Czechoslovak Mathematical Journal

Yin Xi Huang

Decaying positive entire solutions of the p-Laplacian

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 2, 205–220

Persistent URL: http://dml.cz/dmlcz/128523

DECAYING POSITIVE ENTIRE SOLUTIONS OF THE p-LAPLACIAN

YIN XI HUANG, Memphis

(Received January 16, 1992)

1. INTRODUCTION

We consider the following prototype of nonlinear equations

$$\Delta_p u + \varphi(x)|u|^{\lambda-1}u = 0 \tag{1.1}$$

in $\mathbb{R}^N$, where $1 < p < N$, $\lambda < p-1$, $N \geqslant 3$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\varphi(x) > 0$ is continuous. Our objective is to obtain conditions on φ such that (1.1) has a global decaying positive solution and further to describe the asymptotic behavior of the solution precisely.

For the case $p = 2$, (1.1) is the Emden-Fowler equation and has been intensively studied. Detailed global existence results and classification of solutions have been obtained by many authors. We refer to Kawano, Satsuma and Yotsutani [10], Kusano and Swanson [12, 13, 14], Li and Ni [16], Ni [17] and references therein. Much study has appeared recently for the case $p \neq 2$. Properties of global solutions of equations of the type (1.1) have been investigated by Ni and Serrin [18], and Friedman and Veron [6]. For $\varphi \equiv 1$ and $\lambda > p-1$, Guedda and Veron [9], Bidaut-Veron [3] studied local and global behavior of solutions of (1.1). For λ in certain range, classification of solutions is achieved. For the case where $1 < p \leqslant 2, N \geqslant 3$, existence of bounded positive solutions bounded away from zero for equation of the form $\Delta_p u + f(x,u) = 0$ is proved by Kura [11] via the subsolution-supersolution technique.

In this paper, we study the other half range of λ, i.e. $\lambda < p-1$. Suggested by the terminology for the case $p = 2$, we use singular, subhomogeneous, and superhomogeneous to denote the cases where $\lambda < 0$, $0 < \lambda < p-1$, and $\lambda > p-1$ respectively. We say a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ is an entire solution (supersolution, subsolution, respectively) of (1.1) (and its alike) if u satisfies

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v - \varphi|u|^{\lambda-1}uv = (\geqslant,\ \leqslant)0 \tag{1.2}$$

for all $v \in C_0^\infty(\mathbb{R}^N)$, cf. Kura [11, p.7]. Our main results assert the existence of such entire positive solutions which decay to zero, with emphasis on their asymptotic behavior. We will employ the classical Schauder fixed point theorem and the subsolution-supersolution technique established for Δ_p by Kura [11].

We start with equation (1.1) in Section 2. With proper assumptions postulated upon radially symmetric φ, we prove existence of entire positive solutions u such that $|x|^{\frac{N-p}{p-1}}u(x)$ is bounded above and below by positive constants. This result is further generalized to nonradial φ with suitable radial majorant. In Section 3 we study the following mixed subhomogeneous-superhomogeneous equation

$$\Delta_p u + \varphi(|x|)|u|^{\lambda-1}u + \psi(|x|)|u|^{\mu-1}u = 0,$$

where $\lambda < p-1 < \mu$, $\varphi > 0$, $\psi \geqslant 0$. Existence of decaying entire positive solutions is also obtained. In Section 4, we consider existence and asymptotic behavior of entire positive solutions of the more general problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x, u, \nabla u) = 0.$$

Finally we include in Appendix an application of a device due to Allegretto [1] to a general p-Laplace equation. More precisely, we obtain existence of infinitely many positive solutions bounded above and below by positive constants, and generalize Kura's result to the case $p > 2$.

Acknowledgement. The author thanks the referee for his careful refereeing and helpful suggestions and for pointing out an error in the original proof of Theorem 4.1. The author is grateful to Pavel Drábek for some stimulating conversations.

2. Subhomogeneous and singular cases

This section deals with existence of decaying positive solutions of the equation

$$\Delta_p u + f(x, u) = 0 \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where $1 < p < N$, $N \geqslant 3$ and $f(x, u)$ satisfies appropriate assumptions. We note that, by known regularity theory (cf. Tolksdorf [20]), if $p \neq 2$, solutions of (2.1) are in $C^{1+\alpha}$ (some $\alpha > 0$) in general, not in C^2.

We first study the following prototype of (2.1)

$$\Delta_p u + \varphi(|x|)|u|^{\lambda-1}u = 0 \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $\lambda < p-1$ is allowed to assume negative value and φ is continuous. If $y(|x|) > 0$ is an entire solution of (2.2), it is easily checked that y satisfies

$$(2.3)\qquad (r^{N-1}|y'|^{p-2}y')' + r^{N-1}\varphi(r)y^{\lambda}(r) = 0, \quad r \in [0,\infty),$$

with $r = |x|$. We will first concentrate on solvability of equation (2.3).

Throughout this paper we denote $\varrho(t) = \min\{1, t^{\frac{p-N}{p-1}}\}$ for $t \geqslant 0$ and $\mathbb{R}^+ = [0,\infty)$. We introduce the following closed subset Y of $C(\mathbb{R}^+)$:

$$(2.4)\qquad Y := \{y \in C(\mathbb{R}^+) \colon c_1\varrho(t) \leqslant y(t) \leqslant c_2\varrho(t)\},$$

where c_1 and c_2 are positive constants, to be determined *post priori*. We assume that $\varphi(t)$ satisfies the following

$$(2.5)\qquad \varrho_2^{p-1} := \int_0^{\infty} t^{N-1}\varphi(t)\varrho^{\lambda}(t)\,\mathrm{d}t < \infty, \quad \int_0^1 \varphi(t)\,\mathrm{d}t < \infty.$$

On Y, we define an operator $T\colon Y \to C(\mathbb{R}^+)$ by:

$$(2.6)\qquad u(r) = (Ty)(r) = \int_r^{\infty}\Big(\int_0^s (t/s)^{N-1}\varphi(t)y^{\lambda}(t)\,\mathrm{d}t\Big)^{1/(p-1)}\mathrm{d}s.$$

We first verify that T is well defined. Obviously $u(r)$ is a decreasing function of r. We claim that

$$(2.7)\qquad c_2^{-\frac{\lambda}{p-1}}u(0) \leqslant \varrho_1 := \int_0^{\infty}\Big(\int_0^s (t/s)^{N-1}\varphi\varrho^{\lambda}\,\mathrm{d}t\Big)^{1/(p-1)}\mathrm{d}s < \infty.$$

Indeed, we have

$$\begin{aligned}\int_0^{\infty}\Big(\int_0^s (t/s)^{N-1}\varphi\varrho^{\lambda}\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\mathrm{d}s \leqslant{}& \int_0^1\Big(\int_0^s \varphi\varrho^{\lambda}\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\mathrm{d}s\\ &+ \int_1^{\infty} s^{-\frac{N-1}{p-1}}\Big(\int_0^{\infty} t^{N-1}\varphi\varrho^{\lambda}\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\mathrm{d}s.\end{aligned}$$

(2.7) then follows from assumption (2.5) and the fact that $p < N$. Hence T is well defined for $y \in Y$.

It is our purpose to prove that T maps Y into itself and is continuous and compact, which will be achieved by a series of lemmas. Consequently the Schauder fixed point theorem implies existence of fixed point of T in Y, which is a solution of (2.3). First we have

Lemma 2.1. *For u given by (2.6), we have*
(i) $(r^{N-1}|u'|^{p-2}u')' = -r^{N-1}\varphi(r)y^{\lambda}(r)$, *and*
(ii) $\frac{u(r)}{r^{(p-N)/(p-1)}} \to \frac{p-1}{N-p}\left(\int_0^\infty t^{N-1}\varphi(t)y^{\lambda}(t)\,\mathrm{d}t\right)^{\frac{1}{p-1}}$ *as $r \to \infty$.*

This lemma follows from direct calculation and the proof is thus omitted.

Lemma 2.2. *$T\colon Y \to C(\mathbb{R}^+)$ is continuous and compact.*

Proof. For $u = Ty$ with $y \in Y$, we have

$$|u'(r)| = \left(\int_0^r (t/r)^{N-1}\varphi(t)y^{\lambda}(t)\,\mathrm{d}t\right)^{\frac{1}{p-1}}.$$

Observe that, for $r \in [0,1)$,

$$|u'(r)| \leqslant \left(\int_0^1 \varphi(t)y^{\lambda}(t)\,\mathrm{d}t\right)^{\frac{1}{p-1}} < \infty,$$

while for $r \geqslant 1$,

$$|u'(r)| \leqslant r^{-\frac{N-1}{p-1}}\left(\int_0^\infty t^{N-1}\varphi(t)\varrho^{\lambda}(t)\,\mathrm{d}t\right)^{\frac{1}{p-1}} c_2^{\frac{\lambda}{p-1}} < \infty,$$

i.e. T maps Y into a bounded subset of $C^1(\mathbb{R}^+)$ and consequently T is compact.

Since integration is a continuous operation, we conclude that T is continuous. The lemma is proved. □

Next we estimate $u = Ty$. Observe that

$$u(r) \leqslant c_2^{\frac{\lambda}{p-1}} \int_r^\infty \left(\int_0^s (t/s)^{N-1}\varphi\varrho^{\lambda}\,\mathrm{d}t\right)^{\frac{1}{p-1}} \mathrm{d}s. \tag{2.8}$$

Since $\lambda < p-1$, we can choose $c_2 > 0$ so large that

$$c_2^{1-\frac{\lambda}{p-1}} \geqslant \varrho_1.$$

From (2.8) we derive that

$$u(r) \leqslant c_2^{\frac{\lambda}{p-1}} \cdot c_2^{1-\frac{\lambda}{p-1}} = c_2 = c_2\varrho(r)$$

for $r \in [0,1]$.

For $r > 1$, we obtain

$$\begin{aligned} u(r) &\leqslant c_2^{\frac{\lambda}{p-1}} \int_r^\infty s^{-\frac{N-1}{p-1}}\left(\int_0^s t^{N-1}\varphi\varrho^{\lambda}\,\mathrm{d}t\right)^{\frac{1}{p-1}} \mathrm{d}s \\ &\leqslant c_2^{\frac{\lambda}{p-1}} \frac{p-1}{N-p} r^{\frac{p-N}{p-1}} \varrho_2. \end{aligned}$$

By choosing $c_2 > 0$ so large that

$$c_2^{1-\frac{\lambda}{p-1}} \geqslant \max(\varrho_1,\ \tfrac{p-1}{N-p}\varrho_2,\ 1), \tag{2.9}$$

we conclude that $u(r) \leqslant c_2\varrho(r)$ for all $r \in \mathbb{R}^+$.

On the other hand, for $y \in Y$, we also have

$$u(r) = Ty(r) \geqslant c_1^{\frac{\lambda}{p-1}} \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s.$$

It then follows that

$$u(r) \geqslant c_1^{\frac{\lambda}{p-1}} \int_1^\infty \Big(\int_0^s (t/s)^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s$$

for $r \in [0,1]$, and

$$\begin{aligned} u(r) &\geqslant c_1^{\frac{\lambda}{p-1}} \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s \\ &\geqslant c_1^{\frac{\lambda}{p-1}} \tfrac{p-1}{N-p} r^{\frac{p-N}{p-1}} \Big(\int_0^1 t^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}} \end{aligned}$$

for $r > 1$. Again by (2.5) we can choose $c_1 > 0$ so small that $u(r) \geqslant c_1\varrho(r)$ for $r \in \mathbb{R}^+$. Indeed, suffice it to take $c_1^{1-\frac{\lambda}{p-1}}$ as the minimum of the following three quantities:

$$\int_1^\infty \Big(\int_0^s (t/s)^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s,\ \tfrac{p-1}{N-p}\Big(\int_0^1 t^{N-1}\varphi\varrho^\lambda \,\mathrm{d}t\Big)^{\frac{1}{p-1}},\ \text{and } 1.$$

For c_1 and c_2 so chosen, we have

Lemma 2.3. *T maps Y into itself.*

Now by applying the Schauder fixed point theorem, we obtain the following

Theorem 2.4. *Under assumption* (2.5), *equation* (2.2) *admits a radial positive solution $y(r)$ satisfying $c_1\varrho(r) \leqslant y(r) \leqslant c_2\varrho(r)$.*

Remark 2.1. For $\varphi(t) \sim t^{-\alpha}$ with $\alpha > N - (N-p)\lambda/(p-1)$, (2.5) certainly is satisfied. By our assumption, necessarily we have $\alpha > p$. We note that for the case $p = 2$, $\alpha > 2$ (for $N \geqslant 3$) is a common restriction. See also Ni [17].

Returning to equation (2.1), we now assume that $f(x,u) = \varphi(x)|u|^{\lambda-1}u$, and denote

$$\overline{\varphi}(r) = \max_{|x|=r} \varphi(x), \quad \underline{\varphi}(r) = \min_{|x|=r} \varphi(x).$$

We further assume that $\overline{\varphi}$ satisfies (2.5). We then have

Theorem 2.5. *Under the above assumptions, equation* (2.1) *has an entire positive solution $y(x)$ such that $\varepsilon_1 \varrho(|x|) \leqslant y(x) \leqslant \varepsilon_2 \varrho(|x|)$ for some ε_1, $\varepsilon_2 > 0$.*

By well established theorems, see, e.g. Theorem 4.7 of Kura [11], if $\underline{y} \leqslant \overline{y}$ is a pair of sub-supersolution of (2.3) (see (1.2) for definition), then (2.3) has a solution y such that $\underline{y} \leqslant y \leqslant \overline{y}$. Now we can prove Theorem 2.5 as follows.

Proof of Theorem 2.5. Suffice it to construct a pair of sub-supersolution for (2.1). We observe that, by our assumption, both $\underline{\varphi}$ and $\overline{\varphi}$ satisfy (2.5). Theorem 2.4 then implies that there exist $\underline{y}$ and $\overline{y}$ such that

$$\Delta_p \overline{y} + \overline{\varphi}\overline{y}^\lambda = 0,$$

and

$$\Delta_p \underline{y} + \underline{\varphi}\ \underline{y}^\lambda = 0.$$

It then follows that $\overline{y}$ (resp., $\underline{y}$) is a supersolution (resp., subsolution) of (2.1). Since $\lambda < p-1$, we deduce that $\varepsilon \underline{y}$ is also a subsolution of (2.1) for any $0 < \varepsilon < 1$. We then take ε so small that $\varepsilon \underline{y} \leqslant \overline{y}$, which is possible by the asymptotic behavior of $\underline{y}$ and $\overline{y}$, derived from Theorem 2.4. This leads to the conclusion of the theorem. □

Remark 2.2. The case $0 < \lambda < p-1$ corresponds to the sublinear case when $p = 2$. We note that existence of decaying positive solutions has been obtained for the sublinear case by many authors. For more details, we refer to Kusano and Swanson [13], Kawano, Satsuma and Yotsutani [10], Li and Ni [16], and the survey paper by Ni [17]. In our consideration, negative value of λ is allowable, i.e. the singular case is included. Kusano and Swanson [12], Dalmasso [4] and Edelson [5] studied the singular equations for the case $p = 2$. Our results extend theirs to the more general case.

The technique we employed in the previous section is not amenable to deal with the case $\lambda > p-1$. However, for the case

$$f(x,u) = \varphi(|x|)|u|^{\lambda-1}u + \psi(|x|)|u|^{\mu-1}u$$

with $\varphi > 0$, $\psi \geqslant 0$, $\lambda < p-1 < \mu$, which is called the mixed sublinear-superlinear case for $p=2$, our method can induce similar result and this is our objective of this section.

We consider the following equation

$$\Delta_p u + \varphi(|x|)|u|^{\lambda-1}u + \psi(|x|)|u|^{\mu-1}u = 0 \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $\varphi > 0$, $\psi \geqslant 0$ are continuous, $\lambda < p-1 < \mu$. Assume that φ satisfies (2.5) and ψ satisfies

$$\varrho_3^{p-1} := \int_0^\infty t^{N-1}\psi\varrho^\mu \, \mathrm{d}t < \infty, \quad \int_0^1 \psi \, \mathrm{d}t < \infty. \tag{3.2}$$

Following the treatment of Section 2, we define an operator $J\colon Y \to C(\mathbb{R}^+)$ by

$$v(r) = (Jy)(r) = \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\big(\varphi y^\lambda + \psi y^\mu\big)\, \mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s. \tag{3.3}$$

We also have

Lemma 3.1. *For v defined by (3.2),*
(i) $(r^{N-1}|v'|^{p-2}v')' = -r^{N-1}(\varphi y^\lambda + \psi y^\mu)$,
(ii) $\frac{v(r)}{r^{(p-N)/(p-1)}} \to \frac{p-1}{N-p}\big(\int_0^\infty t^{N-1}\big(\varphi y^\lambda + \psi y^\mu\big)\, \mathrm{d}t\big)^{\frac{1}{p-1}}$ *as $r \to \infty$, and*
(iii) *J is continuous and compact.*

Next we show that, for properly chosen $c_1 > 0$ and $c_2 > 0$, J maps Y into itself. Indeed, for any $y \in Y$, we have

$$\begin{aligned} v(r) &= \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\big(\varphi y^\lambda + \psi y^\mu\big)\, \mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s \\ &\leqslant 2^{\frac{1}{p-1}}\Big(c_2^{\frac{\lambda}{p-1}} \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\varphi\varrho^\lambda \, \mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s \\ &\quad + c_2^{\frac{\mu}{p-1}} \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\psi\varrho^\mu \, \mathrm{d}t\Big)^{\frac{1}{p-1}} \mathrm{d}s\Big). \end{aligned}$$

For $r > 1$, calculation shows that

$$
\begin{aligned}
v(r) &\leqslant 2^{\frac{1}{p-1}} \tfrac{p-1}{N-p} r^{\frac{p-N}{p-1}} \Big(c_2^{\frac{\lambda}{p-1}} \Big(\int_0^\infty t^{N-1} \varphi \varrho^\lambda \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \\
&\quad + c_2^{\frac{\mu}{p-1}} \Big(\int_0^\infty t^{N-1} \psi \varrho^\mu \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \Big) \\
&= 2^{\frac{1}{p-1}} \tfrac{p-1}{N-p} r^{\frac{p-N}{p-1}} \big(c_2^{\frac{\lambda}{p-1}} \varrho_2 + c_2^{\frac{\mu}{p-1}} \varrho_3 \big).
\end{aligned}
\tag{3.4}
$$

Denote $g(t) = t^{\frac{\lambda}{p-1}-1} \varrho_2 + t^{\frac{\mu}{p-1}-1} \varrho_3$. It follows that at

$$
c_2 = \Big(\frac{(p-1-\lambda)\varrho_2}{(\mu-p+1)\varrho_3} \Big)^{\frac{p-1}{\mu-\lambda}},
\tag{3.5}
$$

function $g(t)$ achieves its minimum value

$$
\gamma = \varrho_2^{\frac{\mu-p+1}{\mu-\lambda}} \varrho_3^{\frac{p-1-\lambda}{\mu-\lambda}} \Big(\Big(\frac{p-1-\lambda}{\mu-p+1} \Big)^{\frac{\mu-p+1}{\mu-\lambda}} + \Big(\frac{\mu-p+1}{p-1-\lambda} \Big)^{\frac{p-1-\lambda}{\mu-\lambda}} \Big).
\tag{3.6}
$$

Assuming

$$
\gamma \leqslant 2^{-\frac{1}{p-1}} \tfrac{N-p}{p-1},
\tag{3.7}
$$

we then derive from (3.4) that $v(r) \leqslant c_2 \varrho(r)$ for $r > 1$. For $r \in [0,1]$, we observe that

$$
\begin{aligned}
v(r) &\leqslant 2^{\frac{1}{p-1}} \Big(c_2^{\frac{\lambda}{p-1}} \int_0^\infty \Big(\int_0^s (t/s)^{N-1} \varphi \varrho^\lambda \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \mathrm{d}s \\
&\quad + c_2^{\frac{\mu}{p-1}} \int_0^\infty \Big(\int_0^s (t/s)^{N-1} \psi \varrho^\mu \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \mathrm{d}s \Big).
\end{aligned}
$$

For c_2 given by (3.5), we assume that

$$
\begin{aligned}
&\int_0^\infty \Big(\int_0^s (t/s)^{N-1} \varphi \varrho^\lambda \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \mathrm{d}s \leqslant 2^{-1-\frac{1}{p-1}} c_2^{1-\frac{\lambda}{p-1}}, \\
&\int_0^\infty \Big(\int_0^s (t/s)^{N-1} \psi \varrho^\mu \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \mathrm{d}s \leqslant 2^{-1-\frac{1}{p-1}} c_2^{1-\frac{\mu}{p-1}},
\end{aligned}
\tag{3.8}
$$

then $v(r) \leqslant c_2$ for $r \in [0,1]$. We thus conclude that $v(r) \leqslant c_2 \varrho(r)$ for all $r \in \mathbb{R}^+$.

We now claim that, for $c_1 > 0$ small enough, for $r \in [0, \infty)$, $c_1 \varrho(r) \leqslant v(r)$. Indeed, since

$$
v(r) \geqslant c_1^{\frac{\lambda}{p-1}} \int_r^\infty \Big(\int_0^s (t/s)^{N-1} \varphi \varrho^\lambda \, \mathrm{d}t \Big)^{\frac{1}{p-1}} \mathrm{d}s,
$$

by the analysis of the previous section, we conclude that there exists a small $c_1 \leqslant c_2$ such that $c_1 \varrho(r) \leqslant v(r)$ for $r \in \mathbb{R}^+$. Thus we have proved

Lemma 3.2. *Assume that* (2.5) *and* (3.2) *hold. For c_2 defined by* (3.5), *assume moreover that* (3.7) *and* (3.8) *hold. Then for $c_1 > 0$ small enough, J maps Y into itself, where Y is defined by* (2.4).

We note that, if we replace φ and ψ by $\varepsilon\varphi$ and $\varepsilon\psi$ ($\varepsilon > 0$) respectively, c_2 defined by (3.5) does not change at all. Observe that ϱ_2 and ϱ_3 are increasing functions of φ and ψ, whence so is γ. Consequently by choosing $\varepsilon > 0$ small we can always have (3.7) and (3.8) satisfied. Thus we can state

Theorem 3.3. *Assume that* (2.5) *and* (3.2) *hold. Then there exists an $\varepsilon_0 > 0$, for any $\varepsilon \in (0, \varepsilon_0]$, the problem*

$$\Delta_p u + \varepsilon\varphi(|x|)|u|^{\lambda-1}u + \varepsilon\psi(|x|)|u|^{\mu-1}u = 0 \quad \text{in } \mathbb{R}^N$$

has an entire positive radial solution y_ε such that $c_1'\varrho(r) \leqslant y_\varepsilon(r) \leqslant c_2\varrho(r)$ for some $c_1' > 0$, where c_2 is given by (3.5).

Remark 3.1. $\varphi(t) \sim t^{-\alpha}$ with $\alpha > N - (N-p)\lambda/(p-1)$ and $\psi(t) \sim t^{-\delta}$ with $\delta > N - (N-p)\mu/(p-1)$ satisfy (2.5) and (3.2) respectively.

Remark 3.2. For $p = 2$, Kusano and Trench [15] studied a mixed sublinear-superlinear problem and obtained existence of radially symmetric positive solutions. Recently the same problem has been treated by Allegretto and Huang [2], and Furusho [7] when nonradially symmetric functions are involved. Our study here extends that of Kusano and Trench. However, for φ and ψ nonsymmetric, extension of results in Allegretto and Huang [2] and Furusho [7] is not obvious, and any results in this direction will be of interest.

4. General case

In this section we are concerned with the more general equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x, u, \nabla u) = 0 \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

Existence of bounded positive solutions bounded away from zero has been obtained by Kura [11]. See also Appendix. Here we will follow in principle the treatment of Kusano and Swanson [13], invoking the subsolution-supersolution method justified for Δ_p by Kura [11].

First we state the following basic assumptions:

(H1) $f\colon \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$ is locally Hölder continuous with Hölder exponent $\theta \in (0,1)$.

(H2) f satisfies a Nagumo type condition, i.e., for any $R > 0$, and any positive constants k, $K > 0$, there exists a positive constant C, depending on R, k, K, such that

$$f(x,u,\xi) \leqslant C \cdot (1 + |\xi|^{p-1}) \tag{4.2}$$

for $x \in B_R = \{x \in \mathbb{R}^N : |x| \leqslant R\}$, $k \leqslant u \leqslant K$, and $\xi \in \mathbb{R}^N$.

(H3) There exist locally θ-Hölder continuous functions $\underline{f}$ and $\overline{f}\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, such that $\underline{f}(t,1,0) > 0$ for some $t \in [0,1]$ and

$$\underline{f}(|x|,u,|\xi|) \leqslant f(x,u,\xi) \leqslant \overline{f}(|x|,u,|\xi|) \tag{4.3}$$

for all $(x,u,\xi) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N$.

(H4) For $t \geqslant 0$, $u > 0$, $v \geqslant 0$, $\underline{f}(t,u,v)$, $\overline{f}(t,u,v)$ are nondecreasing in both u and v for all $t > 0$, and for all $\alpha > 0$,

$$\overline{f}(t,\alpha u,\alpha v) = \alpha^{\lambda}\overline{f}(t,u,v), \quad \underline{f}(t,\alpha u,\alpha v) = \alpha^{\lambda}\underline{f}(t,u,v), \tag{4.4}$$

for $\lambda \in [0, p-1)$.

Remark 4.1. Nagumo type condition is needed for the subsolution-supersolution method. See also condition (4.1) of Kura [11] for a more general requirement. We note that (4.2) is more stringent than (H2) in Kusano and Swanson [13]. (4.4) can also be replaced by the similar condition (H4) given in [13]. We also remark that functions of the form $f(x,u,\nabla u) = q(x)|u|^{\gamma}|\nabla u|^{\delta}$ are precluded by (H3), where γ and δ are some positive constants.

We denote $t_* = \max\{1,t\}$ in the sequel.

Theorem 4.1. *Assume that* (H1)–(H4) *hold and that either*

(i) $p > 2$ *and for some* $\varepsilon \in (p-2, N-2)$,

$$\int_0^\infty t^{1+\varepsilon}\overline{f}(t,1,t^{-\frac{1}{p-1}})\,dt < \infty \tag{4.5}$$

or

(ii) *for some* $\varepsilon \in (p-1, N-1)$,

$$\int_0^\infty t^{\varepsilon}\overline{f}(t,1,t^{-\frac{\varepsilon}{p-1}})\,dt < \infty. \tag{4.5'}$$

Then (4.1) *has an entire positive decaying solution.*

Proof. (i) We define in this case two functions as follows:

$$y(r) = \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\overline{f}(t, c_2, c_2 t_*^{-\frac{1}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\,\mathrm{d}s, \tag{4.6}$$

$$z(r) = \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\underline{f}(t, c_1\varrho(t), 0)\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\,\mathrm{d}s, \tag{4.7}$$

where $0 < c_1 < c_2$ are two constants to be determined later.

It is easily checked that, $y(0) < \infty$, y is decreasing in r, and for r large,

$$\begin{aligned} y(r) &\leqslant c_2^{\frac{\lambda}{p-1}} \int_r^\infty s^{-\frac{1+\varepsilon}{p-1}} \Big(\int_0^s (t/s)^{N-2-\varepsilon} t^{1+\varepsilon}\overline{f}(t, 1, t_*^{-\frac{1}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\,\mathrm{d}s \\ &\leqslant c_2^{\frac{\lambda}{p-1}} \frac{p-1}{\varepsilon-p+2} r^{-\frac{\varepsilon-p+2}{p-1}} \Big(\int_0^\infty t^{1+\varepsilon}\overline{f}(t, 1, t_*^{-\frac{1}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}, \end{aligned}$$

hence $y(r) \to 0$ as $r \to \infty$.

Observe that, using the similar estimates as above,

$$\begin{aligned} |y'(r)| &\leqslant c_2^{\frac{\lambda}{p-1}} \Big(\int_0^1 \overline{f}(t, 1, 1)\,\mathrm{d}t\Big)^{\frac{1}{p-1}}, \quad r \in [0, 1], \\ |y'(r)| &\leqslant c_2^{\frac{\lambda}{p-1}} r^{-\frac{1+\varepsilon}{p-1}} \Big(\int_0^\infty t^{1+\varepsilon}\overline{f}(t, 1, t_*^{-\frac{1}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}} \\ &\leqslant c_2^{\frac{\lambda}{p-1}} r^{-\frac{1}{p-1}} \Big(\int_0^\infty t^{1+\varepsilon}\overline{f}(t, 1, t_*^{-\frac{1}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}, \quad r > 1. \end{aligned}$$

Hence $|y'(r)| < \infty$ by (H4) and (4.5). We thus conclude that we can choose c_2 large enough, such that

$$y(r) \leqslant c_2, \quad |y'(r)| \leqslant c_2 r^{-\frac{1}{p-1}}, \quad r \in \mathbb{R}^+. \tag{4.8}$$

For c_2 so chosen, we obtain, by (H4),

$$\begin{aligned} (r^{N-1}|y'|^{p-2}y')' &= -r^{N-1}\overline{f}(r, c_2, c_2 t_*^{-\frac{1}{p-1}}) \\ &\leqslant -r^{N-1}\overline{f}(r, y(r), |y'(r)|) \\ &\leqslant -r^{N-1} f(r, y(r), |y'(r)|). \end{aligned}$$

Consequently y is a supersolution of (4.1).

Similar arguments show that, for $c_1 < c_2$ small enough, we have

$$z(r) \geqslant c_1\varrho(r), \quad r \in \mathbb{R}^+. \tag{4.9}$$

Moreover, we have

$$\begin{aligned}(r^{N-1}|z'|^{p-2}z')' &= -r^{N-1}\underline{f}(r, c_1\varrho(r), 0)\\ &\geqslant -r^{N-1}\underline{f}(r, z(r), |z'(r)|)\\ &\geqslant -r^{N-1}f(r, z(r), |z'(r)|).\end{aligned}$$

Thus z is a subsolution of (4.1).

It follows from the definitions of y and z, $c_1 < c_2$ and (H4) that $z(r) \leqslant y(r)$ for all $r \in \mathbb{R}^+$. Theorem 4.5 of Kura [11] again implies the existence of a solution $u(x)$ such that

$$z(|x|) \leqslant u(x) \leqslant y(|x|).$$

The theorem is proved.

(ii) For this case, we define $z(r)$ as in (4.7), and define $y(r)$ as

$$y(r) = \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\overline{f}(t, c_2, c_2 t_*^{-\frac{\varepsilon}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\,\mathrm{d}s. \tag{4.6$'$}$$

One easily checks that the similar estimates hold. In particular, in the place of (4.8) one has

$$y(r) \leqslant c_2, \quad |y'(r)| \leqslant c_2 r^{-\frac{\varepsilon}{p-1}}, \quad r \in \mathbb{R}^+. \tag{4.8$'$}$$

Then the same arguments as in (i) are applicable here and the existence of a positive solution follows. □

Corollary 4.1. *Assume that f is independent of ∇u. If for some $\varepsilon \in (p-1, N-1)$,*

$$\int_0^\infty t^{\varepsilon}\overline{f}(t, 1)\,\mathrm{d}t < \infty,$$

then (4.1) has an entire positive decaying solution.

Theorem 4.2. *Assume that (H1)–(H4) hold and that*

$$\int_0^\infty t^{N-1}\overline{f}(t, \varrho(t), t_*^{\frac{1-N}{p-1}})\,\mathrm{d}t < \infty. \tag{4.10}$$

Then (4.1) has an entire positive solution which behaves as $\varrho(|x|)$ at infinity.

Proof. We define a new function

$$w(r) = \int_r^\infty \Big(\int_0^s (t/s)^{N-1}\overline{f}(t, c_3\varrho(t), c_3 t_*^{\frac{1-N}{p-1}})\,\mathrm{d}t\Big)^{\frac{1}{p-1}}\,\mathrm{d}s. \tag{4.11}$$

Following the same lines of arguments as in the proof of Theorem 4.1 we can prove that, for $c_3 > c_1$ properly chosen, $z(r)$, given by (4.7), and $w(r)$ form a pair of sub-supersolution. The theorem then follows. □

Remark 4.2. When reduced to $p = 2$, condition (4.5) is more stringent than condition (5) of Kusano and Swanson [13]. The reason is that we cannot change the order of integration in (4.6) for the general case $p \neq 2$.

Remark 4.3. In Kusano and Swanson [14], existence of radial entire solutions of quasilinear equations of the form $\nabla \cdot [g(|\nabla u|)\nabla u] = \lambda f(|x|, u)$ is considered. However, we note that our problem is precluded from their consideration since no suitable functions a and b for our case exist to satisfy conditions (2.1) and (2.2) there.

Appendix: Bounded solutions

We briefly sketch here the procedure of Allegretto [1] utilized for the p-Laplacian to obtain solutions bounded above and below by positive constants.

Let $\eta(x) = (1 + |x|^p)^{1/(p-1)}$, $q > N/p$, $p' = p/(p-1)$, and denote by $L_\eta^{p'}(D)$ the associated weighted $L^{p'}$ space defined on the domain D, with norm $\|\varphi\|_{L_\eta^{p'}(D)}^{p'} = \int_D \eta|\varphi|^{p'}$. For any $x \in \mathbb{R}^N$, let $B_i(x)$ (resp. B_i) be the ball in $\mathbb{R}^N$ centered at x (resp. at the origin) with radius i, and

$$H(\varphi, q, i, D) = \sup_{x \in D}\Big(\|\varphi\|_{L^q(B_i(x))}^{1/(p-1)}\Big). \tag{A.1}$$

We first have the following *a priori* estimate:

Lemma A.1. *Let $0 \leqslant f \in L_{loc}^q(\mathbb{R}^N) \cap L_\eta^{p'}(\mathbb{R}^N)$. Then for any $r > 0$, the problem*

$$-\Delta_p u = f(x) \quad \text{in} \quad B_r, \quad u = 0 \quad \text{on} \quad |x| = r$$

has a nonnegative solution $u_r \in W_0^{1,p}(B_r)$ and u_r satisfies

$$\|u_r\|_{L^\infty(B_r)} \leqslant C\Big(\|f\|_{L_\eta^{p'}}^{1/(p-1)} + H(f, q, 2, \mathbb{R}^N)\Big), \tag{A.2}$$

where C is independent of f, r and u_r.

This lemma can be proved using Theorem 1 of Serrin [19] and the following Hardy type inequality:

$$\Big(\int_{\mathbb{R}^N} |\varphi|^p(1 + |x|^p)^{-1}\,\mathrm{d}x\Big)^{1/p} \leqslant \frac{p}{N-p}\Big(\int_{\mathbb{R}^N} |\nabla\varphi|^p\,\mathrm{d}x\Big)^{1/p}, \tag{A.3}$$

for $1 < p < N$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Now we can state

Theorem A.1. *Let $f(x,u)$ be such that $f(x,u) \in L^q_{loc}(\mathbb{R}^N) \cap L^{p'}_\eta(\mathbb{R}^N)$ for any $u \in L^\infty(\mathbb{R}^N)$ and $F(x,a) = \sup_{0 \leqslant u \leqslant a} |f(x,u)|$ for $a > 0$. Assume*

$$\lim_{a \to 0} \frac{M(F(x,a))}{a} = 0 \quad \text{or} \quad \lim_{a \to +\infty} \frac{M(F(x,a))}{a} = 0, \tag{A.4}$$

where $M(v) := \|v\|^{1/(p-1)}_{L^{p'}_\eta} + H(v,q,2,\mathbb{R}^N)$. Then the problem $-\Delta_p u = f(x,u)$ has infinitely many solutions in $\mathbb{R}^N$ which are bounded above and below by positive constants.

Sketch proof. Without loss of generality, we assume $f(x,u) \geqslant 0$ for $u \geqslant 0$. (For the case where $f(x,u)$ changes sign for $u \geqslant 0$, we simply replace in the following proof $f(x,u)$ by $|f(x,u)|$, cf. Allegretto [1]).

For $0 < \varepsilon < 1$, $r > 0$ and $a > 0$, define

$$T(u) = (1-\varepsilon)a + (-\Delta_p)^{-1}(f(x,u))$$

on

$$K = L^p(B_r) \cap \{u\colon (1-\varepsilon)a \leqslant u \leqslant a \text{ a.e.}\}.$$

We note that $-\Delta_p(T(u)) = f(x,u)$ even though Δ_p is nonlinear. It is well known that $(-\Delta_p)^{-1}$ is a positive continuous compact operator on K, hence so is T. By assumption (A.4), one easily checks that T maps K into itself if a is sufficiently small (resp. large for the case $a \to +\infty$). Thus T has a fixed point in K by the Schauder fixed point theorem, i.e. for any $r > 0$, there exists a u_r such that $-\Delta_p u_r = f(x,u_r)$, $(1-\varepsilon)a \leqslant u_r \leqslant a$. To prove that such u_r converges to a solution u as $r \to \infty$, we follow the scheme of Allegretto [1].

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, and $\varphi \equiv 1$ on B_τ for some $\tau > 0$. We have

$$\|\nabla(\varphi u_r)\|^p_{L^p} \leqslant c\Big(\int |\nabla\varphi|^p \cdot u_r^p \,\mathrm{d}x + \int |\nabla u_r|^p \cdot \varphi^p \,\mathrm{d}x\Big). \tag{A.5}$$

We note that the first term on the right hand side can be bounded by a constant independent of u_r and r. To estimate the second term, we observe that for r large such that $\operatorname{supp}\varphi \subset B_r$,

$$\int |\nabla u_r|^{p-2}\nabla u_r \nabla(\varphi^p u_r)\,\mathrm{d}x = \int f(x,u)\varphi^p u_r \,\mathrm{d}x.$$

Therefore,

(A.6)

$$\begin{aligned}\int |\nabla u_r|^p\varphi^p \,\mathrm{d}x &\leqslant \int |f(x,u)|\varphi^p a \,\mathrm{d}x + p\int |\nabla u_r|^{p-1}\varphi^{p-1}\nabla\varphi \,\mathrm{d}x\\ &\leqslant c\int |F(x,a)|\varphi^p \,\mathrm{d}x + \delta^{p'} p\int |\nabla u_r|^p\varphi^p \,\mathrm{d}x + \delta^{-p} p\int |\nabla\varphi|^p \,\mathrm{d}x.\end{aligned}$$

Taking δ such that $\delta^{p'}p = 1/2$ and combining (A.6) with (A.5), we conclude that $\{\varphi u_r\}$ is bounded in $W^{1,p}$. First letting $r \to \infty$ and then $\tau \to \infty$ we conclude that there exists a $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ such that $\|u_r - u\|_{L^p(B_R)} \to 0$ for any $R > 0$. Analogously we estimate

$$\|\nabla(\varphi(u_r - u_t))\|^p_{L^p} \leqslant c\Big(\int |\nabla\varphi|^p \cdot |u_r - u_t|^p \,\mathrm{d}x + \int |\nabla(u_r - u_t)|^p \cdot \varphi^p \,\mathrm{d}x\Big).$$

Multiplying $\varphi^p(u_r - u_t)$ to $-\Delta_p u_r = f(x, u_r)$ and $-\Delta_p u_t = f(x, u_t)$, integrating by parts and subtracting the resulted equations, we obtain

$$\begin{aligned}\int \big(|\nabla u_r|^{p-2}\nabla u_r - |\nabla u_t|^{p-2}\nabla u_t\big)\nabla(\varphi^p(u_r - u_t)) \,\mathrm{d}x \\ = \int \big(f(x, u_r) - f(x, u_t)\big)\varphi^p(u_r - u_t) \,\mathrm{d}x.\end{aligned}$$

Thus

$$\begin{aligned}\int |\nabla(u_r - u_t)|^p \varphi^p \,\mathrm{d}x \leqslant \int |f(x, u_r) - f(x, u_t)|\varphi^p |u_r - u_t| \,\mathrm{d}x \\ + p\int \big(|\nabla u_r|^{p-1} + |\nabla u_t|^{p-1}\big)\varphi^{p-1}|\nabla\varphi| \,\mathrm{d}x.\end{aligned}$$

Estimating as in (A.6) we conclude

$$\|\nabla(\varphi(u_r - u_t))\|^p_{L^p} \leqslant C_1\|u_r - u_t\|^p_{L^p(\operatorname{supp}\varphi)} + C_2\|u_r - u_t\|_{L^p(\operatorname{supp}\varphi)}.$$

Consequently we obtain the convergence of $\{\varphi u_r\}$ in $W^{1,p}$, and the limit function u is the required solution. We iterate the procedure by replacing a by $(1 - \varepsilon)a$ for the case $a \to 0$ and $(1 - \varepsilon)a$ by a for the case $a \to \infty$, and thus obtain infinitely many solutions. This completes the proof. □

Example. For $f(x, u) = (1 + |x|^\alpha)|u|^{\delta-1}u$, with $\delta > p - 1$ or $0 < \delta < p - 1$, and $\alpha < -N - 1 + N/p$, equation $-\Delta_p u = f(x, u)$ has infinitely many solutions bounded above and below by positive constants. We remark that here we lift the restriction $1 < p \leqslant 2$ posted in Kura [11].

References

[1] *W. Allegretto*: On positive L^∞ solutions of a class of elliptic systems. Math. Z. *191* (1986), 479–484.

[2] *W. Allegretto and Y.X. Huang*: Existence of positive global solutions of mixed sublinear-superlinear problems. Can. J. Math. *40* (1988), 1222–1242.

[3] *M.-F. Bidaut-Veron*: Local and global behavior of solutions of quasilinear equations of Emden-Fowler type. Arch. Rat. Mech. Anal. *107* (1989), 293–324.

[4] *R. Dalmasso*: Solutions positives globales d'équations elliptiques semi-linéaires singulières. Bull. Sc. Math. Ser. 2, *112* (1988), 65–76.

[5] *A.L. Edelson*: Entire solutions of singular elliptic equations. J. Math. Anal. Appl. *139* (1989), 523–532.

[6] *A. Friedman and L. Veron*: Singular solutions of some quasilinear elliptic equations. Arch. Rat. Mech. Anal. *96* (1986), 259–287.

[7] *Y. Furusho*: On decaying entire positive solutions of semilinear elliptic equations. Japan. J. Math. *14* (1988), 97–118.

[8] *D. Gilbarg and N.S. Trudinger*: Elliptic Partial Differential Equations of Second Order, 2nd edition. Springer-Verlag, N.Y., 1983..

[9] *M. Guedda and L. Veron*: Local and global properties of solutions of quasilinear elliptic equations. J. Diff. Equa. *76* (1988), 159–189.

[10] *N. Kawano, J. Satsuma and S. Yotsutani*: Existence of positive entire solutions of an Emden-type elliptic equation. Funkcialaj Ekvacioj *31* (1988), 121–145.

[11] *T. Kura*: The weak supersolution-subsolution method for second order quasilinear elliptic equations. Hiroshima Math. J. *19* (1989), 1–36.

[12] *T. Kusano and C.A. Swanson*: Entire positive solutions of singular semilinear elliptic equations. Japan. J. Math. *11* (1985), 145–155.

[13] *T. Kusano and C.A. Swanson*: Decaying entire positive solutions of quasilinear elliptic equations. Mh. Math. *101* (1986), 39–51.

[14] *T. Kusano and C.A. Swanson*: Radial entire solutions of a class of quasilinear elliptic equations. J. Diff. Eq. *83* (1990), 379–399.

[15] *T. Kusano and W.F. Trench*: Global existence of solutions of mixed sublinear-superlinear differential equations. Hiroshima Math. J. *16* (1986), 597–606.

[16] *Y. Li and W.M. Ni*: On conformal scalar curvature equations in R^n. Duke Math. J. *57* (1988), 895–924.

[17] *W.M. Ni*: Some aspects of semilinear elliptic equations in R^n. Nonlinear Diffusion Equations and Their Equalibrium States, Vol. 2 W.M. Ni (L.A. Peletier and J. Serrin, eds.). 1988, pp. 171–205.

[18] *W.M. Ni and J. Serrin*: Nonexistence theorems for singular solutions of quasilinear partial differential equations. Comm. Pure Appl. Math. *38* (1986), 379–399.

[19] *J. Serrin*: Local behavior of solutions of quasilinear equations. Acta Math. *111* (1964), 247–302.

[20] *P. Tolksdorf*: On the Dirichlet problem for quasilinear equations in domains with conical boundary points. Comm. P.D.E. *8* (1983), 773–817.

Author's address: Department of Mathematical Science, University of Memphis, Memphis, TN 38152, U. S. A.