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ON OPERATORS INDUCED BY WEAKLY 2-SINGULAR KERNELS

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In this note we give an estimate for the Weyl numbers of weakly 2-singular integral operators acting on $L_\infty(0,1)$. The result obtained here are related to those in [2], [3], [6, (3.a)] and [7].

In the following, all definitions concerning operators are adopted from [9] and [10].

Let $\mathcal{L}(E,F)$ denote the set of all (bounded linear) operators from the Banach space E into the Banach space F, which is a Banach space with the norm

$$\|T\| = \|T\colon E \to F\| := \sup\{\|Tx\|\colon \|x\| \leqslant 1\}.$$

For $1 \leqslant s \leqslant r < \infty$, an operator $T \in \mathcal{L}(E,F)$ is called absolutely (r,s)-summing, $T \in \Pi_{r,s}(E,F)$, if there exists a constant $c \geqslant 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^r\right)^{1/r} \leqslant c \sup\left\{\left(\sum_{i=1}^{n} |a(x_i)|^s\right)^{1/s} : \|a\| \leqslant 1, a \in E'\right\}$$

for every finite family of elements $x_1, \ldots, x_n \in E$. Then $\pi_{r,s}(T) := \inf c$ defined an ideal norm on $\Pi_{r,s}(E,F)$.

The n-th Weyl number of $T \in \mathcal{L}(E,F)$ is defined by

$$x_n(T) := \sup\{a_n(TS)\colon S \in \mathcal{L}(l_2,E), \|S\| \leqslant 1\},$$

where a_n are the approximation numbers. Then [9, (2.7.3)]

$$n^{1/q} x_n(T) \leqslant \pi_{q,2}(T) \quad \text{for all } T \in \Pi_{q,2}(E,F).$$

Let $2 \leqslant q < \infty$. A Banach space E is said to be of (Rademacher) cotype q if there exists a constant $k \geqslant 0$ such that

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqslant k \int_0^1 \left\|\sum_{i=1}^{n} r_i(t) x_i\right\| \,\mathrm{d}t$$

for all finite families of elements $x_1, \ldots, x_n \in E$, where r_i denotes the i-th Rademacher function. We put $K_q(E) := \inf k$.

Let (X, μ) be a measure space. For any measurable scalar-valued function f defined on X, the non-increasing rearrangement f^* is given by $f^*(t) := \inf\{c > 0\colon \lambda_f(c) \leqslant t\}$ for $t \geqslant 0$, where $\lambda_f(c) := \mu\{x \in X\colon |f(x)| > c\}$. The Lorentz function space $L_{2,1}(X, \mu)$ consists of all (equivalence classes of) measurable scalar-valued functions f such that

$$\|f\|_{2,1} := \int_0^\infty t^{-1/2} f^*(t)\, \mathrm{d}t$$

is finite. In this way we obtain a linear space which is complete with respect to the quasi-norm $\|.\|_{2,1}$. Since there exist equivalent norms, $L_{2,1}(X, \mu)$ even becomes a Banach space. For further information we refer to [1], [5], [8], [11] and [13]. We denote by $L_\infty(X, \mu)$ the set of all (equivalence classes of) measurable scalar-valued functions f which are essentially bounded on X, being a Banach space with the norm

$$\|f\|_\infty := \text{ess} - \sup\{|f(x)|\colon x \in X\}.$$

In the following we only consider the case when (X, μ) is the unit interval equipped with the Lebesgue measure, and the corresponding functions spaces are denoted by $L_{2,1}(0,1)$ and $L_\infty(0,1)$, but we can obtain an analogous result for suitable subsets of $\mathbb{R}^N$. Finally, for a compact Hausdorff space K, $C(K)$ denotes the Banach space of all continuous scalar-valued functions on K with the usual supremum norm.

Theorem. *Let K be defined on the unit square $[0,1] \times [0,1]$ a weakly 2-singular kernel of the form*

$$K(x,y) = \frac{L(x,y)}{|x-y|^{1/2}} \quad \text{if } x \neq y,$$

where K is measurable and $l \in L_{2,1}(0,1)$ with $l(y) := \sup_{x \in [0,1]} |L(x,y)|$. Then for every $q > 2$ the operator $T_K\colon L_\infty(0,1) \to L_\infty(0,1)$ defined by

$$T_K f(x) = \int_0^1 K(x,y) f(y)\, \mathrm{d}y$$

is absolutely $(q,2)$-summing and there is a constant $c_q > 0$ such that

$$\begin{aligned} n^{1/q} x_n\big(T_K\colon L_\infty(0,1) \to L_\infty(0,1)\big) &\leqslant \pi_{q,2}\big(T_K\colon L_\infty(0,1) \to L_\infty(0,1)\big) \\ &\leqslant 2(\sqrt{2}) c_q \|l\|_{2,1} K_q\big(L_{2,1}(0,1)\big) \end{aligned}$$

for $n = 1, 2, \ldots$.

Proof. For every $q > 2$ the Lorentz space $L_{2,1}(0,1)$ is of cotype q (see [4]), therefore the identity map of $L_{2,1}(0,1)$, denoted by $I_{2,1}$, is absolutely $(q,1)$-summing and $\pi_{q,1}(I_{2,1}) \leqslant K_q\big(L_{2,1}(0,1)\big)$. Then the multiplication operator $M_l\colon L_\infty(0,1) \to L_{2,1}(0,1)$, $f \to f \cdot l$, satisfies $M_l \in \Pi_{q,1}\big(L_\infty(0,1), L_{2,1}(0,1)\big)$ and since $L_\infty(0,1)$ can be identified with some Banach space $C(K)$, from [12, (§21)] we obtain $M_l \in \Pi_{q,2}\big(L_\infty(0,1), L_{2,1}(0,1)\big)$, and there is a constant $c_q > 0$ such that

$$
\begin{aligned}
\pi_{q,2}\big(M_l\colon L_\infty(0,1) \to L_{2,1}(0,1)\big) &\leqslant c_q \pi_{q,1}\big(M_l\colon L_\infty(0,1) \to L_{2,1}(0,1)\big)\\
&\leqslant c_q K_q\big(L_{2,1}(0,1)\big)\|M_l\colon L_\infty(0,1) \to L_{2,1}(0,1)\|\\
&\leqslant 2c_q \|l\|_{2,1} K_q\big(L_{2,1}(0,1)\big).
\end{aligned}
$$

For $x \in (0,1)$ let $g_x(y) := |x-y|^{-1/2}$. Then

$$\sup_{t>0} t^{1/2} g_x^*(t) = \sup_{y>0} y[\lambda_{g_x}(y)]^{1/2} \leqslant \sqrt{2}.$$

Put

$$\overline{K}(x,y) = \begin{cases} \frac{K(x,y)}{l(y)} & \text{if } l(y) > 0\\ 0 & \text{if } l(y) = 0. \end{cases}$$

For $f \in L_{2,1}(0,1)$, using that

$$\int_0^1 g_x(y)|f(y)|\,\mathrm{d}y \leqslant \int_0^\infty g_x^*(t) f^*(t)\,\mathrm{d}t$$

we obtain $\|T_{\overline{K}}\colon L_{2,1}(0,1) \to L_\infty(0,1)\| \leqslant \sqrt{2}$. Factorizing T_K as

$$L_\infty(0,1) \xrightarrow{M_l} L_{2,1}(0,1) \xrightarrow{T_{\overline{K}}} L_\infty(0,1)$$

we finally have

$$
\begin{aligned}
n^{1/q} x_n\big(T_K\colon L_\infty(0,1) \to L_\infty(0,1)\big) &\leqslant \pi_{q,2}\big(T_K\colon L_\infty(0,1) \to L_\infty(0,1)\big)\\
&\leqslant 2\big(\sqrt{2}\big) c_q \|l\|_{2,1} K_q\big(L_{2,1}(0,1)\big)
\end{aligned}
$$

for $n = 1, 2, \ldots$. □

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