_n}{(q)_n} {}_3\varphi_2[q^{-n}, q^{1+\alpha+\beta+n}, q^{\xi}; q^{1+\alpha}, q^p; x].$$

Further, the q-polynomial due to Al-Salam and Carlitz [7] is defined by

$$(5) \qquad U_n^{(a)}(x) = x^n \left(\frac{1}{x}\right)_n {}_1\varphi_1 \begin{bmatrix} q^{-n}; & -a \\ xq^{1-n}; & q \end{bmatrix}.$$

For $a = -1$ this polynomial gives the q-analogue of the Hermite polynomial.

Besides, the reader is referred to the papers by Jackson [12] and Khan [14] for q-Laguerre polynomials and Abdi [1] and Ismail [11] for q-Bessel polynomials.

3. Results used

Some of the results of Khan [16] required in this paper are listed below:

$$(1) \qquad T_{k,q}^n \, {}_r\varphi_s^{(q)}[(a_r); (b_s); x] = x^n (q^k)_n \, {}_{r+1}\varphi_{s+1}^{(q)}[(a_r), n+k; (b_s), k; x],$$

$$(2) \qquad T_{k,q}^n = x^n (1-q)^n \prod_{j=0}^{n-1} ([k+j] + q^{k+j} x D_g)$$

$$= x^n (1-q)^n \prod_{j=0}^{n-1} x^{-1} (1-q)^{-1} T_{k+j,q},$$

$$(3) \qquad F(T_{k,q})\{x^{\alpha} f(x)\} = x^{\alpha} F(T_{k+\alpha,q}) f(x),$$

$$(4) \qquad T_{k,q}^n \{u(x) v(x)\} = \sum_{r=0}^{n} \binom{n}{r}_q q^{kr} T_{k,q}^{n-r} v(q^r x) T_{0,q}^r u(x).$$

4. Operational representations

Here we give certain operational formulae and derive certain results for q-Laguerre polynomials. Besides, certain operational representations of some other q-polynomials will also be obtained.

Using (3.2) the following equivalent forms are obtained.

(1) $\{x(1-q^{\alpha})+q^{\alpha}T_{k,q}\}^n f(x) = T^n_{k+\alpha,q} f(x)$

$$= x^n(1-q)^n \prod_{j=0}^{n-1} x^{-1}(1-q)^{-1}T_{k+\alpha+j,q} f(x),$$

(2) $\{q^{\alpha}(1+x)T_{k,q}+x(1-q^{\alpha})-x^2q^{\alpha}\}^n f(x)$

$$= x^n(1-q)^n \prod_{j=0}^{n-1} \Big\{x(1+x)q^{k+\alpha+j}D_q - \frac{xq^{k+\alpha+j}}{1-q} + [k+\alpha+j]\Big\} f(x),$$

and

(3) $$\prod_{j=0}^{n-1} \Big\{q^{\alpha}T_{k,q} + x(1-q^{\alpha}) - \frac{x^2q^j}{1-q}\Big\} f(x)$$

$$= x^n(1-q)^n \prod_{j=0}^{n-1} \Big\{xq^{k+\alpha+j}D_q - \frac{xq^j}{1-q} + [k+\alpha+j]\Big\} f(x).$$

Formulae (4.2) and (4.3) are obtained by applying (4.1) to $e_q(-x)f(x)$ and $E_q(x)f(x)$, respectively.

Now the left hand side of (4.2) can also be written as

$$E_q(-x)T^n_{k+\alpha,q}\{e_q(-x)f(x)\} = x^{-\alpha}E_q(-x)T^n_{k,q}\{x^{\alpha}e_q(-x)f(x)\}.$$

Thus, we get the identity

(4) $T^n_{k,q}\{x^{\alpha}e_q(-x)f(x)\}$

$$= x^{\alpha+n}(q)_n e_q(-x) \sum_{r=0}^{n} \frac{(1+x)_r}{(q)_r x^r}\, {}_qL^{(\alpha+r)}_{n-r}(xq^{n+\alpha+k-1},1)T^r_{0,q} f(x).$$

Similarly, we have

(5) $T^n_{k,q}\{x^{\alpha}E_q(x)f(x)\}$

$$= x^{\alpha+n}(q)_n E_q(xq^n) \sum_{r=0}^{n} \frac{q^{r(k+r)}}{(q)_r x^r}\, {}_qL^{(\alpha+k-1)}_{n-r}(xq^n)T^r_{0,q} f(x).$$

Next, considering the operator ${}_0\varphi_1[-, q^{\alpha+k}; -tT_{k,q}]$, we obtain

$$(6) \qquad {}_0\varphi_1[-, q^{\alpha+k}; -tT_{k,q}]x^{\alpha+n} = x^{\alpha+n}{}_1\varphi_1[q^{k+\alpha+n}, q^{k+\alpha}; -xt].$$

One can also easily obtain the operational formulae

$$(7) \qquad {}_0\varphi_1[-; q^{\alpha+k}; T_{k,q}]\Big\{\frac{x^\alpha}{(1-xt)_{k+\alpha}}\Big\} = \frac{x^\alpha}{(1-xt)_{k+\alpha}}e_q\Big(\frac{x}{[1-xtq^{k+\alpha}]}\Big)$$

and

$$(8) \qquad {}_0\varphi_1\begin{bmatrix} ; & T_{k,q} \\ q^{k+\alpha}; & q \end{bmatrix}\Big\{\frac{x^\alpha}{(1-xt)_{k+\alpha}}\Big\} = \frac{x^\alpha}{(1-xt)_{k+\alpha}}E_q\Big(\frac{-xq}{[1-xtq^{k+\alpha}]}\Big).$$

As this stage we consider the following q-polynomials:

(A) *q-Laguerre Polynomials.* We shall obtain certain formulae and operational representations of q-Laguerre polynomials. Putting $f(x) = 1$ in (4.4) and (4.5) and taking different values of α and k, we get a number of operational representations for the q-Laguerre polynomials ${}_qL_n^{(\alpha)}(x, 1)$ and ${}_qL_n^{(\alpha)}(x)$, e.g.,

$$(9) \qquad T_{k,q}^n\{x^\alpha e_q(-x)\} = x^{\alpha+n}(q)_n e_q(-x)_q L_n^{(\alpha+k-1)}(xq^{n+\alpha+k-1}, 1),$$

$$(10) \qquad T_{k,q}^n\{x^\alpha E_q(x)\} = x^{\alpha+n}(q)_n E_q(xq^n)_q L_n^{(\alpha+k-1)}(xq^n)$$

are obtained by taking $f(x) = 1$ in (4.4) and (4.5).

By a simple change of variable, we also note that

$$(11) \qquad T_{k,q}^n\{x^\alpha e_q(-\lambda x)\} = x^{\alpha+n}(q)_n e_q(-\lambda x)_q L_n^{(\alpha+k-1)}(\lambda xq^{n+\alpha+k-1}, 1)$$

and

$$(12) \qquad T_{k,q}^n\{x^\alpha E_q(\lambda x)\} = x^{\alpha+n}(q)_n E_q(\lambda xq^n)_q L_n^{(\alpha+k-1)}(\lambda xq^n).$$

Now (4.11) and (4.12) can also be written as

$$(13) \quad \{q^\alpha(1+\lambda x)T_{k,q} + x(1-q^\alpha) - \lambda x^2 q^\alpha\}^n \cdot 1 = x^n(q)_n\, {}_qL_n^{(\alpha+k-1)}(\lambda xq^{n+\alpha+k-1}, 1)$$

and

$$(14) \qquad \{q^\alpha T_{k,q} + x(1-q^\alpha) - \lambda x^2\}^n \cdot 1 = x^n(q)_n\, {}_qL_n^{(\alpha+k-1)}(\lambda xq^n).$$

Further, (4.9) gives

$$T_{k,q}^m\{x^{\alpha+n}e_q(-x)_q L_n^{(\alpha+k-1)}(xq^{n+\alpha+k-1}, 1)\} = T_{k,q}^m\Big[\frac{1}{(q)_n}T_{k,q}^n\{x^\alpha e_q(-x)\}\Big].$$

Hence

$$T_{k,q}^m\{x^{\alpha+n}e_q(-x)_qL_n^{(\alpha+k-1)}(xq^{n+\alpha+k-1},1)\} \\ = \frac{(q)_{m+n}}{(q)_n}x^{\alpha+m+n}e_q(-x)_qL_{m+n}^{(\alpha+k-1)}(xq^{m+n+\alpha+k-1},1). \tag{15}$$

Similarly, (4.10) gives

$$T_{k,q}^m\{x^{\alpha+n}E_q(xq^n)_qL_n^{(\alpha+k-1)}(xq^n)\} \\ = \frac{(q)_{m+n}}{(q)_n}x^{\alpha+m+n}E_q(xq^{m+n})_qL_{m+n}^{(\alpha+k-1)}(xq^{m+n}). \tag{16}$$

Using the q-analogue of Kummer's transform (4.6) yields

$$_0\varphi_1[-;q^{\alpha+k};-tT_{k,q}]x^{\alpha+n} = x^{\alpha+n}e_q(-xt)_1\varphi_1\left[\begin{matrix} q^{-n}; & xtq^{n+\alpha+k-1} \\ q^{k+\alpha}; & q \end{matrix}\right]$$

which can alternatively be written as

$$_0\varphi_1[-;q^{\alpha+k};-tT_{k,q}]x^{\alpha+n} \\ = \frac{(q)_n}{(q^{k+\alpha})_n}x^{\alpha+n}e_q(-xt)_qL_n^{(\alpha+k-1)}(xtq^{n+\alpha+k-1},1). \tag{17}$$

Similarly,

$$_0\varphi_1\left[\begin{matrix} ; & -tT_{k,q} \\ q^{k+\alpha}; & q \end{matrix}\right]x^{\alpha+n} = \frac{(q)_nx^{\alpha+n}}{(q^{k+\alpha})_n}E_q(xtq^{1+n})_qL_n^{(\alpha+k-1)}(xtq^{1+n}). \tag{18}$$

Also, we have

$$\left(1+\frac{t}{T_{k,q}}\right)_n x^{-\alpha-k} = \frac{x^{-\alpha-k}(q)_n}{(q^{1+\alpha})_n}{}_qL_n^{(\alpha)}(tq^{\alpha+n}/x,1). \tag{19}$$

As an immediate consequence of the Leibniz formula (3.4) and the formula (4.9) we get

$$_qL_n^{(\alpha+\beta+k)}(xq^{n+\alpha+\beta+k},1) \\ = \sum_{r=0}^{n}\binom{\beta+r}{r}_q q^{r(k+\alpha)}(1+x)_r\ {}_qL_{n-r}^{(\alpha+k-1)}(xq^{\alpha+n+k-1},1), \tag{20}$$

and using (3.4) and (4.10) we obtain

$$_qL_n^{(\alpha+\beta+k)}(xq^n) = \sum_{r=0}^{n}\binom{\beta+r}{r}_q q^{r(k+\alpha)}{}_qL_{n-r}^{(\alpha+k-1)}(xq^n). \tag{21}$$

Formula (4.20) is obtained by putting $u = x^{1+\beta}$ and $v = x^{\alpha} e_q(-x)$ in (3.4), while (4.21) is obtained by putting $u = x^{1+\beta}$ and $v = x^{\alpha} E_q(x)$ in (3.4). On the other hand, if we put $u = x^{\beta} E_q(\mu, x)$, $v = x^{\alpha} e_q(-\lambda x)$ in (3.4) and then employ (4.11) and (4.12), we get the following addition-like theorem, involving the q-Laguerre polynomials ${}_qL_n^{(\alpha)}(x, 1)$ and ${}_qL_n^{(\alpha)}(x)$:

$$(22)\qquad {}_qL_n^{(\alpha+\beta+k-1)}([\lambda+\mu]xq^{n+\alpha+\beta+k-1}, 1) = \sum_{r=0}^{n} \frac{q^{r(k+\alpha)}(1+\lambda x)_r}{(1-\mu x)_r} {}_qL_{n-r}^{(\alpha+k-1)}(\lambda x q^{n+\alpha+k-1}, 1)\, {}_qL_r^{(\beta-1)}(\mu x q^r).$$

From (4.13) and the shift rule (3.3) we have the following formula:

$$(23)\qquad \frac{(q)_{m+n}}{(q^{k+\alpha})_n (q)_m} {}_qL_{m+n}^{(\alpha+k-1)}(xq^{m+n+\alpha+k-1}, 1) = \sum_{r=0}^{n} \binom{n}{r}_q \frac{(-x)^r q^{r(r+\alpha+k-1)}}{(q^{k+\alpha})_r} {}_qL_m^{(n+r+\alpha+k-1)}(xq^{m+n+r+\alpha+k-1}, 1).$$

Similarly, we obtain

$$(24)\qquad \frac{(q)_{m+n}}{(q)_m (q^{k+\alpha})_n} {}_qL_{m+n}^{(\alpha+k-1)}(xq^{m+n}) = \sum_{r=0}^{n} \frac{(q^{-n})_r x^r q^{rn}}{(q)_r (q^{k+\alpha})_r} {}_qL_m^{(n+r+\alpha+k-1)}(xq^m).$$

(B) *q-Bessel Polynomials.* Here we shall give three operational representations for q-Bessel Polynomials. One can obtain many others

$$(25)\qquad T^n_{c+n,q} e_q(q^{n+1}/x) = \frac{(q)_n}{(q^c)_n}(-1)^n q^{\frac{1}{2}n(n+1)} e_q(q/x) J(q; c, n; x).$$

To obtain (4.25), $e_q(q^{n+1}/x)$ is replaced by its equivalent infinite series and $T^n_{c+n,q}$ is operated on the variable x of the series. We then use the q-analogue of Kummer's transform and finally the resulting finite ${}_1\varphi_1$ series is written in reverse order.

Similarly, we also have

$$(26)\qquad T^n_{c,q} e_q\left(\frac{1}{x}\right) = \frac{(q)_n(-x)^n}{(xq)_n (q^{1-c})_n} q^{\frac{1}{2}n(n+1)-nc} e_q\left(\frac{1}{x}\right) J(q; c-n, n; xq^{n+1})$$

and

$$(27)\qquad T^n_{c-n,q} e_q\left(\frac{1}{x}\right) = \frac{(q)_n (q^{1-c})_n (-x)^n q^{\frac{1}{2}n(3n+1)-nc}}{(xq)_n (q^{1-c})_{2n}} e_q\left(\frac{1}{x}\right) J(q; c-2n, n; xq^{n+1}).$$

(C) *q-Jacobi Polynomials.* We give here the following operational representations for the q-Jacobi polynomials $J_n(q,\alpha,\beta;x)$ due to Hahn [10] and the q-Jacobi polynomials $P_{n,q}^{(\alpha,\beta)}(x)$:

$$T_{a,q}^n(1-xq^{1-a-n})_{n+b-a-1} = \frac{(xq^{1-a})_\infty (q^{b+n-1})_n(-x)^n}{(xq^{b-2a})_\infty q^{(1/2)n(n-1)+na}} J_n(q,a,b;x) \tag{28}$$

and

$$T_{a+1,q}^n(1-xq^{-n})_{b+n} = x^n(1-x)_b(q)_n P_{n,q}^{(a,b)}(x). \tag{29}$$

Also, we have

$$T_{a+1,q}^n(1-x)_b = \frac{(q)_n(1-xq^n)_\infty x^n}{(1-xq^b)_\infty} P_{n,q}^{(a,b-n)}(xq^n) \tag{30}$$

and

$$T_{1+a-n,q}^n(1-x)_b = \frac{(q)_n(1-xq^n)_\infty x^n}{(1-xq^b)_\infty} P_{n,q}^{(a-n,b)}(xq^n). \tag{31}$$

Relations (4.30) and (4.31) can alternatively be written as follows:

$$\begin{aligned} &T_{a,q}^n(1-x)_b \\ &\quad= \frac{(q^{a+b})_n(1-xq^b)_\infty(-x)^n}{q^{\frac{1}{2}n(n-1)+na}(1-xq^b)_\infty} J_n(q,a,1+a+b-n;xq^{n+a-1}), \end{aligned} \tag{32}$$

$$\begin{aligned} &T_{a-n,q}^n(1-x)_b \\ &\quad= \frac{(q^{a+b})_n(1-xq^n)_\infty(-x)^n q^{-na+n(n+1)/2}}{(1-xq^b)_\infty} J_n(q,a-n,1+a+b-n;xq^{a-1}). \end{aligned} \tag{33}$$

(D) *Generalized q-Rice Polynomials.* Using (3.1) we have the following operational representation for the generalized q-Rice polynomials $H_{n,q}^{(\alpha,\beta)}(\xi,p,x)$:

$$T_{1+\alpha,q}^{n+\beta}\ {}_2\varphi_1[q^{-n},q^\xi;q^p;x] = x^{n+\beta}(q^{1+\alpha+n})_\beta(q)_n H_{n,q}^{(\alpha,\beta)}(\xi,p,x). \tag{34}$$

If we put $\alpha=0=\beta$, (4.34) reduces to

$$T_{1,q}^n\ {}_2\varphi_1[q^{-n},q^\xi;q^p;x] = x^n(q)_n H_{n,q}(\xi,p,x). \tag{35}$$

(E) *A q-polynomial of Al-Salam and Carlitz.* One can easily obtain the following operational representation for $U_n^{(a)}(x)$:

$$(1-x)(-1)^n q^{n(n-1)/2} e_q\Big(\frac{1}{a}T_{1-n,q,xa}\Big) G_n(aq^{-1},q) = U_n^{(a)}(x) \tag{37}$$

where $G_n(x,q)$ is the Szegö polynomial defined by

$$G_n(x,q) = \sum_{r=0}^{n} \binom{n}{r}_q q^{r(r-n)}x^r. \tag{38}$$

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